Spurious Regressions with Stationary Gegenbauer Processes and Harmonic Processes *

Yixiao Sun
Department of Economics
University of California, San Diego

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Abstract

This paper studies the spurious regressions among stationary Gegenbauer processes, stationary harmonic processes and deterministic trigonometric series. We find the spurious regression can occur between two stationary Gegenbauer processes, as long as their generalized fractional differencing parameters sum up to a value greater than 0.5 and their spectral densities have poles at the same location. The spurious regression may also be present between a stationary Gegenbauer process and a stationary harmonic process, or between a stationary Gegenbauer process and a deterministic trigonometric series, as long as the poles of the discrete Fourier transforms or the spectral densities of underlying processes are located at the same frequency. Our findings suggest that it is the strong persistence and cyclical comovement that cause the spurious effect. Our theoretical results are supported by simulations.

1   Introduction

Since the first Monte Carlo study by Granger and Newbold (1974), much effort has been taken to understand the nature of spurious regressions. Phillips (1986) developed an asymptotic theory for a regression between $I(1)$ processes showing that the asymptotic distributions of the conventional statistics are quite different from those derived under the assumption of stationarity. He proved that the usual t statistic does not have a limiting distribution but diverges as the sample size increases, and that the R^2 has a non-degenerate limiting distribution while the DW statistic converges to zero. Extending Phillips' (1986) approach, Durlauf and Phillips (1988) studied the spurious regression between an $I(1)$ process and a linear trend, while Marmol (1995, 1998) examined the spurious regression between two nonstationary $I(d)$ processes with $d > 1/2$. It has long been believed that it is the deterministic or stochastic trending behavior that causes spurious effects. However, Tsay and Chung (2000) found that a spurious effect can arise in a regression between two stationary $I(d)$ processes, as long as their orders of integration sum up to a value greater than 0.5. Based on this, they suggested that it is long memory or strong dependence, rather than deterministic or stochastic trending, that causes the spurious effects.

This paper explores the possible existence of a spurious relationship between two stationary Gegenbauer processes. This class of process generalizes the popular fractionally integrated processes and is capable of capturing long-range dependence, as well as the periodic cyclical pattern. It has been shown that the autocovariances of a Gegenbauer process decay hyperbolically and sinusoidally, a feature that is manifested in a number of financial and economic time series. One of the purposes of this paper is to extend the theoretical analysis of the spurious regression from $I(d)$ processes to the class of Gegenbauer process. We find that the spurious regression can occur between two stationary Gegenbauer processes. This provides further evidence that the spurious effect is not unique to nonstationary processes, as suggested by Tsay and Chung (2000). We show that strong dependence is not enough to give rise to the spurious effect. Another indispensable condition is that the two processes have spectral densities with poles at the same location. In other words, the two processes are required to have common cyclical movements. The cyclical comovement of economic variables is the rule rather than exception. As suggested by Lucas (1977), the main regularities observed in cyclical fluctuations of economic time series are in their comovement. The strong persistence and cyclical comovement in many economic time series suggest that the spurious effect may be present more often than we previously believed.

This paper also investigates the possible existence of a spurious relationship between a stationary Gegenbauer process and a stationary harmonic process. A harmonic process is defined to be a stochastic trigonometric series with a superimposed white noise. A salient feature of this type of process is that the spectral density function contains a spike at the jump point, or at the location of the pole. Our findings show that the spurious effect can occur between a stationary Gegenbauer process and a harmonic process, as long as the poles of their spectral densities are located at the same frequency. As a corollary, the spurious relationship may be present between a Gegenbauer process and a deterministic sine or cosine wave. This result is comparable to the spurious regression between a stationary fractional process and a polynomial trend. Instead of using a polynomial trend, we use a trigonometric series to reflect the cyclical behavior in the dependent variable.

Our theoretical results are supported by Monte Carlo simulations. The rejection probability for testing the null of zero slope coefficient based on the usual t-test increases with the sample size and the persistence of the dependent and/or independent variables.

The remainder of the paper is organized as follows. Section 2 briefly describes the harmonic process and the Gegenbauer process. Section 3 investigates the spurious regression between two Gegenbauer processes. Section 4 develops a theory for the spurious regression between a Gegenbauer process and a harmonic process, and that between a Gegenbauer process and a deterministic trigonometric series. A simulation study is presented in Section 5. Section 6 concludes. All proofs are given in the appendix.

2 Harmonic Processes and Gegenbauer Processes

Many economic time series exhibit the periodic cyclical behavior. The search for cyclical components, their estimation and testing, is of undoubted interest. A well-known model capable of generating such a periodic behavior is the harmonic process defined by

$$x_t = \rho \cos(\psi t + \theta) + \varepsilon_t, \tag{1}$$

where ρ and ψ are constants, θ is a uniform random variable on $[-\pi, \pi]$, ε_t is $iid(0, \sigma_\varepsilon^2)$ and is independent of θ. This definition is the same as that in Priestley (1981, p. 147) and Fuller (1996, p. 145) except that we consider only one cosine term for simplicity and we add a white noise sequence to the cosine wave, as in Brockwell and Davis (1991, p. 334). The harmonic process as so defined is generated by a stochastic trigonometric series with a superimposed white noise. From the historical point of

view, harmonic processes were probably the first to be considered in time series analysis. Early studies attempt to describe a time series as the sums of sine and cosine waves whose amplitudes and frequencies are chosen so as to give the best fit to the data. More recently, harmonic processes have been used extensively in detecting hidden frequencies in a time series. Different techniques have been proposed for this purpose. See, for example, Hannan (1973), Chen (1988), Quinn (1989) and Kavalieris and Hannan (1994).

The harmonic process in (1) can be written as

$$x_t = A \cos \psi t + B \sin \psi t + \varepsilon_t \tag{2}$$

with $A = \rho \cos \theta$ and $B = -\rho \sin \theta$. It is easy to see that A and B are uncorrelated random variables with mean zero and variance $\sigma^2 = \rho^2/2$. With this, x_t can be shown to be covariance stationary with mean zero and covariance

$$\gamma(j) = E x_t x_{t+j} = 1/2\rho^2 \cos \psi j + \sigma_\varepsilon^2 1\{j = 0\}, \tag{3}$$

where $1\{\cdot\}$ is the indicator function. The essential feature to be noted is that the autocovariance function is a cosine wave and never dies out. This is a very important feature that is responsible for the spurious regression between a stationary Gegenbauer process and a harmonic process. Given the form of the autocovariance function, it is readily seen that the spectral density of x_t has a spike at $\lambda = \psi$. Specifically,

$$f(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi} + \frac{\rho^2}{4} \left(\delta \left(\lambda + \psi \right) + \delta \left(\lambda - \psi \right) \right), \tag{4}$$

where $\delta \left(\cdot \right)$ is the Dirac δ-function satisfying $\delta \left(x \right) = 0$ for $x \neq 0$ and $\int_{-\infty}^{\infty} \delta \left(x \right) g(x) dx = g(0)$ for any well-behaved function $g \left(\cdot \right)$. To be consistent with the terminology used hereafter, we also say that the spectral density has a pole at $\lambda = \psi$.

