TAOLS: Adaptive F and t Tests for Cointegration and Multicointegration

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Abstract

We develop a unified framework for estimation and inference in cointegrated systems using the Transformed and Augmented Ordinary Least Squares (TAOLS) method. The approach does not require an *a priori* assumption about whether the underlying relationship exhibits conventional (nonsingular) cointegration or multicointegration (singular cointegration). We first establish an asymptotic F-test framework for inference under multicointegration that extends existing results previously developed for conventional cointegration. We then show that F-tests designed specifically for either nonsingular cointegration or multicointegration are valid only when the assumed structure holds, exhibiting severe size distortions otherwise. To address this limitation, we propose an adaptive testing procedure that combines these two Wald statistics through a data-driven weighting scheme. The resulting F-test achieves reliable size control and competitive power across a wide range of data-generating processes regardless of the underlying cointegration structure. We apply the proposed approach to examine the cointegrating relationship between U.S. housing starts and completions.

Keywords: Asymptotic F-test, Asymptotic t-test, Cointegration, Multicointegration, TAOLS

JEL Codes: C12, C13, C22

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1 Introduction

Many economic relationships exhibit disequilibria that accumulate over time and influence long-run adjustment dynamics. In such settings, deviations from equilibrium may build up and generate new constraints, thereby inducing additional equilibrium relations. Modeling these dynamics therefore requires econometric frameworks capable of capturing equilibrium relations not only among nonstationary variables but also among accumulated disequilibria.

Multicointegration provides a framework for analyzing such phenomena. Since its introduction by Granger and Lee (1989, 1990), the concept has attracted considerable attention in econometrics. Multicointegration extends the conventional cointegration framework of Engle and Granger (1987) by allowing equilibrium errors themselves to accumulate into new long-run relationships. This feature is particularly relevant for economic stock-flow systems, such as inventories, public debt, and uncompleted housing, as well as physical systems such as the Earth's heat content, where persistent imbalances accumulate to generate long-run equilibrium constraints (cf. Engsted and Haldrup (1999), Lee (1996), Bruns et al. (2020)). Formally, let x_t and y_t be integrated $I(1)$ processes that are cointegrated with a stationary equilibrium error $v_t := y_t - bx_t$. Multicointegration arises when the cumulative sum $\sum_{\tau=1}^{t} v_\tau$ is itself cointegrated with one or both of the original processes, introducing a second layer of long-run equilibrium relationships.

Empirical implementations of multicointegration have traditionally relied on the vector error correction model (VECM) framework for second-order integrated processes (Johansen (1992)). Estimation and inference typically follow the parametric maximum likelihood approach developed by Johansen (1997) and further studied in Paruolo (2000), Johansen (2006), and Boswijk (2010). While this framework provides asymptotically efficient estimators under correct specification, it assumes Gaussian errors and is highly sensitive to the specification of short-run dynamics. Misspecification of the error distribution or the dynamics may lead to unreliable inference.

Recent work has sought to relax these restrictions using semiparametric approaches. In particular, the triangular $I(1)$ framework developed by Kheifets and Phillips (2023) and Phillips and Kheifets (2024) provides a flexible alternative that avoids explicit modeling of short-run dynamics. Within this framework, these two studies have proposed the Fully Modified OLS and Trend IV (TIV) methods, both of which dispense with Gaussian assumptions and leave short-run dynamics unspecified. Building on these contributions and related work (e.g., Phillips (2014) and Hwang and Sun (2018)), Sun et al. (2025) introduce the transformed and augmented OLS (TAOLS) method in a multicointegrated setting and provide an initial investigation of the approach.

Despite the substantial literature, several econometric challenges remain. Existing inference procedures still largely rely on parametric vector autoregressive (VAR) or VECM frameworks that impose strong parametric assumptions. While semiparametric methods alleviate some of these restrictions, formal inference in these frameworks remains underdeveloped. In particular, available testing procedures typically assume that the underlying cointegration regime—conventional cointegration or multicointegration—is known *a priori*. In practice, however, this regime is rarely known. Misspecifying the regime can lead to substantial size distortions and misleading inference.

This paper addresses these gaps by contributing to the econometric analysis of multicointegration in three key ways:

First, we extend the low-frequency inference framework of Hwang and Sun (2018) from conventional cointegration to multicointegration. This produces inference procedures robust to high-frequency contamination and misspecification of short-run dynamics, offering a semiparametric alternative to parametric VECM approaches.

Second, we provide a comprehensive study of the TAOLS method in a multicointegrated system, extending the initial findings of Sun et al. (2025). We establish that the TAOLS estimator is asymptotically mixed normal and that the resulting Wald and t-statistics are asymptotically F- and t-distributed. As a result, standard critical values can be used for inference even in the presence of second-order stochastic trends.

Third, we develop an adaptive testing procedure that remains valid when the cointegration regime is unknown. The procedure combines Wald statistics constructed under the competing hypotheses of conventional cointegration and multicointegration using a data-driven adaptive weight. The resulting statistic remains asymptotically F-distributed, and the test achieves correct size under both regimes while maintaining power comparable to the best regime-specific test.

To understand the first contribution of this paper, it is useful to consider the two-step transformation that defines the TAOLS approach. The method begins by augmenting the multicointegration regression that governs $I(2)$ variables with the first differences of the $I(1)$ regressors to eliminate long-run endogeneity. Subsequently, basis functions, such as sine and cosine, are applied to all variables to extract low-frequency components and filter out short-term fluctuations. Theoretically, this low-frequency filtering is essential for removing second-order bias. TAOLS is simply OLS applied to the transformed and augmented regression—hence the name. By focusing on low-frequency components, TAOLS is robust to high-frequency contamination and misspecification of short-run dynamics.

The TAOLS estimators of all parameters are asymptotically mixed normal and free of second-order asymptotic bias. Despite the presence of second-order stochastic trends characteristic of $I(2)$ variables, a highly nonstandard feature, the transformed and augmented regression effectively functions as a classical linear normal regression, allowing classical test theory to be directly applied and asymptotically justified. This simplification makes TAOLS-based tests particularly attractive for applied researchers. Once the transformed variables—which are straightforward to compute—are obtained, standard statistical software can immediately calculate test statistics and p-values without specialized programming.

A key extension of this theory, and the paper's second contribution, is to show that this classical test theory applies to the entire parameter vector of the multicointegration regression, whereas Sun et al. (2025) establish results only for the cointegration parameter. This extension enables complete inference for multicointegrated systems.

The asymptotic F and t tests extend the methodology of Hwang and Sun (2018) from conventional cointegration to multicointegration. However, the tests developed here and those in Hwang and Sun (2018) are based on different regression specifications and both require *a priori* knowledge of whether a second layer of cointegration is present. These tests perform well when the assumed cointegration structure holds but may exhibit severe size distortions otherwise. For instance, tests designed for multicointegration tend to over-reject when multicointegration is absent, whereas tests based on conventional cointegration may under-reject when multicointegration is present.

To address this problem and as the third and main contribution of this paper, we construct an adaptive Wald statistic that combines the two tests using a data-dependent weight. Depending on the true regime, the weight converges to zero or one so that the combined statistic asymptotically reduces to the appropriate regime-specific test while retaining the convenient F distribution.

Monte Carlo simulations show that the adaptive procedure achieves near-nominal size across a wide range of designs while maintaining power comparable to the best individual test. This robustness makes the procedure particularly attractive for empirical applications where the cointegration regime is uncertain.

To illustrate the practical relevance of the proposed methods, we revisit the long-run relationship between U.S. housing starts and completions. Using TAOLS estimation and the adaptive test, we find evidence of a persistent stock of uncompleted housing units across the full sample and decade-specific subsamples. The multicointegration-based test consistently rejects the null hypothesis that all housing starts are eventually completed. In contrast, the test assuming conventional cointegration is more sensitive to tuning parameters and sample windows. The adaptive test behaves similarly to the latter test, although notable differences emerge in some cases. The feedback parameter is significantly different from zero in all periods, supporting inventory-driven feedback dynamics.

The remainder of the paper is organized as follows. Section 2 introduces the model and its underlying assumptions. Section 3 establishes the asymptotic properties of the TAOLS estimators and associated F- and t-tests. Section 4 presents the adaptive testing procedure and proves its asymptotic validity. Section 5 evaluates finite-sample performance through Monte Carlo experiments, and Section 6 illustrates the method using U.S. housing data. The final section concludes the paper, with all proofs relegated to the appendix. The supplementary appendix extends the results of the paper to accommodate polynomial trends.

2 Multicointegration and Assumptions

Consider the triangular cointegration system

$$y_t = x_t'\beta_0 + u_{0,t}, \tag{1}$$

$$x_t = x_{t-1} + u_{x,t}, \tag{2}$$

for $t = 1, \ldots, T$, where y_t is the main variable of interest and x_t is a vector of d_x covariates. Both x_t and y_t are observed, and we have time series data $\{x_t, y_t\}_{t=1}^T$.[1] The term $u_{0,t}$ represents the equilibrium error, which is unobservable but assumed to be stationary, weakly dependent, and mean-zero, with a positive spectral density. That is, $u_{0,t}$ is integrated of order zero, denoted as $u_{0,t} \sim I(0)$. Similarly, $u_{x,t} \sim I(0)$. Hence, both y_t and x_t are integrated $I(1)$ processes, but their linear combination, $y_t - x_t'\beta_0$, is a stationary $I(0)$ process, representing the first cointegration relationship in the system.

To capture the accumulation of past disequilibria, define

$$U_{0,t} = \sum_{\tau=1}^{t} u_{0,\tau} \text{ for } t = 1, \ldots, T, \text{ and } U_{0,0} := 0.$$

By construction, $\{U_{0,t}\}_{t=1}^T$ is an $I(1)$ process representing the stock of disequilibria accumulated from the equilibrium errors $u_{0,t}$. We further assume that $U_{0,t}$ and x_t are cointegrated:

$$U_{0,t} = (x_t - x_0)'\gamma_0 + e_{0,t} - e_{0,0} \text{ for } t = 1, \ldots, T, \tag{3}$$

[1] For notational convenience, we also assume that x_0 is observed so that we can calculate Δx_t for $t = 1, \ldots, T$.

where $\{e_{0,t} \in \mathbb{R}\}$ is stationary with a positive long-run variance. This defines the second cointegration relationship, in which the accumulated disequilibria constrain x_t. Together, these two cointegration relationships define the multicointegration model.

Define the cumulative series $Y_t = \sum_{\tau=1}^{t} y_\tau$ and $X_t = \sum_{\tau=1}^{t} x_\tau$. Since y_t and x_t are $I(1)$, the cumulative processes Y_t and X_t are integrated of order two ($I(2)$). Combining the cointegration relationships in (1) and (3) leads to the multicointegration representation for the $I(2)$ variables Y_t and X_t:

$$Y_t = \alpha_0 + X_t'\beta_0 + x_t'\gamma_0 + e_{0,t}, \tag{4}$$

where $\alpha_0 = -e_{0,0} - x_0'\gamma_0$ captures the effect of the initial values. Here, β_0 is the cointegration parameter, and γ_0 is the multicointegration parameter. Both β_0 and γ_0 are unknown and are the parameters of interest.

Let $u_t := (u_{0,t}, u_{x,t}')'$ and denote its long-run variance (LRV) matrix by

$$\Omega = \begin{bmatrix} \Omega_{00} & \Omega_{0x} \\ \Omega_{x0} & \Omega_{xx} \end{bmatrix} \in \mathbb{R}^{(d_x+1)\times(d_x+1)}. \tag{5}$$

We assume that $\Omega_{xx} > 0$ (i.e., Ω_{xx} is positive definite), implying that x_t is a full-rank unit-root process. Let

$$\Omega_{00\cdot x} := \Omega_{00} - \Omega_{0x}\Omega_{xx}^{-1}\Omega_{x0}$$

be the conditional long-run variance of $\{u_{0,t}\}$ given $\{u_{x,t}\}$. The second layer of cointegration in (3) implies that $\Omega_{00\cdot x} = 0$. Consequently, the long-run covariance matrix Ω is rank deficient (singular). Intuitively, the long-run variation in $\{u_{0,t}\}$ is completely explained by that in $\{u_{x,t}\}$, so no independent long-run variation remains once $\{u_{x,t}\}$ is given. For this reason, multicointegration is sometimes referred to as singular cointegration. Our formulation of multicointegration is semiparametric, as the short-run dynamics are left completely unspecified and no distributional assumptions are imposed.