A second process capable of generating the cyclical pattern, as well as long range dependence, is the Gegenbauer process (Gray, Zhang and Woodward 1989, 1994; Chung 1996a, 1996b) defined as

$$(1 - 2\cos(\varphi)L + L^2)^d x_t = \varepsilon_t, \tag{5}$$

where d is the generalized fractional differencing parameter and $\varepsilon_t = iid(0, \sigma_\varepsilon^2)$. The moving average representation of the above Gegenbauer process involves the Gegenbauer polynomial:

$$(1 - 2\cos(\varphi)L + L^2)^{-d} = \sum_{j=0}^{\infty} c_j L^j, \tag{6}$$

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where $c_0 = 1, c_1 = 2d \cos \varphi$ and

$$c_j(d, \varphi) = 2 \cos \varphi \left((d-1)/j + 1\right) c_{j-1}(d, \varphi) - (2(d-1)/j + 1) c_{j-2}(d, \varphi). \quad (7)$$

Using the Gegenbauer representation, the process x_t can then be formally written as

$$x_t = \sum_{j=0}^{\infty} c_j(d, \varphi) \varepsilon_{t-j}. \quad (8)$$

The Gegenbauer process is stationary if $0 < \varphi < \pi$ and $d < 1/2$ and is invertible if $0 < \varphi < \pi$ and $d > -1/2$. When $\varphi = 0, \pi$, the Gegenbauer process is stationary if $d < 1/4$ and is invertible if $d > -1/4$. Note that when $\cos \varphi = 1$, the Gegenbauer process reduces to a fractionally integrated process $I(2d)$. In this paper, we focus on the case that $0 < \varphi < \pi$ so that the Gegenbauer process exhibits some periodic cyclical behavior. In this case, the autocovariance function for a stationary Gegenbauer process can be approximated by

$$\gamma(j) \sim K j^{2d-1} \cos (\varphi j) \quad (9)$$

for some constant K (Chung, 1996a), where the symbol $\sim$ means that the ratio of the two sides tends to 1, as $j \to \infty$. The autocovariance function thus resembles a hyperbolically damped cosine wave. In view of (3), the autocovariance functions for the harmonic process and the Gegenbauer process share a similar cyclical pattern.

It is easy to show that the spectral density of a Gegenbauer process is

$$f(\lambda) = \frac{\sigma^2}{2\pi} |4 \sin(\frac{\lambda + \varphi}{2}) \sin(\frac{\lambda - \varphi}{2})|^{-2d}. \quad (10)$$

Therefore, when $0 < d < 1/2$, we have

$$f(\lambda) \sim C|\lambda^2 - \varphi^2|^{-2d} \; as \; \lambda \to \varphi, \quad (11)$$

for some positive constant C. The spectral density function has a pole at $\lambda = \varphi$. Therefore, both the harmonic process and the Gegenbauer process have poles in their spectral densities. We will show that the pole properties lead to the spurious effect between a Gegenbauer process and a harmonic process in Section 4.

3 Spurious Regression Between Stationary Gegenbauer Processes

In this section, we consider the spurious regression between two stationary Gegenbauer processes. Suppose that x_t and y_t follow independent Gegenbauer processes

such that

$$(1 - 2\cos\varphi_x L + L^2)^{d_x} x_t \;=\; \varepsilon_{xt}, \tag{12}$$

$$(1 - 2\cos\varphi_y L + L^2)^{d_y} y_t \;=\; \varepsilon_{yt}, \tag{13}$$

where $0 < \varphi_x, \varphi_y < \pi, |d_x| < 1/2, |d_y| < 1/2$ and $\varepsilon_{xt}, \varepsilon_{yt}$ are $iid(0, \sigma_{\varepsilon x}^2)$ and $iid(0, \sigma_{\varepsilon y}^2)$, respectively.

Consider regressing y_t on a constant and x_t,

$$y_t = \widehat{\alpha} + \widehat{\beta} x_t + \widehat{u}_t, t = 1, ..., T. \tag{14}$$

The ordinary least squares estimate of β is given by

$$\widehat{\beta} = \frac{\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^{T}(x_t - \bar{x})^2}, \tag{15}$$

where $\bar{x} = \sum_{t=1}^{T} x_t/T$ and $\bar{y} = \sum_{t=1}^{T} y_t/T$. The t-statistic is $\widehat{t}_{\beta,OLS} = \widehat{\beta}/\widehat{\sigma}_{\beta,OLS}$ where

$$\widehat{\sigma}_{\beta,OLS} = \widehat{\sigma}_u \left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right)^{1/2}, \tag{16}$$

$\widehat{\sigma}_u^2 = \sum_{t=1}^{T} \widehat{u}_t^2/T$ and $\widehat{u}_t = (y_t - \bar{y}) - \widehat{\beta}(x_t - \bar{x})$. The R^2 and the DW (Durbin-Waston) statistic are defined in the usual way, i.e.

$$R^2 = \frac{\widehat{\beta}^2 \sum_{t=1}^{T}(x_t - \bar{x})^2}{\sum_{t=1}^{T}(y_t - \bar{y})^2}, \text{ and } DW = \frac{\sum_{t=2}^{T}(\widehat{u}_t - \widehat{u}_{t-1})^2}{\sum_{t=1}^{T} \widehat{u}_t^2}. \tag{17}$$

The next theorem presents the asymptotic behaviors of $\widehat{\beta}, \widehat{t}_{\beta,OLS}, R^2$ and the DW statistic.

Theorem 1 *Let x_t and y_t be the time series defined by (12) and (13), respectively. If $d_x, d_y \in (0, 1/2)$ such that $d_x + d_y > 1/2$ and $\varphi_x = \varphi_y = \varphi \in (0, \pi)$, then*
(a) $\widehat{\beta} = O_p(T^{d_x+d_y-1})$,
(b) $\widehat{t}_{\beta,OLS} = O_p(T^{d_x+d_y-1/2})$,
(c) $R^2 = O_p(T^{2d_x+2d_y-2})$,
(d) $RW = 2 - 2\rho_y(1) + o_p(1)$,
where $\rho_y(1)$ is the lag-1 autocorrelation of y_t.

The most significant result in the above theorem is that the t-statistic diverges at the rate of $T^{d_x+d_y-1/2}$. The larger the sum of d_x and d_y is, the faster the t-statistic diverges. The divergence of the t-statistic implies that the rejection probability for

testing the null of $\beta = 0$ based on the usual t-test approaches 1 as the sample size increases. This result reflects the spurious effect in the t-test. Since both the dependent and independent variables are stationary and ergodic, the spurious effect is not expected. This finding suggests that it is strong dependence, instead of nonstationarity, that is potentially responsible for the spurious effect.

Although the t-statistic diverges as in the spurious regression studied before, the behaviors of other statistics resemble those derived in the conventional case of no spurious effect. For example, the OLS estimator $\widehat{\beta}$ of the slope coefficient and the R^2 converge to zero in probability. The DW statistic converges to $2 - 2\rho_y(1)$, a limit that is obtained in the case of AR(1) errors. However, the convergence rate of $\widehat{\beta}$ is much slower than the usual $T^{-1/2}$ rate. The speed of convergence depends on the order of magnitude of $\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y})$, which in turn depends on the persistence of x_t and y_t. The more persistent x_t and y_t are, the more slowly $\widehat{\beta}$ converges to zero, and the faster the t-statistic diverges.

It should be noted that no matter how persistent the underlying time series are, the divergence rate of the t-statistic is slower than $\sqrt{T}$, which is the corresponding divergence rate for spurious regressions between $I(1)$ processes or nonstationary $I(d)$ processes. See Phillips (1986), Marmol (1995, 1998), and Tsay and Chung (2000). When the underlying time series are not very persistent, the divergence rate can be very slow. In particular, when $d_x + d_y = 0.5$, we get $t_{\widehat{\beta}} = O_p(\log^{1/2} T)$ as shown in the following theorem. The proof is similar to that of Theorem 1 and is omitted.

Theorem 2 *Let x_t and y_t be the time series defined by (12) and (13), respectively. If $d_x, d_y \in (0, 1/2)$ such that $d_x + d_y = 1/2$ and $\varphi_x = \varphi_y = \varphi \in (0, \pi)$, then*
   (a) $\widehat{\beta} = O_p(T^{-1/2} \log^{1/2} T)$,
   (b) $\widehat{t}_{\beta, OLS} = O_p(\log^{1/2} T)$,
   (c) $R^2 = O_p(T^{-1} \log T)$,
   (d) $RW = 2 - 2\rho_y(1) + o_p(1)$.

An important condition in Theorems 1 and 2 is that $\varphi_x = \varphi_y$. If $\varphi_x \neq \varphi_y$, then there is no spurious effect even if both processes are persistent enough such that $d_x + d_y \geq 1/2$. In fact, the proofs of Theorems 1 and 2 rely crucially on the fact that $\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y}) = O_p(T^{d_x + d_y})$ when $d_x + d_y > 1/2$ and $\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y}) = O_p(T^{d_x + d_y} \log^{1/2} T)$ when $d_x + d_y = 1/2$. Thus, the order of $\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y})$ is quite different from the case when both x_t and y_t are weakly dependent processes. However, when $\varphi_x \neq \varphi_y$, $\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y}) = O_p(T^{1/2})$, which can be proved as

follows:

$$
var\left(\sum_{t=1}^{T} x_t y_t\right) = \sum_{t=1}^{T} E x_t^2 y_t^2 + 2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} E x_t x_{t-s} y_t y_{t-s}
$$

$$
\sim \quad T\sigma_x^2 \sigma_y^2 + 2 K_x K_y \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_x+2d_y-2} \cos\varphi_x s \cos\varphi_y s
$$

$$
\sim \quad T\sigma_x^2 \sigma_y^2 + K_x K_y \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_x+2d_y-2} \left(\cos((\varphi_x+\varphi_y)s) + \cos((\varphi_x-\varphi_y)s)\right)
$$

$$
= \quad O(T), \tag{18}
$$

where the last line follows from $\sum_{s=1}^{\infty} s^\alpha \cos\beta s < \infty$ for any $\alpha \in (-\infty, 0)$ and $\beta \in (0, 2\pi)$. Using this result and following the argument similar to the proof of Theorem 1, we can establish the following theorem. Details are omitted.

Theorem 3 *Let x_t and y_t be the time series defined by (12) and (13), respectively. If $d_x, d_y \in (0, 1/2)$ such that $d_x + d_y > 1/2$ and $\varphi_x, \varphi_y \in (0, \pi)$ with $\varphi_x \neq \varphi_y$, then*
   (a) $\widehat{\beta} = O_p(T^{-1/2})$,
   (b) $\widehat{t}_{\beta,OLS} = O_p(1)$,
   (c) $R^2 = O_p(T^{-1})$,
   (d) $RW = 2 - 2\rho_y(1) + o_p(1)$.

Theorem 3 shows all the statistics behave as if x_t and y_t are weakly dependent processes. Therefore, regressions between two stationary Gegenbauer processes with different pole locations do not suffer from the spurious effect. This suggests that the strong persistence is not sufficient for the existence of a spurious relationship. Both the strong dependence and cyclical co-movement are required to generate a spurious regression. From a broad perspective, these requirements are consistent with the spurious regression between two independent $I(d)$ processes, because $I(d)$ processes are persistent and their spectral densities, or generalized spectral densities (Solo 1992), have poles at the same location $\lambda = 0$. In fact, Theorems 1, 2, and 3 can be easily extended to the case $\varphi = 0$ so that they incorporate the spurious regressions between stationary fractional processes as special cases. We do not pursue this extension here as we choose to sacrifice generality for clarity.