Our multicointegration framework features a triangular $I(1)$ system of cointegration (Phillips (1991)), as given in equations (1) and (2). In this system, the presence of the second layer of cointegration leads to the singularity of Ω. This is in contrast to the VAR formulation of Engsted and Johansen (1999), which, as noted by Phillips and Kheifets (2024), imposes a full-rank restriction on Ω and therefore rules out multicointegrated $I(1)$ systems in the VAR framework. As a result, Engsted and Johansen (1999) have to embed multicointegration in a VAR system for $I(2)$ variables.

Finally, if Ω is of full rank (nonsingular), which rules out a second layer of cointegration, equations (1) and (2) define conventional cointegration, also referred to as nonsingular cointegration. Augmenting (1) with Δx_t to remove the endogeneity yields $y_t = x_t'\beta_0 + \Delta x_t'\gamma_0 + u_{0\cdot x,t}$.

Integrating this equation leads to the integrated and modified (IM) model of Vogelsang and Wagner (2014):

$$Y_t = \alpha_0 + X_t'\beta_0 + x_t'\gamma_0 + U_{0\cdot x,t}. \tag{6}$$

The multicointegration model in (4) is fundamentally different from this formulation. In the IM model, x_t arises from integrating the auxiliary regressor Δx_t to eliminate second-order bias, and γ_0 is typically treated as a nuisance parameter. In contrast, in (4) x_t governs a second equilibrium relation (3), and γ_0 represents the multicointegration parameter. From a statistical perspective, the IM model has a full-rank LRV matrix Ω and nonstationary $I(1)$ errors, whereas (4) features a reduced-rank Ω and stationary $I(0)$ errors. The difference in the integration orders of the error processes leads to a different asymptotic theory.

Assumption 1 *(i)* $x_0 = o_p(\sqrt{T})$.

(ii) The following Functional Central Limit Theorem (FCLT) holds:

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor T\cdot\rfloor}\begin{pmatrix} e_{0,t} \\ u_{x,t}\end{pmatrix} \Rightarrow \begin{pmatrix} B_e(\cdot) \\ B_x(\cdot)\end{pmatrix} \equiv \mathrm{BM}(\mathbf{\Omega}^{(ex)}), \tag{7}$$

where

$$\mathbf{\Omega}^{(ex)} = \begin{bmatrix} \Omega_{ee} & \Omega_{ex} \\ \Omega_{xe} & \Omega_{xx}\end{bmatrix}$$

is positive definite and $\mathrm{BM}(\mathbf{\Omega}^{(ex)})$ *represents Brownian motion with variance matrix* $\mathbf{\Omega}^{(ex)}$.

In the above, we have assumed that $\mathbf{\Omega}^{(ex)}$ is positive definite, and this assumption implies the absence of additional higher-order cointegration. We allow the possibility that $\Omega_{ex} \neq 0$ so that there may be long-run correlation between $u_{x,t}$ and $e_{0,t}$. To remove this possible long-run correlation, we define $\delta_0 := \Omega_{xx}^{-1}\Omega_{xe}$, and let

$$e_{0\cdot x,t} = e_{0,t} - u_{x,t}'\delta_0 = e_{0,t} - \Delta x_t'\delta_0.$$

Then, there is no long-run correlation between $u_{x,t}$ and $e_{0\cdot x,t}$. Furthermore, the long-run variance of $\{e_{0\cdot x,t}\}$ is given by

$$\Omega_{ee\cdot x} := \Omega_{ee} - \Omega_{ex}\Omega_{xx}^{-1}\Omega_{xe},$$

which is positive under the assumption that $\mathbf{\Omega}^{(ex)}$ is positive definite.

Augmenting (4) with Δx_t, we obtain:

$$Y_t = \alpha_0 + X_t'\beta_0 + x_t'\gamma_0 + \Delta x_t'\delta_0 + e_{0\cdot x,t}. \tag{8}$$

Given the sample $\{(Y_t, X_t, x_t, \Delta x_t)\}_{t=1}^{T}$, we are interested in estimating and conducting inference on the cointegration and multicointegration parameters β_0 and γ_0 in the above model.

3 TAOLS and Asymptotic F and t Tests under Multicointegration

Building on the methods proposed by Hwang and Sun (2018), Sun (2023), and Sun et al. (2025), our approach to estimation begins by employing basis functions to transform (8). These transformations allow us to extract the long-term trend component of the underlying time series, facilitating the analysis of long-run co-movement patterns.

We now describe the transformation step in detail. Let $\{\phi_i(r)\}_{i=1}^{\infty}$ denote a complete orthonormal basis for $L^2[0,1]$, such as the Fourier basis

$$\{1, \sqrt{2}\sin(2\pi r), \sqrt{2}\cos(2\pi r), \ \sqrt{2}\sin(4\pi r), \sqrt{2}\cos(4\pi r), \ldots\}.$$

From this set, we select the first K basis functions, where K is a fixed tuning parameter chosen *a priori*. For each $i \in [K] := \{1, \ldots, K\}$, we define the transformed variables as follows:

$$\begin{aligned}
V_{Y,i} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T} Y_t\phi_i\left(\frac{t}{T}\right), \ V_{X,i} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} X_t\phi_i\left(\frac{t}{T}\right), \\
V_{x,i} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\phi_i\left(\frac{t}{T}\right), \ V_{\Delta x,i} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} \Delta x_t\phi_i\left(\frac{t}{T}\right), \\
V_{\ell,i} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T} \phi_i\left(\frac{t}{T}\right), \ V_{e\cdot x,i} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_{0\cdot x,t}\phi_i\left(\frac{t}{T}\right),
\end{aligned} \tag{9}$$

where $V_{\ell,i}$ corresponds to the transformation of the constant regressor $\ell_t = 1$ for all $t \in [T]$.

When the Fourier basis is used, these transformed variables correspond to the real and imaginary parts of the standard complex Fourier transform. They capture low-frequency variations in the data, with energy concentrated at frequencies near zero. In other words, they represent long-term trends while filtering out short-term fluctuations. An alternative and commonly used orthonormal basis for $L^2[0,1]$ is the shifted sine basis $\{\sqrt{2}\sin\left[(i-\frac{1}{2})\pi r\right] : i = 1, 2, \ldots\}$. In practice, the choice between the Fourier basis and the shifted sine basis has a negligible effect on performance. Following Phillips and Kheifets (2024) and Sun et al. (2025), we adopt the shifted sine basis in our simulation and empirical studies. Notably, we do not use complex Fourier transforms based on complex exponentials. By avoiding complex numbers and their associated intricacies, the proposed method is more accessible and appealing to applied researchers.

Using these transformed variables, we obtain the equation:

$$V_{Y,i} = V_{\ell,i}\alpha_0 + V'_{X,i}\beta_0 + V'_{x,i}\gamma_0 + V'_{\Delta x,i}\delta_0 + V_{e\cdot x,i} \tag{10}$$

for $i \in [K]$. This equation can be interpreted as a cross-sectional regression involving K data points. We assume that K is sufficiently large so that the number of observations exceeds the number of regressors (i.e., $K > 3d_x + 1$).

Our estimation and inference are based on the transformed regression equation (10), treating $V_{e\cdot x,i}$ as the regression error. Following Hwang and Sun (2018), we refer to (10) as the Transformed and Augmented (TA) regression, and the corresponding OLS estimator as the TAOLS estimator.

One might ask why TAOLS is preferred over simply applying Ordinary Least Squares to equation (8). There are three main reasons. First, OLS based on (8) exhibits second-order asymptotic bias, complicating inference, whereas TAOLS eliminates this bias. Second, TAOLS enables the effortless application of standard statistical tests, such as Wald and t tests, relying on well-known reference distributions. Third, and most importantly, the transformed variables capture the long-run, low-frequency variations of the underlying time series, offering a more direct approach to uncovering long-run relationships. Since TAOLS does not use high-frequency variations, it is robust to high-frequency contamination in the data. See, for example, Müller and Watson (2018) for a recent application of basis transforms.

To formally present the TAOLS estimator, define $V_Y = (V_{Y,1}, \ldots, V_{Y,K})' \in \mathbb{R}^{K\times 1}$, $V_\ell = (V_{\ell,1}, \ldots, V_{\ell,K})' \in \mathbb{R}^{K\times 1}$, and similarly define $V_X \in \mathbb{R}^{K\times d_x}$, $V_x \in \mathbb{R}^{K\times d_x}$, $V_{\Delta x} \in \mathbb{R}^{K\times d_x}$, and $V_{e\cdot x} \in \mathbb{R}^{K\times 1}$. Then, we have

$$V_Y = V_\ell \alpha_0 + V_X \beta_0 + V_x \gamma_0 + V_{\Delta x}\delta_0 + V_{e\cdot x}.$$

Given an observation matrix $Z \in \mathbb{R}^{n\times d_z}$, define $P_Z = Z\left(Z'Z\right)^{-1} Z'$ and $Q_Z = I_n - P_Z$ where I_n is the $n \times n$ identity matrix. Both P_Z and Q_Z are projection matrices. In the absence of multicollinearity, the TAOLS estimator of $\theta_0 = \left(\alpha_0', \beta_0', \gamma_0', \delta_0'\right)'$ is then given by

$$\hat{\theta}_{\text{TAOLS}} := (\hat{\alpha}'_{\text{TAOLS}}, \hat{\beta}'_{\text{TAOLS}}, \hat{\gamma}'_{\text{TAOLS}}, \hat{\delta}'_{\text{TAOLS}})' = \left(V'_{\ell,X,x,\Delta x} V_{\ell,X,x,\Delta x}\right)^{-1} V'_{\ell,X,x,\Delta x} V_Y,$$

where $V_{\ell,X,x,\Delta x} = (V_\ell, V_X, V_x, V_{\Delta x})$. By the Frisch–Waugh–Lovell theorem, the TAOLS estimator $\hat{\beta}_{\text{TAOLS}}$ of β_0 satisfies

$$\hat{\beta}_{\text{TAOLS}} - \beta_0 = \left(V_X' Q_{[V_\ell, V_x, V_{\Delta x}]} V_X\right)^{-1} V_X' Q_{[V_\ell, V_x, V_{\Delta x}]} V_{e\cdot x},$$

and the TAOLS estimator $\hat{\gamma}_{\text{TAOLS}}$ of γ_0 satisfies

$$\hat{\gamma}_{\text{TAOLS}} - \gamma_0 = \left(V_x' Q_{[V_\ell, V_X, V_{\Delta x}]} V_x\right)^{-1} V_x' Q_{[V_\ell, V_X, V_{\Delta x}]} V_{e\cdot x}.$$

We make the following assumptions on the basis functions, which are similar to the corresponding assumptions in Hwang and Sun (2018).

Assumption 2 *(i) For every $i \in [K]$, $\phi_i(\cdot)$ is continuously differentiable. (ii) The functions $\{\phi_i(\cdot)\}_{i=1}^{K}$ are orthonormal in $L^2[0,1]$.*

Under Assumption 1(ii), we have

$$\begin{pmatrix} \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor T\cdot \rfloor} e_{0\cdot x,t} \\ \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor T\cdot \rfloor} u_{x,t} \end{pmatrix} \Rightarrow \begin{pmatrix} \Omega_{ee\cdot x}^{1/2} W_{e\cdot x}(\cdot) \\ \Omega_{xx}^{1/2} W_x(\cdot) \end{pmatrix},$$

where $W_{e\cdot x}(\cdot)$ is the standard one-dimensional Brownian motion, $W_x(\cdot)$ is the standard d_x-dimensional Brownian motion, and $W_{e\cdot x}(\cdot)$ and $W_x(\cdot)$ are independent. Combining this with Assumptions 1(i) and 2(i), we obtain the joint convergence:

$$\begin{aligned}
\frac{1}{T^2} V_X &\Rightarrow \int_0^1 \boldsymbol{\phi}(r) W_X'(r)\, dr \cdot \Omega_{xx}^{1/2} := \xi_X \text{ for } W_X(r) = \int_0^r W_x(s)\, ds, \\
\frac{1}{T} V_x &\Rightarrow \int_0^1 \boldsymbol{\phi}(r) W_x(r)\, dr \cdot \Omega_{xx}^{1/2} := \xi_x, \\
V_{\Delta x} &\Rightarrow \int_0^1 \boldsymbol{\phi}(r)\, dW_x'(r) \cdot \Omega_{xx}^{1/2} := \eta_{\Delta x}, \\
\frac{1}{\sqrt{T}} V_\ell &= \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{\phi}\left(\frac{t}{T}\right) \Rightarrow \int_0^1 \boldsymbol{\phi}(r)\, dr := \xi_\ell, \\
V_{e\cdot x} &\Rightarrow \Omega_{ee\cdot x}^{1/2} \int_0^1 \boldsymbol{\phi}(r)\, dW_{e\cdot x}(r) := \Omega_{ee\cdot x}^{1/2} \eta_{e\cdot x},
\end{aligned}$$

where

$$\boldsymbol{\phi}(r) = (\phi_1(r), \ldots, \phi_K(r))'.$$ [2]