The requirement that the spectral densities have poles at the same location seems to suggest that the spurious regression may not occur very often. However, this is not the case. First, many economic time series, after removal of the trend in the mean and seasonal components, have spectral densities which have peaks at the origin (Granger, 1966). Therefore, the adjusted time series may have poles at the

origin. Second, cyclical co-movements are not uncommon. In fact, they have been so prevalent that they have acquired the status of "stylized facts." See, for example, Lucas (1977). Therefore, economic time series may have spectral densities with poles at the frequencies corresponding to the seasonal effect or the business cycle effect. It is thus quite plausible that the spurious effect occurs more often than previously thought.

4 Spurious Regression Between Stationary Gegenbauer Process and Harmonic Process

In this section, we investigate the possible spurious relationship between a stationary Gegenbauer process and a harmonic process and that between a stationary Gegenbauer process and a deterministic trigonometric series.

Suppose that x_t follows a harmonic process and y_t follows a Gegenbauer process such that

$$x_t = A\cos\varphi_x t + B\sin\varphi_x t + \varepsilon_{xt}, \qquad (19)$$
$$y_t = (1 - 2\cos\varphi_y L + L^2)^{-d_y}\varepsilon_{yt}, \qquad (20)$$

where $A = \rho\cos\theta$, $B = -\rho\sin\theta$, $\theta \sim uniform[-\pi, \pi]$, $0 < \varphi_x, \varphi_y < \pi$, $\varepsilon_{xt}, \varepsilon_{yt}$ are $iid(0, \sigma_{\varepsilon x}^2)$ and $iid(0, \sigma_{\varepsilon y}^2)$ respectively, and ε_{xt}, ε_{yt} and θ are mutually independent.

As in Section 3, we consider regressing y_t on a constant and x_t,

$$y_t = \widehat{\alpha} + \widehat{\beta}x_t + \widehat{u}_t, t = 1, ..., T. \qquad (21)$$

The next theorem presents the asymptotic behaviors of $\widehat{\beta}, \widehat{t}_{\beta,OLS}, R^2$ and the DW statistic.

Theorem 4 *Let x_t and y_t be the time series generated by (19) and (20), respectively. If $d_y \in (0, 1/2)$, $\varphi_x = \varphi_y = \varphi \in (0, \pi)$, then*
    (a) $\widehat{\beta} = O_p(T^{d_y - 1/2})$,
    (b) $t_{\widehat{\beta},OLS} = O_p(T^{d_y})$,
    (c) $R^2 = O_p\left(T^{2d_y - 1}\right)$,
    (d) $DW = 2 - 2\rho_y(1) + o_p(1)$.

The most important result in the above Theorem is that the t-statistic diverges at the rate of T^{d_y}. Therefore, the usual t-statistic goes to infinity as long as $d_y > 0$. In comparison with Theorem 1, there is no requirement on the degree of persistence in y_t. The divergence rate is also larger than that obtained for two stationary Gegenbauer

processes. This is a little surprising because one would expect that the spurious effect may occur more easily between two similar processes. The theorem shows that this is not the case.

We should point out again that the poles must be at the same location to produce spurious effects. The following theorem establishes results similar to Theorem 3.

Theorem 5 *Let x_t and y_t be the time series generated by (19) and (20), respectively. If $d_y \in (0, 1/2)$, $\varphi_x, \varphi_y \in (0, \pi)$ and $\varphi_x \neq \varphi_y$, then*
    (a) $\widehat{\beta} = O_p(T^{-1/2})$,
    (b) $t_{\widehat{\beta}, OLS} = O_p(1)$,
    (c) $R^2 = O_p\left(T^{-1}\right)$,
    (d) $DW = 2 - 2\rho_y(1) + o_p(1)$.

We now investigate the possible existence of a spurious relationship between a stationary Gegenbauer process and a deterministic sine and cosine wave. More specifically, we use the same setup, except that the data generating process for x_t is replaced by

$$x_t = A\cos\varphi_x t + B\sin\varphi_x t, \tag{22}$$

where A and B are deterministic constants. It turns out that the results in Theorem 4 remain valid.

Theorem 6 *Let x_t and y_t be the time series generated by (22) and (20), respectively. If $d_y \in (0, 1/2)$ and $\varphi_x = \varphi_y = \varphi \in (0, \pi)$, then*
    (a) $\widehat{\beta} = O_p(T^{d_y-1/2})$,
    (b) $t_{\widehat{\beta}, OLS} = O_p(T^{d_y})$,
    (c) $R^2 = O_p\left(T^{2d_y-1}\right)$,
    (d) $DW = 2 - 2\rho_y(1) + o_p(1)$.
If $\varphi_x, \varphi_y \in (0, \pi)$, $\varphi_x \neq \varphi_y$, then
    (e) $\widehat{\beta} = O_p(T^{-1/2})$,
    (f) $t_{\widehat{\beta}, OLS} = O_p(1)$,
    (g) $R^2 = O_p\left(T^{-1}\right)$,
    (h) $DW = 2 - 2\rho_y(1) + o_p(1)$.

Theorem 6 shows that the spurious effect can happen between a stationary Gegenbauer process and a deterministic sine or cosine wave. With the deterministic regressor, our results resemble those of Tsay and Chung (2000), who studied the spurious regression between a stationary long memory process and a linear deterministic trend. The linear trend and the sine or cosine wave share one thing in common. It is well

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known that the discrete Fourier transform (DFT) of a polynomial trend exhibits unboundedness at the zero frequency (Corbae, Ouliaris and Phillips, 2002). In contrast, the DFT of a sine or cosine wave exhibits unboundedness at a given nonzero frequency, say φ_0. On the other hand, an $I(d)$ process for any d has an unbounded spectral density or generalized spectral density at $\lambda = 0$ (the generalized spectral density is defined as the limit of the expectation of the periodogram (Solo, 1992)). In contrast, a Gegenbauer process has an unbounded spectral density at a nonzero frequency, say φ_0 again. These facts suggest that it is the unboundedness at the same location in the DFT (for a deterministic sequence) or the (generalized) spectral density that leads to the spurious regression.

5 A Simulation Study

In this section, we examine the validity of our theoretical results by simulations. We consider three types of regression: the regression between two stationary Gegenbauer processes, the regression between a stationary Gegenbauer process and a stationary harmonic process, and the regression between a stationary Gegenbauer process and a deterministic trigonometric series.

To simulate a Gegenbauer process z_t, we use the moving average representation and truncate the infinite sum after M iterations. More specifically, we generate z_t according to $z_t = \sum_{j=0}^{M} c_j(d, \varphi)\varepsilon_{t-j}$, where ε_t is iid $N(0, 1)$. We take M to be $2T$, where T is the sample size. To reduce the initialization effect, we generate a time series of length $2T$ and trim the first T observations to get the simulated time series. For the harmonic process defined in (19), we take $\rho = 1$ and $\sigma_{\varepsilon x}^2 = 1$. For the trigonometric series defined in (22), we take $A = 1$ and $B = 1$.

We consider sample sizes $T = 100, 500, 1000, 2000, 5000$. For each simulated sample, we calculate the OLS estimate $\widehat{\beta}$ and construct the usual t-statistic. We report the percentage of rejections in 1000 replications, i.e. the percentage of t such that $|t| > 1.96$. Since $x_t \widehat{u}_t$ is autocorrelated, we also construct the heteoscedasiticity and autocorrelation consistent (HAC) variance estimator for $\widehat{\beta}$, i.e.

$$\widehat{\sigma}_{\beta,HAC}^2 = \left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right)^{-1} V \left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right)^{-1}, \tag{23}$$

where

$$V = \sum_{t=1}^{T}(x_t - \bar{x})^2\widehat{u}_t^2 + 2\sum_{j=1}^{S_T}(1 - \frac{j}{S_T + 1}) \sum_{t=j+1}^{T}(x_t - \bar{x})\widehat{u}_t\widehat{u}_{t-j}(x_{t-j} - \bar{x}). \tag{24}$$

The corresponding t-statistic is then given by $\widehat{t}_{\beta,HAC} = \widehat{\beta}/\widehat{\sigma}_{\beta,HAC}$. In the simulation study, we take S_T to be $[T^{1/3}]$, the largest integer that is smaller than $T^{1/3}$.

Table 1 presents the results when x_t and y_t are independent stationary Gegenbauer processes with parameters satisfying $d_x = d_y = d$ and $\varphi_x = \varphi_y = \varphi = \pi/4$. It is apparent that for both $\widehat{t}_{\beta,OLS}$ and $\widehat{t}_{\beta,HAC}$ statistics, the rejection rate increases with the sample size and the sum of d_x and d_y. The table also shows that the increase is not very fast, which is attributable to the slow divergence of the t-statistic. Therefore, the asymptotic results are reflected in this finite sample scenario. It should be emphasized that the spurious effect is obvious even for a small sample size, such as 100. The HAC estimate reduces the rejection rate, but the spurious effect is still apparent.

We also consider the cases that φ takes other values and d_x and d_y take different values. To save space, we do not present those results because the qualitative observations for the case $d_x = d_y = d$, $\varphi = \pi/4$ apply. Simulation results not reported also support the absence of the spurious effect when φ_x is not equal to φ_y.

Table 2 presents the results when y_t is a stationary Gegenbauer process while x_t is either a stationary harmonic process or a deterministic trigonometric series. These results are obtained for $\varphi_x = \varphi_y = \pi/4$. A general feature of these results is that the rejection rate increases with the sample size and the persistence of y_t, regardless of the t-statistic employed and the process that x_t follows. Another notable feature is that the rejection rate is higher when x_t is a deterministic trigonometric series than when x_t is a harmonic process. This may be attributable to the extra noise in the harmonic process. Comparing Table 2 with Table 1, we find the spurious effect is more obvious between a Gegenbauer process and a harmonic process or between a Gegenbauer process and a trigonometric series than between two Gegenbauer processes. Again, the HAC based t-statistic helps reduce the spurious effect but by no means eliminates it.

The results for other values of φ are similar. To save space, we do not report them here. We also find support that there is no spurious effect when φ_x is not equal to φ_y. In short, all of our theoretical results are supported by simulations.

6   Conclusion

In this paper, we provide further examples that the spurious effect can occur between two stationary processes. Our main finding is that the spurious regression can arise between two stationary Gegenbauer processes, as long as the sum of their generalized fractional differencing parameters is greater than 0.5 and their power spectrum have poles at the location. Another equally important finding is that spurious regression

may be present between a stationary Gegenbauer process and a stationary harmonic process or between a stationary Gegenbauer process and a deterministic trigonometric series, as long as the underlying processes share the same hidden periodicity. To the best of our knowledge, we are the first to introduce the deterministic trigonometric series and harmonic processes into the analysis of spurious regressions. The trigonometric series and the harmonic process are just as important, if not more important, than the polynomial trend in modeling a number of financial and economic time series.

This paper helps deepen the understanding of spurious regressions. From this study, we gain some insights that are not available in the classic studies by Phillips (1986) and Durlauf and Phillips (1988) and more recent studies by Marmol (1995, 1998) and Tsay and Chung (2000). Previous studies have led us to believe that it is the nonstationarity or the unboundedness of the spectral density at the zero frequency that causes the spurious effect. Our analysis shows that the unboundedness at a nonzero frequency can also give rise to the spurious effect. In finite samples, the unboundedness may be relaxed to the peakedness. This is supported by Granger, Hyung and Jeon (2001) who found that the spurious regression can occur between two stationary, yet persistent, AR(1) processes. It is well known that spectral densities of persistent AR(1) processes have apparent peaks at the zero frequency. Their study is likely to be extended to the case that the spectral densities of the underlying processes have large peaks at some nonzero frequency.

This paper may be extended by including more general cases. For example, the harmonic process and the trigonometric series may contain more than one sine or cosine term. The innovations that drive the Gegenbauer process may follow a general weakly dependent process, say an ARMA process. There may be more than one regressor in the regression. We expect most, if not all, of the asymptotic results obtained for the simple case continue to hold for more general cases. The paper does not give an expression for the limiting distributions of $\widehat{\beta}$ and the t-statistic. Although lack of the limiting distributions does not prevent us from evaluating the convergence rate of $\widehat{\beta}$ and the divergence rate of the t-statistic, it is still desirable to derive the asymptotic distributions. This paper considers only the stationary Gegenbauer processes. It is not surprising that the spurious effects are present and more dramatic among nonstationary Gegenbauer processes, harmonic processes and deterministic trigonometric series.