Note that the dependence of $\boldsymbol{\phi}(\cdot)$ and $(\xi_X, \xi_x, \eta_{\Delta x}, \xi_\ell, \eta_{e\cdot x})$ on K has been suppressed. It then follows that

$$T^2(\hat{\beta}_{\text{TAOLS}} - \beta_0) \Rightarrow \Omega_{ee\cdot x}^{1/2} \left(\xi_X' Q_{[\xi_\ell, \xi_x, \eta_{\Delta x}]} \xi_X\right)^{-1} \xi_X' Q_{[\xi_\ell, \xi_x, \eta_{\Delta x}]} \eta_{e\cdot x}, \tag{11}$$

and

$$T(\hat{\gamma}_{\text{TAOLS}} - \gamma_0) \Rightarrow \Omega_{ee\cdot x}^{1/2} \left(\xi_x' Q_{[\xi_\ell, \xi_X, \eta_{\Delta x}]} \xi_x\right)^{-1} \xi_x' Q_{[\xi_\ell, \xi_X, \eta_{\Delta x}]} \eta_{e\cdot x}, \tag{12}$$

provided that the limiting matrices are invertible. Notably, $\eta_{e\cdot x}$, which is a functional of $W_{e\cdot x}(\cdot)$, is independent of ξ_X, ξ_x, and $\eta_{\Delta x}$, all of which are functionals of $W_x(\cdot)$. Consequently, conditional on ξ_X, ξ_x, and $\eta_{\Delta x}$, the limiting distributions in (11) and (12) are normal distributions. Therefore, each limiting distribution is a mixed normal distribution.

[2] We denote pathwise Lebesgue integrals by ξ and Itô stochastic integrals by η. Among these, only the stochastic integral $\eta_{e\cdot x}$—the one corresponding to the error term—is intentionally made scale-free. This notational convention is maintained throughout the paper.

Theorem 1 *Let Assumptions 1 and 2 hold. Let $\zeta = [\xi_\ell, \xi_X, \xi_x, \eta_{\Delta x}]$ and assume that $\zeta'\zeta$ is invertible almost surely.*

(a) *For fixed K as $T \to \infty$,*

$$D_T(\hat{\theta}_{\text{TAOLS}} - \theta_0) \Rightarrow \Omega_{ee\cdot x}^{1/2} \left(\zeta'\zeta\right)^{-1} \zeta'\eta_{e\cdot x} \equiv \mathcal{MN}(0, \Omega_{ee\cdot x} \left(\zeta'\zeta\right)^{-1}) := \theta_\infty,$$

where D_T is a diagonal $(3d_x + 1) \times (3d_x + 1)$ matrix with four different blocks:

$$D_T = \text{diag}(\sqrt{T}, T^2 I_{d_x}, T I_{d_x}, I_{d_x}).$$

(b) *For fixed K as $T \to \infty$,*

$$\begin{aligned} T^2(\hat{\beta}_{\text{TAOLS}} - \beta_0) &\Rightarrow \Omega_{ee\cdot x}^{1/2} \left(\xi_X' Q_{[\xi_\ell, \xi_x, \eta_{\Delta x}]} \xi_X\right)^{-1} \xi_X' Q_{[\xi_\ell, \xi_x, \eta_{\Delta x}]} \eta_{e\cdot x} \\ &\equiv \mathcal{MN}(0, \Omega_{ee\cdot x} \left(\xi_X' Q_{[\xi_\ell, \xi_x, \eta_{\Delta x}]} \xi_X\right)^{-1}) := \beta_\infty. \end{aligned} \tag{13}$$

(c) *For fixed K as $T \to \infty$,*

$$\begin{aligned} T\left(\hat{\gamma}_{\text{TAOLS}} - \gamma_0\right) &\Rightarrow \Omega_{ee\cdot x}^{1/2} \left(\xi_x' Q_{[\xi_\ell, \xi_X, \eta_{\Delta x}]} \xi_x\right)^{-1} \xi_x' Q_{[\xi_\ell, \xi_X, \eta_{\Delta x}]} \eta_{e\cdot x} \\ &\equiv \mathcal{MN}(0, \Omega_{ee\cdot x} \left(\xi_x' Q_{[\xi_\ell, \xi_X, \eta_{\Delta x}]} \xi_x\right)^{-1}) := \gamma_\infty. \end{aligned} \tag{14}$$

(d) *For fixed K as $T \to \infty$,*

$$\begin{aligned} \hat{\delta}_{\text{TAOLS}} - \delta_0 &\Rightarrow \Omega_{ee\cdot x}^{1/2} \left(\eta_{\Delta x}' Q_{[\xi_\ell, \xi_X, \xi_x]} \eta_{\Delta x}\right)^{-1} \eta_{\Delta x}' Q_{[\xi_\ell, \xi_X, \xi_x]} \eta_{e\cdot x} \\ &\equiv \mathcal{MN}(0, \Omega_{ee\cdot x} \left(\eta_{\Delta x}' Q_{[\xi_\ell, \xi_X, \xi_x]} \eta_{\Delta x}\right)^{-1}) := \delta_\infty. \end{aligned} \tag{15}$$

Theorem 1(a) shows that the TAOLS estimators of all parameters are asymptotically mixed normal. In particular, Theorems 1(b) and (c) establish this result for the cointegration parameter β_0 and the multicointegration parameter γ_0. The same conclusion holds for the intercept α_0 and the endogeneity parameter δ_0. Theorem 1 extends Theorem 1(b) of Sun et al. (2025), which focuses only on the cointegration parameter and does not include an intercept in the model, unlike our model in Equation (10).

A notable feature of Theorem 1 is that the limiting mixed normal distributions have zero mean. This property is a key advantage of the TAOLS approach. Developed by Hwang and Sun (2018), the method builds on the deterministic trend IV framework of Phillips (2014). Under large-K asymptotics, with $T \to \infty$ and $K \to \infty$ at an appropriate rate, TAOLS and trend IV are asymptotically equivalent. When K is fixed, however, TAOLS effectively converts the model into a classical linear normal regression, allowing standard inference procedures. In particular, we can develop asymptotic F-tests and t-tests, which we now proceed to explore.

We consider testing hypotheses about the cointegration parameters and multicointegration parameters separately. For the former, the null and alternative hypotheses are given by $\mathcal{H}_0 : H\beta_0 = h_0$ and $\mathcal{H}_1 : H\beta_0 \neq h_0$. For the latter, the null and alternative hypotheses are $\mathcal{H}_0 : R\gamma_0 = r_0$ and $\mathcal{H}_1 : R\gamma_0 \neq r_0$. Here, $H,\, R \in \mathbb{R}^{p\times d_x}$ and $h_0,\, r_0 \in \mathbb{R}^{p\times 1}$. We assume that both H and R have full row rank p, so there are p non-reducible restrictions under the null.

To construct the test statistic, we first obtain the residual vector:

$$\hat{V}_{e\cdot x} = V_Y - V_\ell \hat{\alpha}_{\text{TAOLS}} - V_X \hat{\beta}_{\text{TAOLS}} - V_x \hat{\gamma}_{\text{TAOLS}} - V_{\Delta x}\hat{\delta}_{\text{TAOLS}},$$

and then estimate the long-run variance of $\{e_{0\cdot x,t}\}$ by

$$\hat{\Omega}_{ee\cdot x} = \frac{1}{K - 3d_x - 1}||\hat{V}_{e\cdot x}||^2, \tag{16}$$

where $\|\cdot\|$ denotes the Euclidean norm. Note that a degree-of-freedom adjustment has been applied in the above.

Based on $\hat{\Omega}_{ee\cdot x}$, we calculate the Wald statistics as follows:

$$\mathbb{W}_{m,\beta}(K) = \frac{1}{\hat{\Omega}_{ee\cdot x}}(H\hat{\beta}_{\text{TAOLS}} - h_0)' \left[H\left(V_X' Q_{[V_\ell,V_x,V_{\Delta x}]} V_X\right)^{-1} H'\right]^{-1} (H\hat{\beta}_{\text{TAOLS}} - h_0)/p, \tag{17}$$

$$\mathbb{W}_{m,\gamma}(K) = \frac{1}{\hat{\Omega}_{ee\cdot x}}\left(R\hat{\gamma}_{\text{TAOLS}} - r_0\right)' \left[R\left(V_x' Q_{[V_\ell,V_X,V_{\Delta x}]} V_x\right)^{-1} R'\right]^{-1} \left(R\hat{\gamma}_{\text{TAOLS}} - r_0\right)/p. \tag{18}$$

For the special case $p = 1$, we may calculate the t-statistics as:

$$\mathbb{T}_{m,\beta}(K) = \frac{H\hat{\beta}_{\text{TAOLS}} - h_0}{\sqrt{\hat{\Omega}_{ee\cdot x} H\left(V_X' Q_{[V_\ell,V_x,V_{\Delta x}]} V_X\right)^{-1} H'}}, \tag{19}$$

$$\mathbb{T}_{m,\gamma}(K) = \frac{R\hat{\gamma}_{\text{TAOLS}} - r_0}{\sqrt{\hat{\Omega}_{ee\cdot x} R\left(V_x' Q_{[V_\ell,V_X,V_{\Delta x}]} V_x\right)^{-1} R'}}. \tag{20}$$

In the above, the subscript m on the test statistics $\mathbb{W}_m$ and $\mathbb{T}_m$ indicates that they are obtained under multicointegration.

Both statistics follow directly from OLS estimation of the TA regression in Equation (10). Once the transformed variables are obtained, implementation is straightforward using standard statistical software. Packages such as `Stata` report joint tests using the F distribution and single-parameter tests using the t distribution. The following theorem provides a formal justification.

Theorem 2 *Let the assumptions in Theorem 1 hold.*

(*a*) *Under the null* $\mathcal{H}_0 : H\beta_0 = h_0$,

$$\mathbb{W}_{m,\beta}(K) \Rightarrow F_{p,K-3d_x-1}, \tag{21}$$

$$\mathbb{T}_{m,\beta}(K) \Rightarrow t_{K-3d_x-1}, \tag{22}$$

for fixed K *as* $T \to \infty$, *where* $F_{p,K-3d_x-1}$ *is the standard* F *distribution with degrees of freedom* p *and* $K - 3d_x - 1$, *and* t_{K-3d_x-1} *is the standard* t *distribution with degrees of freedom* $K - 3d_x - 1$.

(*b*) *Similarly, under the null* $\mathcal{H}_0 : R\gamma_0 = r_0$,

$$\mathbb{W}_{m,\gamma}(K) \Rightarrow F_{p,K-3d_x-1}, \tag{23}$$

$$\mathbb{T}_{m,\gamma}(K) \Rightarrow t_{K-3d_x-1}, \tag{24}$$

for fixed K *as* $T \to \infty$.

Theorem 2(a) is a variation of the result established in Sun et al. (2025) and is presented here for completeness. Theorem 2(b) introduces a new result that extends the findings in Sun et al. (2025), which does not consider hypothesis testing for γ_0, and Hwang and Sun (2018), which does not consider multicointegration. This theorem provides justification for the standard F and t approximations, which facilitate both straightforward implementation and improved accuracy by explicitly accounting for the estimation errors in $\hat{\delta}_{\text{TAOLS}}$ and $\hat{\Omega}_{ee\cdot x}$—errors typically ignored in the fully modified method (cf. Kheifets and Phillips (2023)). For further discussion of F and t asymptotic theory in other nonstationary and stationary settings, see, for instance, Hwang and Sun (2017), Sun (2023), Sun et al. (2025), Hwang and Sun (2025), and the references therein.

As a special case of the t-test, we can test whether each individual coefficient is zero and use this information to construct confidence intervals. Theorem 1(a) shows that the standard errors (SE) can be computed as the square root of the diagonal elements of $\hat{\Omega}_{ee\cdot x}(V'_{\ell,X,x,\Delta x}V_{\ell,X,x,\Delta x})^{-1}$. An asymptotically valid 95% confidence interval for $\theta_{0,i}$ can be constructed as

$$\hat{\theta}_{\text{TAOLS},i} \pm t^{0.975}_{K-3d_x-1} \cdot \text{SE}(\hat{\theta}_{\text{TAOLS},i}),$$

where $\hat{\theta}_{\text{TAOLS},i}$ is the i-th element of $\hat{\theta}_{\text{TAOLS}}$, and $t^{0.975}_{K-3d_x-1}$ is the 97.5% quantile of the t-distribution with degrees of freedom $K - 3d_x - 1$. This provides a rigorous justification for the use of statistical packages such as Stata, where t-distributions are used to calculate p-values and construct confidence intervals.

The local power properties of the proposed F and t tests can be established using arguments analogous to those in Hwang and Sun (2018). For example, under the local alternative $\mathcal{H}_1 : H\beta_0 = h_0 + \beta_\Delta/T^2$ for some fixed vector β_Δ, we can show that

$$\mathbb{W}_{m,\beta}(K) \Rightarrow F_{p,K-3d_x-1}(\|\lambda\|^2),$$

where

$$\|\lambda\|^2 = \frac{\beta'_\Delta \left\{ H \left(\xi'_X Q_{[\xi_\ell,\xi_x,\eta_{\Delta x}]} \xi_X \right)^{-1} H' \right\}^{-1} \beta_\Delta}{\Omega_{ee\cdot x}},$$

and $F_{p,K-3d_x-1}(\|\lambda\|^2)$ denotes the mixed noncentral F distribution with (random) noncentrality parameter $\|\lambda\|^2$. To conserve space, we omit the formal derivations and additional details.

4 Towards a Unified Asymptotic F and t Theory

The F and t tests developed in the previous sections are asymptotically valid only under multicointegration and become invalid under conventional cointegration. Conversely, the F and t tests of Hwang and Sun (2018) are valid under conventional cointegration but fail under multicointegration. This section develops adaptive F and t tests that remain asymptotically valid under either regime.

4.1 Asymptotic F and t Tests under Conventional Cointegration

This subsection briefly reviews the TAOLS method and the associated asymptotic F and t tests under conventional cointegration, as developed by Hwang and Sun (2018), and lays out the setting for the adaptive procedure introduced in Section 4.3.

As discussed earlier, under conventional cointegration, we retain equations (1) and (2) but assume that Ω is nonsingular. After performing the long-run projection, we obtain the modified model

$$y_t = x'_t\beta_0 + \Delta x'_t\gamma_0 + u_{0\cdot x,t},\ t = 1,\ldots,T, \tag{25}$$

where $\gamma_0 := \Omega_{xx}^{-1}\Omega_{x0}$, $u_{0\cdot x,t} = u_{0,t} - \Delta x'_t\gamma_0$, and the long-run variance $\Omega_{00\cdot x} := \Omega_{00} - \Omega_{0x}\Omega_{xx}^{-1}\Omega_{x0}$ of $\{u_{0\cdot x,t}\}$ is strictly positive. The key distinction between conventional cointegration and multicointegration is whether $\Omega_{00\cdot x}$ is positive or zero.

Applying the basis transformation to all variables in (25) yields

$$V_{y,i} = V'_{x,i}\beta_0 + V'_{\Delta x,i}\gamma_0 + V_{0\cdot x,i}, \tag{26}$$

where $V_{x,i}$ and $V_{\Delta x,i}$ are defined in (9), and $V_{y,i}$ and $V_{0\cdot x,i}$ are defined similarly:

$$V_{y,i} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} y_t \phi_i\left(\frac{t}{T}\right), \; V_{0\cdot x,i} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_{0\cdot x,t} \phi_i\left(\frac{t}{T}\right).$$

The TAOLS estimator of Hwang and Sun (2018) is obtained by applying OLS to (26). Denote the resulting estimators by $\tilde{\beta}_{\text{TAOLS}}$ and $\tilde{\gamma}_{\text{TAOLS}}$, and define the transformed residuals as:

$$\tilde{V}_{0\cdot x,i} = V_{y,i} - V_{x,i}' \tilde{\beta}_{\text{TAOLS}} - V_{\Delta x,i}' \tilde{\gamma}_{\text{TAOLS}}.$$

We can then estimate the long-run variance $\Omega_{00\cdot x}$ of $\{u_{0\cdot x,t}\}$ by

$$\tilde{\Omega}_{00\cdot x} = \frac{1}{K - 2d_x} \sum_{i=1}^{K} \left[\tilde{V}_{0\cdot x,i}\right]^2. \tag{27}$$

Based on the TAOLS estimator and $\tilde{\Omega}_{00\cdot x}$, we construct the Wald and t statistics for testing $\mathcal{H}_0 : H\beta_0 = h_0$ against $\mathcal{H}_1 : H\beta_0 \neq h_0$ as follows:

$$\mathbb{W}_{c,\beta}(K) = \frac{\left(H\tilde{\beta}_{\text{TAOLS}} - h_0\right)' \left[H\left(V_x' Q_{V_{\Delta x}} V_x\right)^{-1} H'\right]^{-1} \left(H\tilde{\beta}_{\text{TAOLS}} - h_0\right)}{p\tilde{\Omega}_{00\cdot x}}, \tag{28}$$

$$\mathbb{T}_{c,\beta}(K) = \frac{H\tilde{\beta}_{\text{TAOLS}} - h_0}{\sqrt{\tilde{\Omega}_{00\cdot x} H\left(V_x' Q_{V_{\Delta x}} V_x\right)^{-1} H'}} \text{ when } p = 1. \tag{29}$$

Similarly, we can construct the Wald and t statistics for testing $\mathcal{H}_0 : R\gamma_0 = r_0$ against $\mathcal{H}_1 : R\gamma_0 \neq r_0$ as:

$$\mathbb{W}_{c,\gamma}(K) = \frac{\left(R\tilde{\gamma}_{\text{TAOLS}} - r_0\right)' \left[R\left(V_{\Delta x}' Q_{V_x} V_{\Delta x}\right)^{-1} R'\right]^{-1} \left(R\tilde{\gamma}_{\text{TAOLS}} - r_0\right)}{p\tilde{\Omega}_{00\cdot x}}, \tag{30}$$

$$\mathbb{T}_{c,\gamma}(K) = \frac{R\tilde{\gamma}_{\text{TAOLS}} - r_0}{\sqrt{\tilde{\Omega}_{00\cdot x} R\left(V_{\Delta x}' Q_{V_x} V_{\Delta x}\right)^{-1} R'}} \text{ when } p = 1. \tag{31}$$

The subscript "c" on the test statistics indicates that they are constructed assuming conventional cointegration (i.e., in the absence of multicointegration).

To establish the asymptotic properties of $\tilde{\Omega}_{00\cdot x}$ and the test statistics under conventional cointegration, we maintain the following assumption.

Assumption 3 *(i)* $x_0 = o_p(\sqrt{T})$;

(ii) The joint FCLT holds:

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor n\cdot \rfloor} \begin{pmatrix} u_{0,t} \\ u_{x,t} \end{pmatrix} \Rightarrow \begin{pmatrix} B_0(\cdot) \\ B_x(\cdot) \end{pmatrix} = \text{BM}(\Omega),$$

where Ω*, as defined in (5), is positive definite.*

Under this assumption, we have

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\left(\begin{array}{c} u_{0\cdot x,t} \\ u_{x,t}\end{array}\right)\Rightarrow\left(\begin{array}{c}\Omega_{00\cdot x}^{1/2}W_{0\cdot x}\left(\cdot\right) \\ \Omega_{xx}^{1/2}W_{x}\left(\cdot\right)\end{array}\right),$$

where $W_{0\cdot x}\left(\cdot\right)$ and $W_{x}\left(\cdot\right)$ are independent Brownian motions.

Theorem 3 *Let Assumptions 2 and 3 hold. If $K>2d_x$ and $\left(\xi_x,\eta_{\Delta x}\right)'\left(\xi_x,\eta_{\Delta x}\right)$ is invertible almost surely, then*

$$\tilde{\Omega}_{00\cdot x}\Rightarrow\Omega_{00\cdot x}\frac{1}{K-2d_x}\eta_{0\cdot x}'Q_{[\xi_x,\eta_{\Delta x}]}\eta_{0\cdot x}.$$

$$\mathbb{W}_{c,\beta}(K)\Rightarrow F_{p,K-2d_x},\ \mathbb{W}_{c,\gamma}(K)\Rightarrow F_{p,K-2d_x},\ \textit{and}$$

$$\mathbb{T}_{c,\beta}(K)\Rightarrow t_{K-2d_x},\ \mathbb{T}_{c,\gamma}(K)\Rightarrow t_{K-2d_x}\ \textit{for } p=1,$$

where $\eta_{0\cdot x}=\int_0^1\boldsymbol{\phi}\left(r\right)dW_{0\cdot x}\left(r\right).$

The limiting distributions of $\mathbb{W}_{c,\beta}(K)$ and $\mathbb{T}_{c,\beta}(K)$ reproduce Theorem 3 of Hwang and Sun (2018) (page 15). The results for $\mathbb{W}_{c,\gamma}(K)$ and $\mathbb{T}_{c,\gamma}(K)$ provide an extension of their theory.

4.2 Asymptotics under Alternative Scenarios

The asymptotic theory in the previous sections assumes *a priori* knowledge of whether the variables are linked by conventional cointegration or multicointegration. In practice, such information is typically unavailable. We therefore examine the behavior of several variance estimators and test statistics under alternative cointegration regimes. These results motivate and inform the structure of the adaptive testing procedure we propose.

The two cointegration regimes can be represented in a unified framework as

$$y_t=x_t'\beta_0+\Delta x_t'\gamma_0+\underbrace{1\left\{\Omega_{00\cdot x}>0\right\}u_{0\cdot x,t}+1\left\{\Omega_{00\cdot x}=0\right\}\Delta e_{0,t}}_{:=z_t}\tag{32}$$

or

$$Y_t=\alpha_0+X_t'\beta_0+x_t'\gamma_0+\Delta x_t'\delta_0+\underbrace{1\left\{\Omega_{00\cdot x}>0\right\}U_{0\cdot x,t}+1\left\{\Omega_{00\cdot x}=0\right\}e_{0\cdot x,t}}_{:=\tilde{Z}_t},\tag{33}$$

where $\delta_0=0\times 1\left\{\Omega_{00\cdot x}>0\right\}+\Omega_{xx}^{-1}\Omega_{xe}\times 1\left\{\Omega_{00\cdot x}=0\right\}$ and $\tilde{Z}_t$ is the cumulative sum of z_t, with an adjustment for the initial value when $\Omega_{00\cdot x}=0$. Equation (32) applies to the level variables (y_t,x_t) and forms the basis for the F and t tests under conventional cointegration, as developed in Hwang and Sun (2018) and reviewed in Section 4.1. Equation (33) applies to the cumulative variables (Y_t, X_t) and underlies the F and t tests under multicointegration described in Section 3.

Under multicointegration ($\Omega_{00\cdot x} = 0$), $z_t = \Delta e_{0,t} \sim I(-1)$, whereas under conventional cointegration ($\Omega_{00\cdot x} > 0$), $z_t = u_{0\cdot x,t} \sim I(0)$. Hence, the stochastic behavior of z_t distinguishes the two regimes. Since z_t is not directly observed, it has to be estimated. Two feasible estimates are given by

$$\hat{z}_t = y_t - x_t'\hat{\beta}_{\text{TAOLS}} - \Delta x_t'\hat{\gamma}_{\text{TAOLS}},$$
$$\tilde{z}_t = y_t - x_t'\tilde{\beta}_{\text{TAOLS}} - \Delta x_t'\tilde{\gamma}_{\text{TAOLS}}.$$

Let $\hat{\vartheta}_{\text{TAOLS}} := (\hat{\beta}_{\text{TAOLS}}', \hat{\gamma}_{\text{TAOLS}}')'$ denote the relevant subvector of the full TAOLS estimator defined in Section 3 and $\tilde{\vartheta}_{\text{TAOLS}} := (\tilde{\beta}_{\text{TAOLS}}', \tilde{\gamma}_{\text{TAOLS}}')'$ be the TAOLS estimator defined in Section 4.1. Under multicointegration, $\hat{\vartheta}_{\text{TAOLS}}$ is asymptotically more efficient, whereas under conventional cointegration, $\tilde{\vartheta}_{\text{TAOLS}}$ is asymptotically more efficient. Since the true regime is unknown, we employ both estimators, leading to two different empirical sequences, $\{\hat{z}_t\}$ and $\{\tilde{z}_t\}$.