Table 1: Spurious Regression between Two Stationary Gegenbauer Processes: Percentage of $|t| > 1.96$

NOBS	$\widehat{t}_{\beta,OLS}$				$\widehat{t}_{\beta,HAC}$			
	d=0.15	d=0.25	d=0.35	d=0.45	d=0.15	d=0.25	d=0.35	d=0.45
100	28.90	35.50	46.60	58.00	25.60	30.70	38.80	51.00
500	34.10	50.90	63.50	72.60	22.40	37.00	47.20	58.50
1000	37.30	53.80	65.30	79.20	22.80	35.10	47.80	65.30
2000	41.70	56.00	70.60	83.80	22.30	37.50	51.10	67.50
5000	44.30	57.90	77.00	87.40	24.10	34.40	54.60	71.20

Table 2: Spurious Regression between a Gegenbauer Process and a Harmonic Processes and that between a Gegenbauer Process and a Trigonometric Series: Percentage of $|t| > 1.96$

NOBS	$\widehat{t}_{\beta,OLS}$				$\widehat{t}_{\beta,HAC}$			
	d_y=0.15	d_y=0.25	d_y=0.35	d_y=0.45	d_y=0.15	d_y=0.25	d_y=0.35	d_y=0.45
				Harmonic Process				
100	37.30	43.40	50.70	54.00	36.90	42.80	49.10	54.50
500	49.10	60.40	67.50	74.50	36.90	42.80	49.10	54.50
1000	56.40	65.60	76.80	81.50	48.20	58.40	69.80	74.60
2000	60.20	71.70	79.70	86.10	51.00	63.40	72.50	79.70
5000	70.20	76.10	84.20	89.10	56.70	65.10	76.50	83.70
				Trigonometric Series				
100	60.30	63.60	66.80	72.90	55.60	57.40	61.90	68.40
500	68.50	75.20	81.90	83.60	55.20	62.60	71.50	76.20
1000	73.30	79.10	84.60	88.10	60.00	66.90	73.10	79.30
2000	79.60	82.30	88.60	92.50	63.20	67.40	76.50	84.50
5000	81.80	89.40	91.70	94.30	63.90	76.10	80.10	86.00

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7 Appendix of Proofs

Proof of Theorem 1. (a) Since x_t and y_t are ergodic, we have

$$\text{plim}\frac{1}{T}\sum_{t=1}^{T}(x_t - \bar{x})^2 = \sigma_x^2 \text{ and plim}\frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})^2 = \sigma_y^2, \tag{25}$$

where σ_x^2 and σ_y^2 are the variances of x_t and y_t, respectively.

To prove (a), it suffices to calculate the order of

$$\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y}) = \sum_{t=1}^{T} x_t y_t - T\bar{x}\bar{y}. \tag{26}$$

Using (9), we have, for some positive constants K_x and K_y,

$$var\left(\sum_{t=1}^{T} x_t y_t\right)$$

$$= \sum_{t=1}^{T} E x_t^2 y_t^2 + 2\sum_{t=1}^{T}\sum_{s=1}^{t-1} E x_t x_{t-s} y_t y_{t-s}$$

$$\sim T\sigma_x^2 \sigma_y^2 + 2K_x K_y \sum_{t=1}^{T}\sum_{s=1}^{t-1} s^{2d_x + 2d_y - 2} \cos^2 \varphi s$$

$$\sim T\sigma_x^2 \sigma_y^2 + K_x K_y \sum_{t=1}^{T}\sum_{s=1}^{t-1} s^{2d_x + 2d_y - 2} (\cos 2\varphi s + 1)$$

$$\sim T\sigma_x^2 \sigma_y^2 + K_x K_y (2d_x + 2d_y - 1)^{-1} \sum_{t=1}^{T} t^{2d_x + 2d_y - 1} + K_x K_y \sum_{t=1}^{T}\sum_{s=1}^{t-1} s^{2d_x + 2d_y - 2} \cos 2\varphi s$$

$$\sim T\sigma_x^2 \sigma_y^2 + K_x K_y (2d_x + 2d_y - 1)^{-1} (2d_x + 2d_y)^{-1} T^{2d_x + 2d_y}$$

$$+ K_x K_y \sum_{t=1}^{T}\sum_{s=1}^{t-1} s^{2d_x + 2d_y - 2} \cos 2\varphi s. \tag{27}$$

Since $2d_x + 2d_y - 2 < 0$, we have $\sum_{s=1}^{t-1} s^{2d_x + 2d_y - 2} \cos 2\varphi s \leq C$ for some constant C that is independent of t. This is because the infinite sum $\sum_{s=1}^{\infty} s^{\alpha} \cos \beta s$ is finite for any $\alpha \in (-\infty, 0)$ and $\beta \in (0, 2\pi)$. See Zygmund (1959, page 70). Therefore, the last term in (27) is $O(T)$. As a consequence,

$$var\left(\sum_{t=1}^{T} x_t y_t\right) \sim K_x K_y (2d_x + 2d_y - 1)^{-1} (2d_x + 2d_y)^{-1} T^{2d_x + 2d_y}. \tag{28}$$

This implies that

$$\sum_{t=1}^{T} x_t y_t = O_p(T^{d_x + d_y}). \tag{29}$$

Following a similar argument, we have

$$\bar{x} = O_p(T^{1/2}) \text{ and } \bar{y} = O_p(T^{1/2}). \tag{30}$$

Combining (29) with (30) yields

$$\sum_{t=1}^{T} (x_t - \bar{x})(y_t - \bar{y}) = O_p(T^{d_x + d_y}). \tag{31}$$

Hence $\widehat{\beta} = O_p(T^{d_x + d_y - 1})$ using (25) and (31).

(b) We calculate the order of $\widehat{\sigma}_u^2$:

$$\begin{aligned}
\widehat{\sigma}_u^2 &= \sum_{t=1}^{T} \left((y_t - \bar{y}) - \widehat{\beta}(x_t - \bar{x}) \right)^2 / T \\
&= T^{-1} \sum_{t=1}^{T} (y_t - \bar{y})^2 + T^{-1} \widehat{\beta}^2 \sum_{t=1}^{T} (x_t - \bar{x})^2 - 2T^{-1} \widehat{\beta} \sum_{t=1}^{T} ((y_t - \bar{y})(x_t - \bar{x})) \\
&= T^{-1} \sum_{t=1}^{T} (y_t - \bar{y})^2 + O(T^{2d_x + 2d_y - 2}) - 2O(T^{2d_x + 2d_y - 2}) \\
&= T^{-1} \sum_{t=1}^{T} (y_t - \bar{y})^2 (1 + o_p(1)). \tag{32}
\end{aligned}$$

Therefore, $\text{plim}_{T \to \infty} \widehat{\sigma}_u^2 = \sigma_y^2$. Combine this with (25) and (31), we have $t_{\widehat{\beta},OLS} = O_p(T^{d_x + d_y - 1/2})$.