Based on each of the two sequences, we construct estimators of the long-run and short-run variances of $\{z_t\}$ as follows:

$$\hat{\Omega}_{zz} = \frac{1}{K - 3d_x - 1}\left\|V_{\hat{z}}\right\|^2, \; \hat{\Sigma}_{zz} = \frac{1}{T - 3d_x - 1}\sum_{t=1}^{T}\hat{z}_t^2,$$
$$\tilde{\Omega}_{zz} = \frac{1}{K - 2d_x}\left\|V_{\tilde{z}}\right\|^2, \; \tilde{\Sigma}_{zz} = \frac{1}{T - 2d_x}\sum_{t=1}^{T}\tilde{z}_t^2,$$

where $V_{\check{z}} = T^{-1/2}\sum_{t=1}^{T}\boldsymbol{\phi}(t/T)\,\check{z}_t$ for $\check{z}_t = \hat{z}_t$ or $\tilde{z}_t$. To establish the asymptotic properties of these variance estimators, we maintain the assumption below.

Assumption 4 *When* $\Omega_{00\cdot x} = 0$, $(e_{0,T}, e_{0,0})' \Rightarrow (e_{0,\infty}, e_{0,0})'$ *jointly with the FCLT in* (7), *and*

$$\frac{1}{T}\sum_{t=1}^{T}(\Delta e_{0,t})^2 \to^p \Sigma_{\Delta e,\Delta e} > 0.$$

When $\Omega_{00\cdot x} > 0$, $(\xi_\ell, \xi_X, \xi_x, \eta_{\Delta x})'(\xi_\ell, \xi_X, \xi_x, \eta_{\Delta x})$ *is invertible almost surely, and*

$$\frac{1}{T}\sum_{t=1}^{T}\left(\Delta x_t', u_{0\cdot x,t}'\right)'\left(\Delta x_t', u_{0\cdot x,t}'\right) \to^p \begin{pmatrix} \Sigma_{u_x,u_x} & \Sigma_{u_x,0\cdot x} \\ \Sigma_{0\cdot x,u_x} & \Sigma_{0\cdot x,0\cdot x} \end{pmatrix} > 0.$$

The following propositions characterize the asymptotic behavior of $\hat{\Omega}_{zz}$ and $\hat{\Sigma}_{zz}$ under both regimes.

Proposition 4 *Consider the case $\Omega_{00\cdot x} = 0$. Let the assumptions in Theorem 1 and Assumption 4 hold. Then*

$$T\hat{\Omega}_{zz} \Rightarrow \frac{1}{K - 3d_x - 1} \|\boldsymbol{\varrho}\|^2 \text{ and } \hat{\Sigma}_{zz} \to^p \Sigma_{\Delta e, \Delta e},$$

where

$$\boldsymbol{\varrho} = \boldsymbol{\phi}(1) e_{0,\infty} - \boldsymbol{\phi}(0) e_{0,0}. \tag{34}$$

Proposition 5 *Consider the case $\Omega_{00\cdot x} > 0$. Let the assumptions in Theorem 3 and Assumption 4 hold. Then*

$$\hat{\Omega}_{zz} \Rightarrow \frac{1}{K - 3d_x - 1} \Omega_{00\cdot x} \left\| \eta_{0\cdot x} - (\xi_x, \eta_{\Delta x}) \left[\xi'_{X,x} Q_{[\xi_\ell, \eta_{\Delta x}]} \xi_{X,x}\right]^{-1} \xi'_{X,x} Q_{[\xi_\ell, \eta_{\Delta x}]} \xi_{0\cdot x} \right\|^2 := \Omega^\infty_{c,I},$$

and

$$\hat{\Sigma}_{zz} \Rightarrow \Sigma_{0\cdot x, 0\cdot x} - 2\Sigma_{0\cdot x, u_x} \gamma^\infty_{c,I} + \left(\gamma^\infty_{c,I}\right)' \Sigma_{u_x, u_x} \gamma^\infty_{c,I} := \Sigma^\infty_{c,I},$$

where $\gamma^\infty_{c,I}$ is the distributional limit of $\hat{\gamma}_{\text{TAOLS}} - \gamma_0 : \hat{\gamma}_{\text{TAOLS}} - \gamma_0 \Rightarrow \gamma^\infty_{c,I}$,[3] *and*

$$\xi_{0\cdot x} = \int_0^1 \boldsymbol{\phi}(r) W_{0\cdot x}(r)\, dr.$$

Furthermore, if $\xi'_{0\cdot x} Q_{[\xi_\ell, \xi_X, \xi_x, \eta_{\Delta x}]} \xi_{0\cdot x} > 0$ almost surely, then

$$\mathbb{W}_{m,\beta}(K) = O_p(1), \mathbb{W}_{m,\gamma}(K) = O_p(1), \mathbb{T}_{m,\beta}(K) = O_p(1), \ \mathbb{T}_{m,\gamma}(K) = O_p(1).$$

Proposition 4 shows that, under multicointegration, $\hat{\Omega}_{zz}$ converges to zero, as expected, since the true long-run variance $\Omega_{\Delta e, \Delta e}$ of $\{z_t := \Delta e_{0,t}\}$ is zero. The contribution of the proposition is to characterize the $1/T$ rate of this convergence. It also establishes that $\hat{\Sigma}_{zz}$ consistently estimates the short-run variance $\Sigma_{\Delta e, \Delta e}$ of $\{z_t\}$. Consequently,

$$T\left(\frac{\hat{\Omega}_{zz}}{\hat{\Sigma}_{zz}}\right) \Rightarrow \frac{1}{K - 3d_x - 1} \frac{\|\boldsymbol{\varrho}\|^2}{\Sigma_{\Delta e, \Delta e}}, \tag{35}$$

so the scaled ratio of the long-run to short-run variance estimators converges to a nondegenerate limit.

[3] In $\Omega^\infty_{c,I}$, $\Sigma^\infty_{c,I}$ and $\gamma^\infty_{c,I}$, the subscript "c" denotes conventional cointegration and the subscript "I" denotes the integrated/cumulative regression in (33) used to estimate the slope coefficients.

Proposition 5 shows that, under conventional cointegration ($\Omega_{00\cdot x} > 0$), both $\hat{\Omega}_{zz}$ and $\hat{\Sigma}_{zz}$ converge in distribution to nondegenerate random variables and are therefore inconsistent for their population counterparts. This inconsistency arises because, when $\Omega_{00\cdot x} > 0$, (33) resembles a spurious regression with $I(1)$ errors, rendering $\hat{\gamma}_{\text{TAOLS}}$ inconsistent. Nevertheless, both estimators are stochastically bounded, and so is their ratio:

$$\frac{\hat{\Omega}_{zz}}{\hat{\Sigma}_{zz}} \Rightarrow \frac{\Omega^{\infty}_{c,I}}{\Sigma^{\infty}_{c,I}}.$$

Comparing this result with (35) reveals a sharp difference in the asymptotic behavior of the ratio of long-run to short-run variance estimators under multicointegration and conventional cointegration, providing a practical basis for distinguishing the two cases.

Proposition 5 further shows that test statistics constructed under multicointegration remain stochastically bounded even when only conventional cointegration holds. This property follows from their self-normalizing structure, which prevents divergence regardless of the underlying cointegration structure. However, the limiting distributions are no longer the classical F or t distributions. Consequently, the F and t tests that are valid under multicointegration cannot be relied upon when multicointegration is absent, as their nominal reference distributions no longer provide valid approximations for inference.

The next two propositions present the asymptotic properties of the variance estimators $\tilde{\Omega}_{zz}$ and $\tilde{\Sigma}_{zz}$. They also show that the test statistics $\mathbb{W}_{c,\beta}(K)$, $\mathbb{T}_{c,\beta}(K)$, $\mathbb{W}_{c,\gamma}(K)$, and $\mathbb{T}_{c,\gamma}(K)$ defined in (28)–(31) are $O_p(1)$ under multicointegration.

Proposition 6 *Consider the case $\Omega_{00\cdot x} = 0$. Let Assumptions 1, 2, and 4 hold, and assume that $(\xi_x, \eta_{\Delta x})'(\xi_x, \eta_{\Delta x})$ is invertible and $\boldsymbol{\varrho}' Q_{[\xi_x, \eta_{\Delta x}]} \boldsymbol{\varrho} > 0$ almost surely, where $\boldsymbol{\varrho}$ is defined in (34). Then*

$$T\tilde{\Omega}_{zz} \Rightarrow \frac{1}{K - 2d_x} \boldsymbol{\varrho}' Q_{[\xi_x, \eta_{\Delta x}]} \boldsymbol{\varrho},$$
$$\mathbb{W}_{c,\beta}(K) = O_p(1), \ \mathbb{T}_{c,\beta}(K) = O_p(1), \ \mathbb{W}_{c,\gamma}(K) = O_p(1), \ \mathbb{T}_{c,\gamma}(K) = O_p(1), \ \textit{and}$$
$$\tilde{\Sigma}_{zz} \to^p \Sigma_{\Delta e, \Delta e}.$$

Proposition 7 *Consider the case $\Omega_{00\cdot x} > 0$. Let the assumptions in Theorem 3 and Assumption 4 hold. Then*

$$\tilde{\Omega}_{zz} \Rightarrow \Omega_{00\cdot x} \frac{1}{K - 2d_x} \eta_{0\cdot x}' Q_{[\xi_x, \eta_{\Delta x}]} \eta_{0\cdot x} := \Omega^{\infty}_{c,L},$$

$$\tilde{\Sigma}_{zz} \Rightarrow \begin{pmatrix} -\gamma^{\infty}_{c,L} \\ 1 \end{pmatrix}' \begin{pmatrix} \Sigma_{u_x, u_x}, & \Sigma_{0\cdot x, u_x} \\ \Sigma'_{0\cdot x, u_x}, & \Sigma_{00\cdot x} \end{pmatrix} \begin{pmatrix} -\gamma^{\infty}_{c,L} \\ 1 \end{pmatrix} := \Sigma^{\infty}_{c,L} > 0,$$

where $\gamma_{c,L}^{\infty}$ *is the distributional limit of* $\tilde{\gamma}_{\text{TAOLS}} - \gamma_0$*:* $\tilde{\gamma}_{\text{TAOLS}} - \gamma_0 \Rightarrow \gamma_{c,L}^{\infty}$.[4]

It follows from Propositions 6 and 7 that under multicointegration

$$T\frac{\tilde{\Omega}_{zz}}{\tilde{\Sigma}_{zz}} \Rightarrow \frac{1}{K-2d_x}\frac{\boldsymbol{\varrho}' Q_{[\xi_x,\eta_{\Delta x}]}\boldsymbol{\varrho}}{\Sigma_{\Delta e,\Delta e}}$$

and under conventional cointegration

$$\frac{\tilde{\Omega}_{zz}}{\tilde{\Sigma}_{zz}} \Rightarrow \frac{\Omega_{c,L}^{\infty}}{\Sigma_{c,L}^{\infty}}.$$

The key implication is that the ratio of long-run to short-run variance estimators behaves differently across regimes: under multicointegration it vanishes at rate $1/T$, whereas under conventional cointegration it converges to a nondegenerate random variable. This distinction provides a simple device for detecting the relevant cointegration structure.

4.3 Adaptive F and t Tests

Our adaptive testing procedure applies to testing both β_0 and γ_0, under single or multiple null restrictions. Without loss of generality, we focus on testing β_0 and illustrate the approach using a Wald statistic. Leveraging the variance ratios $\hat{\Omega}_{zz}/\hat{\Sigma}_{zz}$ and $\tilde{\Omega}_{zz}/\tilde{\Sigma}_{zz}$, we form a weighted average of two test statistics $\mathbb{W}_{c,\beta}$ and $\mathbb{W}_{m,\beta}$. The statistic $\mathbb{W}_{c,\beta}$ is constructed under conventional cointegration in Section 4.1, whereas $\mathbb{W}_{m,\beta}$ is constructed under multicointegration in Section 3. For notational economy, we will drop the subscript β; this should not create any confusion, as our focus throughout this subsection is on testing β_0.

To motivate the weight, let $\kappa \in (0,1)$ be a fixed constant. By Propositions 4 and 5, $T^{\kappa}\hat{\Omega}_{zz}/\hat{\Sigma}_{zz}$ converges to zero in probability under multicointegration, while it diverges to $+\infty$ with probability approaching one under conventional cointegration. Define

$$\hat{a} = \exp\left(-T^{\kappa}\hat{\Omega}_{zz}/\hat{\Sigma}_{zz}\right).$$

Then $\hat{a} \to^p 1$ under multicointegration and $\hat{a} \to^p 0$ under conventional cointegration. Similarly, based on Propositions 6 and 7, define

$$\tilde{a} = \exp\left(-T^{\kappa}\tilde{\Omega}_{zz}/\tilde{\Sigma}_{zz}\right),$$

[4]In $\Omega_{c,L}^{\infty}$, $\Sigma_{c,L}^{\infty}$ and $\gamma_{c,L}^{\infty}$, the subscript "c" denotes conventional cointegration and the subscript "L" denotes the level regression in (32) used to estimate the slope coefficients.

for which $\tilde{a} \to^p 1$ under multicointegration and $\tilde{a} \to^p 0$ under conventional cointegration. Finally, define the weight as the simple average of $\hat{a}$ and $\tilde{a}$:

$$a_T = \frac{\hat{a} + \tilde{a}}{2}.$$

It follows that $a_T \to^p 1$ under multicointegration and $a_T \to^p 0$ under conventional cointegration.

Let $\mathbb{W}_m$ and $\mathbb{W}_c$ denote the Wald statistics derived under multicointegration and conventional cointegration, respectively. The adaptive statistic is defined to be

$$\mathbb{W}_a := a_T \mathbb{W}_m + (1 - a_T)\mathbb{W}_c.$$

The statistic $\mathbb{W}_a$ is asymptotically F distributed regardless of the underlying cointegration regime. To see this, consider the case $\Omega_{00\cdot x} > 0$, for which $a_T \to^p 0$. Furthermore, by Proposition 5, $\mathbb{W}_m = O_p(1)$. Hence

$$\mathbb{W}_a = o_p(1)\,\mathbb{W}_m + \mathbb{W}_c(1 + o_p(1)) \Rightarrow F_{p, K_c - 2d_x}$$

where K_c is the number of basis functions used under conventional cointegration. On the other hand, when $\Omega_{00\cdot x} = 0$, $a_T \to^p 1$ and by Proposition 6, $\mathbb{W}_c = O_p(1)$. Hence

$$\mathbb{W}_a = \mathbb{W}_m(1 + o_p(1)) + o_p(1)\,\mathbb{W}_c \Rightarrow F_{p, K_m - 3d_x - 1}$$

where K_m is the number of basis functions used under multicointegration. Let $K_c = K + 2d_x$ and $K_m = K + 3d_x + 1$ for some integer K, so that the effective degrees of freedom are the same across the two regimes. Then, under the null, $\mathbb{W}_a \Rightarrow F_{p,K}$ in both regimes. Under the alternative hypothesis, $\mathbb{W}_a$ approximates the statistic corresponding to the correctly specified regime, thereby maintaining power close to that of $\mathbb{W}_c$ when conventional cointegration holds and to that of $\mathbb{W}_m$ when multicointegration is present.

We now describe our proposed adaptive test in some detail. For a given tuning parameter K, a fixed constant $\kappa \in (0, 1)$, and a nominal significance level α, the test is implemented as follows.

1. Wald statistic under Conventional Cointegration

    (a) Let $K_c = K + 2d_x$.

    (b) Estimate $V_{y,i} = V_{x,i}'\beta_0 + V_{\Delta x,i}'\gamma_0 + V_{z,i}$ for $i \in [K_c]$ by OLS to obtain $\tilde{\beta}_{\text{TAOLS}}$ and $\tilde{\gamma}_{\text{TAOLS}}$.

(c) Compute

$$\tilde{\Omega}_{zz} = \frac{1}{K}\sum_{i=1}^{K_c}\left(V_{y,i} - V_{x,i}'\tilde{\beta}_{\text{TAOLS}} - V_{\Delta x,i}'\tilde{\gamma}_{\text{TAOLS}}\right)^2,$$

$$\tilde{\Sigma}_{zz} = \frac{1}{T-2d_x}\sum_{t=1}^{T}\left(y_t - x_t\tilde{\beta}_{\text{TAOLS}} - \Delta x_t\tilde{\gamma}_{\text{TAOLS}}\right)^2,$$

and the weighting factor $\tilde{a} = \exp(-T^{\kappa}\tilde{\Omega}_{zz}/\tilde{\Sigma}_{zz})$.

(d) Construct

$$\mathbb{W}_c = \frac{\left(H\tilde{\beta}_{\text{TAOLS}} - h_0\right)'\left[H\left(V_x' Q_{V_{\Delta x}} V_x\right)^{-1}H'\right]^{-1}\left(H\tilde{\beta}_{\text{TAOLS}} - h_0\right)}{p\tilde{\Omega}_{zz}}.$$

2. Wald statistic under Multicointegration

(a) Let $K_m = K + 3d_x + 1$.

(b) Estimate $V_{Y,i} = V_{\ell,i}\alpha_0 + V_{X,i}'\beta_0 + V_{x,i}'\gamma_0 + V_{\Delta x,i}'\delta_0 + V_{z,i}$ for $i \in [K_m]$ by OLS to obtain $\hat{\alpha}_{\text{TAOLS}}, \hat{\beta}_{\text{TAOLS}}, \hat{\gamma}_{\text{TAOLS}}$ and $\hat{\delta}_{\text{TAOLS}}$.

(c) Compute

$$\hat{\Omega}_{zz} = \frac{1}{K}\sum_{i=1}^{K_m}(V_{Y,i} - V_{\ell,i}\hat{\alpha}_{\text{TAOLS}} - V_{X,i}'\hat{\beta}_{\text{TAOLS}} - V_{x,i}'\hat{\gamma}_{\text{TAOLS}} - V_{\Delta x,i}'\hat{\delta}_{\text{TAOLS}})^2,$$

$$\hat{\Sigma}_{zz} = \frac{1}{T-3d_x-1}\sum_{t=1}^{T}(y_t - x_t'\hat{\beta}_{\text{TAOLS}} - \Delta x_t\hat{\gamma}_{\text{TAOLS}})^2,$$

and form the weighting factor $\hat{a} = \exp(-T^{\kappa}\hat{\Omega}_{zz}/\hat{\Sigma}_{zz})$.

(d) Construct

$$\mathbb{W}_m = \frac{(H\hat{\beta}_{\text{TAOLS}} - h_0)'\left[H\left(V_X' Q_{[V_\ell, V_x, V_{\Delta x}]} V_X\right)^{-1}H'\right]^{-1}(H\hat{\beta}_{\text{TAOLS}} - h_0)}{p\hat{\Omega}_{zz}}.$$

3. Combination of the Wald statistics

(a) Compute the weight $a_T : a_T = (\hat{a} + \tilde{a})/2$.

(b) Form the adaptive test statistic: $\mathbb{W}_a = a_T\mathbb{W}_m + (1 - a_T)\mathbb{W}_c$.

4. Decision rule: Let $F_{p,K}^{\alpha}$ denote the $(1-\alpha)$-quantile of the $F_{p,K}$ distribution. Reject the null hypothesis if $\mathbb{W}_a > F_{p,K}^{\alpha}$.

This procedure requires no pre-testing for the presence of multicointegration and delivers correct asymptotic size under both the conventional cointegration regime and the multicointegration regime.

5 Simulation

5.1 Data Generating Processes

Following Phillips and Kheifets (2024), we consider the data generating process

$$y_t = \beta_0 x_t + u_{0,t},$$
$$x_t = x_{t-1} + u_{x,t}, \quad t = 1, \ldots, T,$$

where $x_0 = 0$ and the innovation vector follows an MA(1) process:

$$u_t = \binom{u_{0,t}}{u_{x,t}} = L\epsilon_t + D_1 L\epsilon_{t-1},$$

with L being the lower-triangular Cholesky decomposition of Σ :

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

and $\epsilon_t \sim$ iid $\mathcal{N}(0, I_2)$ for $t = 0, 1, \ldots, T$. The (short-run) variance and long-run variance of u_t are $\Sigma + D_1 \Sigma D_1'$ and

$$\Omega = \begin{pmatrix} \Omega_{00} & \Omega_{0x} \\ \Omega_{x0} & \Omega_{xx} \end{pmatrix} = (I_2 + D_1)L\left[(I_2 + D_1)L\right]',$$

respectively.

The key parameters in the DGP are:

- $\beta_0 = 2$, the true cointegration coefficient;
- $\gamma_0 := \Omega_{xx}^{-1}\Omega_{0x}$, the multicointegration coefficient or the long-run endogeneity coefficient;
- ρ, the correlation between contemporaneous innovations, which controls endogeneity;
- D_1, a general 2×2 moving-average coefficient matrix, which, together with Σ, determines whether the system exhibits conventional cointegration or multicointegration.

Models C0, C1, and C2 are designed so that $\Omega_{00\cdot x} > 0$, corresponding to conventional cointegration. Models M0, M1, M2, and M3 are constructed such that $\Omega_{00\cdot x} = \Omega_{00} - \Omega_{0x}\Omega_{xx}^{-1}\Omega_{x0} = 0$, representing multicointegration.

Under multicointegration, the multicointegration error $e_{0,t}$ (see equation (3)) is given by

$$
\begin{aligned}
e_{0,t} - e_{0,0} &= \sum_{s=1}^{t} \left(u_{0,s} - u_{x,s}'\gamma_0\right) = \sum_{s=1}^{t} \left(L\epsilon_s + D_1 L\epsilon_{s-1}\right)' \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix} \\
&= \sum_{s=1}^{t} \epsilon_s' \left(L + D_1 L\right)' \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix} - \epsilon_t' L' D_1' \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix} + \epsilon_0' L' D_1' \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix} \\
&= -\epsilon_t' L' D_1' \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix} + \epsilon_0' L' D_1' \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix} + \sum_{s=1}^{t} \epsilon_s' L' \left(I_2 + D_1\right)' \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix}.
\end{aligned}
$$

For each summand in the third term above, the mean is zero, and the variance is $\Omega_{00\cdot x}$. Thus, when $\Omega_{00\cdot x} = 0$, as in Models M0–M3, the third term is zero almost surely. Hence, for $t = 0, 1, \ldots$

$$
\begin{aligned}
e_{0,t} &= -\begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix}' D_1 L\epsilon_t \sim iid\ N\left(0, \Omega_{ee}\right) \text{ for} \\
\Omega_{ee} &= \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix}' D_1 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} D_1' \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix}.
\end{aligned}
$$

Given the above linear representations of $e_{0,t}$ and $u_{x,t}$, we can obtain the long-run covariance between $e_{0,t}$ and $u_{x,t}$ as

$$
\begin{aligned}
\Omega_{xe} &= -\begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix}' D_1 L L' \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix}' D_1 L L' D_1' \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -\begin{pmatrix} 1 \\ -\gamma_0 \end{pmatrix}' D_1 \Sigma (I_2 + D_1)' \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{aligned}
$$

Based on this, we can compute $\delta_0 = \Omega_{xx}^{-1}\Omega_{xe}$ and $e_{0\cdot x,t} = e_{0,t} - u_{x,t}'\delta_0$. It can be shown that $e_{0\cdot x,t}$ follows an MA(1) process with long-run variance $\Omega_{ee\cdot x} = \Omega_{ee} - \Omega_{ex}\Omega_{xx}^{-1}\Omega_{xe}$.

Table 1 summarizes the seven DGPs, reporting the corresponding values of D_1 and ρ, as well as the resulting Ω, γ_0, δ_0, $\Omega_{00\cdot x}$, and $\Omega_{ee\cdot x}$ (where applicable).

For each DGP, we evaluate the three F-tests for testing β_0, all using $F_{p,K}$ critical values:

(i) the test of Hwang and Sun (2018) based on $\mathbb{W}_c(K_c)$;

(ii) the test in Section 3 based on $\mathbb{W}_m(K_m)$;

(iii) the adaptive test in Section 4.3 based on $\mathbb{W}_a$, which combines the two statistics $\mathbb{W}_c(K_c)$ and $\mathbb{W}_m(K_m)$ using the data-driven weight a_T.

For the weighting scheme, we consider two alternatives: soft weighting ($a_{\text{soft}} = a_T$) and hard weighting ($a_{\text{hard}} = \mathbb{I}\{a_T > 0.5\}$). The tuning parameter κ varies over the grid $\{0, 0.1, \ldots, 1.0\}$. We find that values of κ near 0.5 yield superior size control compared to extreme values near 0 or 1. Accordingly, we report results for $\kappa \in \{0.4, 0.5, 0.6\}$. All empirical null rejection rates are computed using 10,000 simulation replications.