(c) Using (25) and part (a), we have

$$R^2 = \frac{\widehat{\beta}^2 \sum_{t=1}^{T} (x_t - \bar{x})^2}{\sum_{t=1}^{T} (y_t - \bar{y})^2} = O_p \left(T^{2d_1 + 2d_2 - 2} \right). \tag{33}$$

(d) For the DW statistic, we have

$$\begin{aligned}
DW &= \frac{\sum_{t=2}^{T} (\widehat{u}_t - \widehat{u}_{t-1})^2}{\sum_{t=1}^{T} \widehat{u}_t^2} \\
&= \frac{2 \sum_{t=2}^{T} \widehat{u}_t^2 - \sum_{t=2}^{T} 2\widehat{u}_t \widehat{u}_{t-1}}{\sum_{t=1}^{T} \widehat{u}_t^2} + o_p(1) = 2 - 2 \frac{\sum_{t=2}^{T} \widehat{u}_t \widehat{u}_{t-1}}{\sum_{t=1}^{T} \widehat{u}_t^2} + o_p(1) \\
&= 2 - 2 \frac{\sum_{t=2}^{T} \left(y_t - \bar{y} - \widehat{\beta}(x_t - \bar{x}) \right) \left(y_{t-1} - \bar{y} - \widehat{\beta}(x_{t-1} - \bar{x}) \right)}{\sum_{t=1}^{T} \widehat{u}_t^2} + o_p(1)
\end{aligned}$$

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$$= 2 - 2\frac{\sum_{t=2}^{T}(y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum_{t=1}^{T}\widehat{u}_t^2} - 2\widehat{\beta}^2\frac{\sum_{t=2}^{T}(x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum_{t=1}^{T}\widehat{u}_t^2} \qquad (34)$$

$$+ 2\widehat{\beta}\frac{\sum_{t=2}^{T}(y_t - \bar{y})(x_{t-1} - \bar{x})}{\sum_{t=1}^{T}\widehat{u}_t^2} + 2\widehat{\beta}\frac{\sum_{t=2}^{T}(x_t - \bar{x})(y_{t-1} - \bar{y})}{\sum_{t=1}^{T}\widehat{u}_t^2}$$

$$= 2 - 2\rho_y(1) + O\left(T^{2d_x + 2d_y - 2}\right) = 2 - 2\rho_y(1) + o_p(1),$$

as desired. ∎

Proof of Theorem 4. (a) We first compute the limit of $T^{-1}\sum_{t=1}^{T}(x_t - \bar{x})^2$:

$$\text{plim}\frac{1}{T}\sum_{t=1}^{T}(x_t - \bar{x})^2$$

$$= \text{plim}\frac{1}{T}\sum_{t=1}^{T}(A\cos\varphi t + B\sin\varphi t + \varepsilon_{xt})^2 - \left(\frac{1}{T}\sum_{t=1}^{T}A\cos\varphi t + B\sin\varphi t + \varepsilon_{xt}\right)^2$$

$$= A^2\lim\frac{1}{T}\sum_{t=1}^{T}\cos^2\varphi t + B^2\lim\frac{1}{T}\sum_{t=1}^{T}\sin^2\varphi t + 2AB\lim\frac{1}{T}\sum_{t=1}^{T}\cos\varphi t\sin\varphi t + \sigma_{\varepsilon_x}^2$$

$$- A^2\lim\left(\frac{1}{T}\sum_{t=1}^{T}\cos\varphi t\right)^2 - B^2\lim\left(\frac{1}{T}\sum_{t=1}^{T}\sin\varphi t\right)^2$$

$$- 2AB\left(\frac{1}{T}\sum_{t=1}^{T}\cos\varphi t\right)\left(\frac{1}{T}\sum_{t=1}^{T}\sin\varphi t\right)$$

$$= \frac{1}{2}A^2 + \frac{1}{2}B^2 + \sigma_{\varepsilon x}^2, \qquad (35)$$

where we have used the facts that

$$\lim T^{-1}\sum_{t=1}^{T}\cos^2\varphi t = 1/2, \quad \lim T^{-1}\sum_{t=1}^{T}\sin^2\varphi t = 1/2, \qquad (36)$$

$$\lim T^{-1}\sum_{t=1}^{T}\cos\varphi t\sin\varphi t = 0, \lim T^{-1}\sum_{t=1}^{T}\cos\varphi t = 0, \qquad (37)$$

and $\lim T^{-1}\sum_{t=1}^{T}\sin\varphi t = 0$.

Next, we calculate the order of $\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y}) = \sum_{t=1}^{T}x_t y_t - T\bar{x}\bar{y}$. Using (3) and (9), we have, for some constant K_y,

$$var\left(\sum_{t=1}^{T}x_t y_t\right)$$

$$= \sum_{t=1}^{T}Ex_t^2 y_t^2 + 2\sum_{t=1}^{T}\sum_{s=1}^{t-1}Ex_t x_{t-s}y_t y_{t-s}$$

$$\sim \quad T\sigma_x^2\sigma_y^2 + K_y\rho^2 \sum_{t=1}^{T}\sum_{s=1}^{t-1} s^{2d_y-1}\cos(\varphi s)\cos(\varphi s)$$

$$\sim \quad 1/2 K_y\rho^2 (2d_y+1)^{-1}(2d_y)^{-1} T^{2d_y+1}, \tag{38}$$

where the last line follows from the same argument as in (27). Combine the above results, we get, after some simple algebraic manipulations, $\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y}) = O_p(T^{d_y+1/2})$. As a consequence, $\widehat{\beta} = O_p(T^{d_y+1/2-1}) = O_p(T^{d_y-1/2})$.

(b) Following the same steps as in the proof of Theorem 1(b), we calculate the order of $\widehat{\sigma}_u^2$ below:

$$\widehat{\sigma}_u^2 = T^{-1}\sum_{t=1}^{T}(y_t - \bar{y})^2 + T^{-1}\widehat{\beta}^2 \sum_{t=1}^{T}(x_t - \bar{x})^2 - 2T^{-1}\widehat{\beta}\sum_{t=1}^{T}((y_t - \bar{y})(x_t - \bar{x}))$$

$$= \quad T^{-1}\sum_{t=1}^{T}(y_t - \bar{y})^2 + O(T^{2d_y-1}) - 2O(T^{d_y-1/2-1+d_y+1/2})$$

$$= \quad T^{-1}\sum_{t=1}^{T}(y_t - \bar{y})^2(1 + o_p(1)). \tag{39}$$

Therefore, $\text{plim}_{T\to\infty}\widehat{\sigma}_u^2 = \sigma_y^2$. Combine this with (35) and $\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y}) = O_p(T^{d_y+1/2})$, we have $t_{\widehat{\beta},OLS} = O_p(T^{d_y})$.