Table 1: Data Generating Processes with Key Parameters

DGP	D_1	ρ	$\mathrm{var}(u_t)$	$\Omega := \mathrm{lrvar}(u_t)$	γ_0	δ_0	$\Omega_{00\cdot x}$	$U_{0\cdot x,t}$ or $e_{0,t}$	$\Omega_{ee\cdot x}$
C0	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0	0	1	$U_{0\cdot x,t} \sim I(1)$	–
C1	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	.5	$\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$	.5	0	.75	$U_{0\cdot x,t} \sim I(1)$	–
C2	$\begin{pmatrix} 0.3 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}$	.5	$\begin{pmatrix} 1.37 & 1.23 \\ 1.23 & 2.48 \end{pmatrix}$	$\begin{pmatrix} 2.37 & 2.88 \\ 2.88 & 4.48 \end{pmatrix}$	.64	0	.52	$U_{0\cdot x,t} \sim I(1)$	–
M0	$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$	0	$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	0	0	0	$e_{0,t} \overset{iid}{\sim} N(0,1)$	1
M1	$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$	.5	$\begin{pmatrix} 2 & 0.5 \\ 0.5 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	0	.5	0	$e_{0,t} \overset{iid}{\sim} N(0,1)$	.75
M2	$\begin{pmatrix} 0.3 & 0.4 \\ 5.2 & 0.6 \end{pmatrix}$	.5	$\begin{pmatrix} 1.37 & 3.43 \\ 3.43 & 31.52 \end{pmatrix}$	$\begin{pmatrix} 2.37 & 9.48 \\ 9.48 & 37.92 \end{pmatrix}$	.25	.13	0	$e_{0,t} \overset{iid}{\sim} N(0,.81)$	.17
M3	$\begin{pmatrix} -0.3 & 0.4 \\ 0.7 & -0.6 \end{pmatrix}$	.5	$\begin{pmatrix} 1.13 & 0.28 \\ 0.28 & 1.43 \end{pmatrix}$	$\begin{pmatrix} 0.93 & 0.93 \\ 0.93 & 0.93 \end{pmatrix}$	1	.16	0	$e_{0,t} \overset{iid}{\sim} N(0,1)$	.98

5.2 Simulation Results

The empirical null rejection rates reported in Tables 2–5 provide clear evidence of the different size properties of the F-tests under different cointegration structures. For nonsingular cointegration models (C0–C2), the F-test based on $\mathbb{W}_c$ exhibits accurate size across sample sizes ($T = 100, 200, 400$) and K values ($K = 10$, 20 with $K_c = K + 2$, $K_m = K + 4$). In contrast, it severely under-rejects under multicointegration (M0–M3) with rejection rates often falling below 2% and in extreme cases approaching zero (e.g., M0 with $K = 20$, $T = 200$). This size distortion arises because the auxiliary regression in the nonsingular framework fails to capture the higher-order cointegration structure, leading to overly conservative inference.

The F-test based on $\mathbb{W}_m$ displays near-nominal size in models M0–M3 where multicointegration is present, but substantially over-rejects in conventional cointegration models, particularly when $K = 20$. Rejection rates exceed 60% in some cases (e.g., C0–C2 with $T = 100$), reflecting a severe lack of robustness when the multicointegration structure is absent.

These findings underscore a critical asymmetry: the F-test designed for nonsingular cointegration becomes unreliable in the presence of multicointegration, while the F-test designed under multicointegration becomes unreliable when multicointegration is absent.

The adaptive test $\mathbb{W}_a$ mitigates this sensitivity by combining the two statistics through the data-driven weight a_T. Tables 2–5 show that a_T is small (i.e., close to zero) under nonsingular cointegration (C0–C2) and large (i.e., close to one) under multicointegration (M0–M3), with this pattern becoming more pronounced as the sample size increases. This substantial difference in a_T across the two cointegration regimes provides strong evidence that the proposed procedure effectively adapts to the underlying cointegration structure. Consequently, $\mathbb{W}_a$ maintains rejection rates close to the nominal level across both regimes.

In particular, with $K = 10$ and $\kappa = 0.5$ under soft weighting, the adaptive test outperforms both individual tests in an overall sense. It avoids the severe under-rejection of $\mathbb{W}_c$ under multicointegration and the extreme over-rejection of $\mathbb{W}_m$ under nonsingular cointegration, delivering rejection rates consistently near the nominal 5% level in most scenarios. This adaptability makes the adaptive test a practical and robust choice when the nature of the cointegration relationships is unknown.

Size-adjusted power calculations under the alternative hypotheses $\mathcal{H}_1 : \beta_0 = 2.1$ for conventional cointegration models and $\mathcal{H}_1 : \beta_0 = 2.001$ for multicointegration models are reported in Tables 6–7. The results indicate that the F-test based on the statistic $\mathbb{W}_c$ exhibits higher power than the test based on $\mathbb{W}_m$ under conventional cointegration, whereas the opposite holds for multicointegration, where $\mathbb{W}_m$ displays substantially greater power. This pattern is unsurprising: each test enjoys an optimality property in the regime for which it was designed (see Phillips (2014) and Hwang and Sun (2018) for the test based on $\mathbb{W}_c$, and Phillips and Kheifets (2024) and Sun et al. (2025) for the test based on $\mathbb{W}_m$).

To assess the power of the adaptive test, let $\mathbb{W}_{\text{soft}}$ denote the version based on soft weighting ($a_{\text{soft}} = a_T$), and $\mathbb{W}_{\text{hard}}$ the version based on hard weighting ($a_{\text{hard}} = \mathbb{I}\{a_T > 0.5\}$). When $K = 10$, the size-adjusted power of both $\mathbb{W}_{\text{soft}}$ and $\mathbb{W}_{\text{hard}}$ is generally high across all values of κ, with only a modest decline as κ increases. Both versions exhibit power comparable to that of the better-performing individual F-test. When $K = 20$, however, $\mathbb{W}_{\text{hard}}$ experiences a noticeable power loss in multicointegration models, particularly for small sample sizes and large κ. Overall, $\mathbb{W}_{\text{soft}}$ tends to maintain power that is similar to or superior to that of $\mathbb{W}_{\text{hard}}$.

Considering both size and power performances for testing β_0, the adaptive test $\mathbb{W}_a$ with soft weighting and tuning parameters $K = 10$ and $\kappa = 0.5$ performs best overall. Under this configuration, the corresponding adaptive F-test for testing γ_0 also has more accurate size than the individual F-tests based on $\mathbb{W}_c$ and $\mathbb{W}_m$. Table 8 illustrates this finding, along with other qualitative patterns that are similar to those reported in Tables 2–5. Accordingly, we recommend the adaptive test $\mathbb{W}_a$ with soft weighting, $K = 10$, and $\kappa = 0.5$ for empirical applications.

Table 2: Empirical null rejection rates of the F tests for testing β_0 under soft weighting with 10,000 simulation replications: $K = 10$

				$\kappa = 0.4$		$\kappa = 0.5$		$\kappa = 0.6$	
Model	T	$\mathbb{W}_c$	$\mathbb{W}_m$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$
C0	100	0.052	0.470	0.069	0.013	0.056	0.003	0.053	0.001
C0	200	0.050	0.479	0.059	0.006	0.051	0.001	0.050	0.000
C0	400	0.050	0.478	0.054	0.003	0.050	0.000	0.050	0.000
C1	100	0.049	0.469	0.064	0.013	0.052	0.003	0.049	0.001
C1	200	0.052	0.476	0.061	0.006	0.054	0.001	0.052	0.000
C1	400	0.049	0.481	0.053	0.003	0.050	0.000	0.049	0.000
C2	100	0.050	0.471	0.091	0.036	0.062	0.011	0.053	0.002
C2	200	0.050	0.483	0.072	0.019	0.055	0.004	0.051	0.001
C2	400	0.050	0.478	0.062	0.009	0.052	0.001	0.050	0.000
M0	100	0.008	0.051	0.034	0.759	0.028	0.655	0.022	0.527
M0	200	0.011	0.049	0.042	0.888	0.037	0.821	0.033	0.724
M0	400	0.013	0.050	0.045	0.947	0.042	0.907	0.038	0.842
M1	100	0.020	0.051	0.036	0.759	0.031	0.654	0.027	0.527
M1	200	0.023	0.050	0.043	0.888	0.038	0.822	0.034	0.725
M1	400	0.022	0.048	0.044	0.947	0.042	0.907	0.039	0.842
M2	100	0.046	0.070	0.047	0.759	0.040	0.655	0.033	0.527
M2	200	0.049	0.054	0.045	0.889	0.041	0.822	0.036	0.726
M2	400	0.048	0.053	0.048	0.947	0.045	0.907	0.041	0.842
M3	100	0.008	0.057	0.037	0.754	0.030	0.648	0.024	0.519
M3	200	0.008	0.051	0.044	0.888	0.039	0.820	0.033	0.722
M3	400	0.011	0.050	0.047	0.946	0.044	0.906	0.040	0.839

Table 3: Empirical null rejection rates of the F tests for testing β_0 under soft weighting with 10,000 simulation replications: $K = 20$

				$\kappa = 0.4$		$\kappa = 0.5$		$\kappa = 0.6$	
Model	T	$\mathbb{W}_c$	$\mathbb{W}_m$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$
C0	100	0.051	0.608	0.064	0.006	0.053	0.001	0.051	0.000
C0	200	0.047	0.613	0.052	0.002	0.048	0.000	0.047	0.000
C0	400	0.049	0.609	0.051	0.001	0.049	0.000	0.049	0.000
C1	100	0.049	0.598	0.064	0.006	0.051	0.001	0.050	0.000
C1	200	0.050	0.612	0.055	0.002	0.051	0.000	0.050	0.000
C1	400	0.049	0.615	0.051	0.001	0.049	0.000	0.049	0.000
C2	100	0.047	0.599	0.095	0.022	0.059	0.004	0.049	0.000
C2	200	0.050	0.609	0.074	0.010	0.054	0.001	0.051	0.000
C2	400	0.050	0.611	0.059	0.004	0.050	0.000	0.050	0.000
M0	100	0.002	0.054	0.019	0.534	0.010	0.383	0.005	0.235
M0	200	0.001	0.053	0.036	0.785	0.028	0.669	0.017	0.515
M0	400	0.004	0.046	0.039	0.909	0.034	0.843	0.028	0.737
M1	100	0.011	0.053	0.019	0.533	0.013	0.381	0.011	0.234
M1	200	0.010	0.052	0.036	0.786	0.027	0.669	0.019	0.515
M1	400	0.013	0.048	0.040	0.909	0.035	0.843	0.027	0.737
M2	100	0.049	0.084	0.034	0.532	0.030	0.381	0.032	0.233
M2	200	0.055	0.065	0.043	0.787	0.036	0.670	0.033	0.517
M2	400	0.058	0.053	0.045	0.910	0.040	0.843	0.032	0.738
M3	100	0.003	0.067	0.020	0.519	0.012	0.367	0.007	0.221
M3	200	0.002	0.053	0.034	0.783	0.026	0.665	0.017	0.510
M3	400	0.003	0.047	0.040	0.908	0.032	0.841	0.026	0.734

Table 4: Empirical null rejection rates of the F tests for testing β_0 under hard weighting with 10,000 simulation replications: $K = 10$

				$\kappa = 0.4$		$\kappa = 0.5$		$\kappa = 0.6$	
Model	T	$\mathbb{W}_c$	$\mathbb{W}_m$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$
C0	100	0.052	0.470	0.052	0.000	0.052	0.000	0.052	0.000
C0	200	0.050	0.479	0.050	0.000	0.050	0.000	0.050	0.000
C0	400	0.050	0.478	0.050	0.000	0.050	0.000	0.050	0.000
C1	100	0.049	0.469	0.049	0.000	0.049	0.000	0.049	0.000
C1	200	0.052	0.476	0.052	0.000	0.052	0.000	0.052	0.000
C1	400	0.049	0.481	0.049	0.000	0.049	0.000	0.049	0.000
C2	100	0.050	0.471	0.050	0.001	0.050	0.000	0.050	0.000
C2	200	0.050	0.483	0.050	0.000	0.050	0.000	0.050	0.000
C2	400	0.050	0.478	0.050	0.000	0.050	0.000	0.050	0.000
M0	100	0.008	0.051	0.050	0.971	0.048	0.849	0.040	0.585
M0	200	0.011	0.049	0.049	0.999	0.049	0.985	0.047	0.918
M0	400	0.013	0.050	0.050	1.000	0.050	0.999	0.049	0.980
M1	100	0.020	0.051	0.050	0.971	0.047	0.848	0.043	0.588
M1	200	0.023	0.050	0.050	0.999	0.050	0.986	0.049	0.919
M1	400	0.022	0.048	0.048	1.000	0.048	0.999	0.047	0.979
M2	100	0.046	0.070	0.068	0.969	0.061	0.850	0.053	0.585
M2	200	0.049	0.054	0.054	0.999	0.053	0.986	0.051	0.918
M2	400	0.048	0.053	0.053	1.000	0.052	0.999	0.051	0.982
M3	100	0.008	0.057	0.056	0.967	0.052	0.836	0.039	0.565
M3	200	0.008	0.051	0.051	1.000	0.051	0.986	0.049	0.916
M3	400	0.011	0.050	0.050	1.000	0.050	0.999	0.050	0.981