(c) Using (35), $\text{plim}_{T\to\infty} T^{-1}\sum_{t=1}^{T}(y_t - \bar{y})^2 = \sigma_y^2$, and part (a), we have

$$R^2 = \frac{\widehat{\beta}^2 \sum_{t=1}^{T}(x_t - \bar{x})^2}{\sum_{t=1}^{T}(y_t - \bar{y})^2} = O_p\left(T^{2d_y-1}\right). \tag{40}$$

(d) In view of (34), we have

$$DW \quad = \quad 2 - 2\frac{\sum_{t=2}^{T}(y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum_{t=1}^{T}\widehat{u}_t^2} - 2\widehat{\beta}^2\frac{\sum_{t=2}^{T}(x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum_{t=1}^{T}\widehat{u}_t^2}$$

$$+ 2\widehat{\beta}\frac{\sum_{t=2}^{T}(y_t - \bar{y})(x_{t-1} - \bar{x})}{\sum_{t=1}^{T}\widehat{u}_t^2} + 2\widehat{\beta}\frac{\sum_{t=2}^{T}(x_t - \bar{x})(y_{t-1} - \bar{y})}{\sum_{t=1}^{T}\widehat{u}_t^2}$$

$$= \quad 2 - 2\rho_y(1) + O\left(T^{2d_y-1}\right) = 2 - 2\rho_y(1) + o_p(1), \tag{41}$$

which completes the proof of the theorem. ■

Proof of Theorem 6. The proof is similar to that of Theorem 4. To save space, we only calculate the order of $var(\sum_{t=1}^{T} x_t y_t)$ when $\varphi_x = \varphi_y = \varphi$. First,

$$var\left(\sum_{t=1}^{T} A\cos(\varphi t)y_t\right)$$

$$
= \sum_{t=1}^{T} E A^2 \cos^2\left(\varphi t\right) y_t^2 + 2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} A^2 \cos\left(\varphi t\right) \cos\left(\varphi\left(t-s\right)\right) y_t y_{t-s}
$$

$$
\sim \ 1/2 A^2 T \sigma_y^2 + 2 K_y A^2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \cos\left(\varphi t\right) \cos\left(\varphi\left(t-s\right)\right) \cos \varphi s
$$

$$
\sim \ 1/2 A^2 T \sigma_y^2 + K_y A^2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \left(\cos\left(\varphi\left(2t-s\right)\right) \cos \varphi s + \cos^2\left(\varphi s\right)\right)
$$

$$
\sim \ 1/2 A^2 T \sigma_y^2 + K_y A^2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \left(\frac{\cos\left(2\varphi t\right) + \cos 2\varphi\left(t-s\right)}{2} + \frac{1 + \cos\left(2\varphi s\right)}{2}\right)
$$

$$
\sim \ 1/2 A^2 T \sigma_y^2 + 1/2 K_y A^2 \left(2d_y + 1\right)^{-1} \left(2d_y\right)^{-1} T^{2d_y+1} + O(T)
$$

$$
\sim \ 1/2 K_y A^2 \left(2d_y + 1\right)^{-1} \left(2d_y\right)^{-1} T^{2d_y+1}. \tag{42}
$$

Similarly,

$$
var\left(\sum_{t=1}^{T} B \sin\left(\varphi t\right) y_t\right)
$$

$$
= \sum_{t=1}^{T} E B^2 \sin^2\left(\varphi t\right) y_t^2 + 2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} B^2 \sin\left(\varphi t\right) \sin\left(\varphi\left(t-s\right)\right) y_t y_{t-s}
$$

$$
\sim \ 1/2 B^2 T \sigma_y^2 + 2 K_y B^2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \sin\left(\varphi t\right) \sin\left(\varphi\left(t-s\right)\right) \cos \varphi s
$$

$$
\sim \ 1/2 B^2 T \sigma_y^2 + K_y B^2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \sin\left(\varphi t\right) \left(\sin \varphi\left(t-2s\right) + \sin \varphi t\right)
$$

$$
\sim \ 1/2 B^2 T \sigma_y^2 + K_y B^2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \left(\frac{\cos 2\varphi s - \cos 2\varphi\left(t-s\right)}{2} + \frac{1 - \cos 2\varphi t}{2}\right)
$$

$$
\sim \ 1/2 K_y B^2 \left(2d_y + 1\right)^{-1} \left(2d_y\right)^{-1} T^{2d_y+1}. \tag{43}
$$

Finally,

$$
cov\left(\sum_{t=1}^{T} A \cos\left(\varphi t\right) y_t, B \sin\left(\varphi t\right) y_t\right)
$$

$$
= \ AB \sum_{t=1}^{T} \cos\left(\varphi t\right) \sin\left(\varphi t\right) E y_t^2 + 2 \sum_{t=1}^{T} \sum_{s=1}^{t-1} AB \cos\left(\varphi t\right) \sin\left(\varphi\left(t-s\right)\right) E y_t y_{t-s}
$$

$$
= \ o(T) + 2 A B K_y \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \cos\left(\varphi t\right) \sin\left(\varphi\left(t-s\right)\right) \cos \varphi s
$$

$$
= \ o(T) + A B K_y \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \left(\sin\left(\varphi(2t-s)\right) - \sin\left(\varphi s\right)\right) \cos \varphi s
$$

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$$
\begin{aligned}
&= \; o(T) + ABK_y \sum_{t=1}^{T} \sum_{s=1}^{t-1} s^{2d_y-1} \left(1/2 \sin\left(2\varphi t\right) + 1/2 \sin(2\varphi(t-s)) - 1/2 \sin\left(2\varphi s\right)\right) \\
&= \; O(T). \hspace{8cm} (44)
\end{aligned}
$$

Therefore $var(\sum_{t=1}^{T} x_t y_t) \sim CT^{2d_y+1}$ for some constant C. Parts (a), (b), (c) and (d) of the theorem now follow from the same arguments as in the proof of Theorem 4. Similarly, we can prove Parts (e), (f), (g) and (h) of the theorem. ∎

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