Table 5: Empirical null rejection rates of the F tests for testing β_0 under hard weighting with 10,000 simulation replications: $K = 20$

				$\kappa = 0.4$		$\kappa = 0.5$		$\kappa = 0.6$	
Model	T	$\mathbb{W}_c$	$\mathbb{W}_m$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$
C0	100	0.051	0.608	0.051	0.000	0.051	0.000	0.051	0.000
C0	200	0.047	0.613	0.047	0.000	0.047	0.000	0.047	0.000
C0	400	0.049	0.609	0.049	0.000	0.049	0.000	0.049	0.000
C1	100	0.049	0.598	0.049	0.000	0.049	0.000	0.049	0.000
C1	200	0.050	0.612	0.050	0.000	0.050	0.000	0.050	0.000
C1	400	0.049	0.615	0.049	0.000	0.049	0.000	0.049	0.000
C2	100	0.047	0.599	0.047	0.000	0.047	0.000	0.047	0.000
C2	200	0.050	0.609	0.050	0.000	0.050	0.000	0.050	0.000
C2	400	0.050	0.611	0.050	0.000	0.050	0.000	0.050	0.000
M0	100	0.002	0.054	0.043	0.613	0.020	0.211	0.006	0.032
M0	200	0.001	0.053	0.053	0.994	0.051	0.922	0.036	0.560
M0	400	0.004	0.046	0.046	1.000	0.046	0.998	0.044	0.961
M1	100	0.011	0.053	0.041	0.608	0.025	0.211	0.014	0.033
M1	200	0.010	0.052	0.052	0.994	0.051	0.921	0.037	0.563
M1	400	0.013	0.048	0.048	1.000	0.048	0.998	0.046	0.961
M2	100	0.049	0.084	0.061	0.606	0.048	0.210	0.049	0.035
M2	200	0.055	0.065	0.064	0.994	0.061	0.923	0.056	0.569
M2	400	0.058	0.053	0.053	1.000	0.053	0.998	0.051	0.961
M3	100	0.003	0.067	0.043	0.571	0.018	0.184	0.006	0.028
M3	200	0.002	0.053	0.053	0.995	0.049	0.914	0.035	0.550
M3	400	0.003	0.047	0.047	1.000	0.047	0.998	0.046	0.961

Table 6: Size-adjusted power with 10,000 simulation replications: $K = 10$

				$\kappa = 0.4$		$\kappa = 0.5$		$\kappa = 0.6$	
Model	T	$\mathbb{W}_c$	$\mathbb{W}_m$	$\mathbb{W}_{\text{soft}}$	$\mathbb{W}_{\text{hard}}$	$\mathbb{W}_{\text{soft}}$	$\mathbb{W}_{\text{hard}}$	$\mathbb{W}_{\text{soft}}$	$\mathbb{W}_{\text{hard}}$
				$\mathcal{H}_1 : \beta_0 = 2.1$					
C0	100	0.899	0.718	0.885	0.899	0.895	0.899	0.898	0.899
C0	200	0.994	0.899	0.993	0.994	0.994	0.994	0.994	0.994
C0	400	1.000	0.975	1.000	1.000	1.000	1.000	1.000	1.000
C1	100	0.936	0.766	0.921	0.936	0.932	0.936	0.934	0.936
C1	200	0.997	0.919	0.996	0.997	0.997	0.997	0.997	0.997
C1	400	1.000	0.978	1.000	1.000	1.000	1.000	1.000	1.000
C2	100	0.999	0.944	0.998	0.999	0.998	0.999	0.999	0.999
C2	200	1.000	0.990	1.000	1.000	1.000	1.000	1.000	1.000
C2	400	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000
				$\mathcal{H}_1 : \beta_0 = 2.001$					
M0	100	0.060	0.169	0.160	0.167	0.156	0.160	0.150	0.139
M0	200	0.244	0.705	0.702	0.705	0.698	0.696	0.692	0.657
M0	400	0.763	0.982	0.982	0.982	0.982	0.980	0.981	0.967
M1	100	0.036	0.214	0.200	0.212	0.188	0.197	0.165	0.154
M1	200	0.103	0.755	0.750	0.754	0.746	0.746	0.738	0.697
M1	400	0.618	0.988	0.988	0.988	0.987	0.986	0.987	0.971
M2	100	0.195	0.975	0.977	0.953	0.975	0.851	0.970	0.626
M2	200	0.788	1.000	1.000	0.999	1.000	0.993	1.000	0.968
M2	400	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
M3	100	0.062	0.189	0.184	0.185	0.176	0.177	0.165	0.147
M3	200	0.208	0.693	0.691	0.693	0.688	0.685	0.682	0.641
M3	400	0.727	0.983	0.983	0.983	0.983	0.982	0.982	0.969

Note: $\mathbb{W}_{\text{soft}}$ denotes $\mathbb{W}_a$ with soft weighting, and $\mathbb{W}_{\text{hard}}$ denotes $\mathbb{W}_a$ with hard weighting.

Table 7: Size-adjusted power with 10,000 simulation replications: $K = 20$

Model	T	$\mathbb{W}_c$	$\mathbb{W}_m$	$\kappa = 0.4$ $\mathbb{W}_{\text{soft}}$	$\kappa = 0.4$ $\mathbb{W}_{\text{hard}}$	$\kappa = 0.5$ $\mathbb{W}_{\text{soft}}$	$\kappa = 0.5$ $\mathbb{W}_{\text{hard}}$	$\kappa = 0.6$ $\mathbb{W}_{\text{soft}}$	$\kappa = 0.6$ $\mathbb{W}_{\text{hard}}$
				$\mathcal{H}_1 : \beta_0 = 2.1$					
C0	100	0.922	0.740	0.911	0.922	0.920	0.922	0.922	0.922
C0	200	0.997	0.912	0.997	0.997	0.997	0.997	0.997	0.997
C0	400	1.000	0.981	1.000	1.000	1.000	1.000	1.000	1.000
C1	100	0.954	0.787	0.947	0.954	0.953	0.954	0.954	0.954
C1	200	0.999	0.930	0.999	0.999	0.999	0.999	0.999	0.999
C1	400	1.000	0.984	1.000	1.000	1.000	1.000	1.000	1.000
C2	100	1.000	0.953	0.999	1.000	0.999	1.000	1.000	1.000
C2	200	1.000	0.993	1.000	1.000	1.000	1.000	1.000	1.000
C2	400	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
				$\mathcal{H}_1 : \beta_0 = 2.001$					
M0	100	0.057	0.188	0.173	0.153	0.163	0.096	0.143	0.064
M0	200	0.192	0.752	0.749	0.749	0.744	0.697	0.729	0.459
M0	400	0.735	0.988	0.988	0.988	0.987	0.986	0.986	0.957
M1	100	0.038	0.247	0.201	0.177	0.165	0.090	0.116	0.048
M1	200	0.042	0.794	0.788	0.790	0.782	0.733	0.758	0.461
M1	400	0.461	0.992	0.992	0.992	0.992	0.990	0.992	0.959
M2	100	0.075	0.974	0.976	0.618	0.966	0.265	0.916	0.107
M2	200	0.582	1.000	1.000	0.998	1.000	0.957	1.000	0.796
M2	400	0.987	1.000	1.000	1.000	1.000	1.000	1.000	0.999
M3	100	0.069	0.216	0.198	0.172	0.185	0.096	0.155	0.072
M3	200	0.156	0.744	0.736	0.740	0.731	0.686	0.717	0.439
M3	400	0.662	0.989	0.989	0.989	0.989	0.987	0.989	0.959

Note: $\mathbb{W}_{\text{soft}}$ denotes $\mathbb{W}_a$ with soft weighting, and $\mathbb{W}_{\text{hard}}$ denotes $\mathbb{W}_a$ with hard weighting.

Table 8: Empirical null rejection rates of the F tests for testing γ_0 under soft weighting with 10,000 simulation replications: $K = 10$

				$\kappa = 0.4$		$\kappa = 0.5$		$\kappa = 0.6$	
Model	T	$\mathbb{W}_c$	$\mathbb{W}_m$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$	$\mathbb{W}_a$	$\bar{a}$
C0	100	0.048	0.277	0.053	0.013	0.050	0.003	0.048	0.001
C0	200	0.047	0.285	0.049	0.006	0.048	0.001	0.047	0.000
C0	400	0.051	0.284	0.053	0.003	0.051	0.000	0.051	0.000
C1	100	0.050	0.282	0.054	0.013	0.051	0.003	0.050	0.001
C1	200	0.051	0.280	0.053	0.006	0.051	0.001	0.051	0.000
C1	400	0.051	0.278	0.052	0.003	0.051	0.000	0.051	0.000
C2	100	0.047	0.294	0.055	0.036	0.050	0.011	0.047	0.002
C2	200	0.054	0.292	0.059	0.019	0.054	0.004	0.054	0.001
C2	400	0.054	0.281	0.056	0.009	0.054	0.001	0.054	0.000
M0	100	0.055	0.051	0.040	0.759	0.039	0.655	0.039	0.527
M0	200	0.052	0.048	0.041	0.888	0.039	0.821	0.036	0.724
M0	400	0.052	0.050	0.047	0.947	0.045	0.907	0.043	0.842
M1	100	0.047	0.051	0.041	0.759	0.037	0.654	0.035	0.527
M1	200	0.039	0.048	0.042	0.888	0.040	0.822	0.037	0.725
M1	400	0.043	0.050	0.048	0.947	0.046	0.907	0.042	0.842
M2	100	0.014	0.109	0.072	0.759	0.058	0.655	0.042	0.527
M2	200	0.018	0.069	0.058	0.889	0.053	0.822	0.045	0.726
M2	400	0.021	0.059	0.055	0.947	0.051	0.907	0.045	0.842
M3	100	0.073	0.064	0.055	0.754	0.055	0.648	0.057	0.519
M3	200	0.058	0.051	0.044	0.888	0.043	0.820	0.042	0.722
M3	400	0.055	0.057	0.053	0.946	0.051	0.906	0.048	0.839

6 Empirical Application

Engsted and Haldrup (1999) and Phillips and Kheifets (2024) study the long-run relationship between U.S. housing starts and housing completions using different modeling approaches. Engsted and Haldrup (1999) employ a parametric Gaussian VAR-based framework that explicitly allows for multicointegration, whereas Phillips and Kheifets (2024) adopt a semiparametric $I(1)$ triangular system, as in the present study, but use a trend instrumental variable (TIV) estimator and assume that the cointegration structure is known *a priori*. In contrast, we reexamine this relationship using the semiparametric TAOLS estimator, which leaves the short-run dynamics and innovation distribution completely unspecified and relies exclusively on low-frequency components, thereby mitigating sensitivity to high-frequency noise.[5] More importantly, the adaptive F- and t-tests derived from TAOLS automatically adjust to the unknown cointegration structure, offering a simple and robust testing procedure that has no direct counterpart in the existing literature.

The multicointegration model for the housing market is specified as follows:

$$\begin{aligned} y_t - x_t\beta_0 &= u_{0,t}, \\ U_t - x_t\gamma_0 &= e_t, \quad t = 1, \dots, T, \end{aligned}$$

where y_t denotes housing units completed, x_t denotes housing units started, and

$$U_t = \sum_{j=1}^{t} u_{0,j} = Y_t - X_t\beta_0, \text{ for } X_t = \sum_{j=1}^{t} x_j, \; Y_t = \sum_{j=1}^{t} y_j.$$

Here, if $\beta_0 = 1$, then $-U_t$ is the cumulative inventory of uncompleted units. The second equation then captures the feedback from the accumulated inventory to new starts. Incorporating a constant and adding the first difference Δx_t to address long-run endogeneity, we obtain

$$Y_t = \alpha_0 + X_t\beta_0 + \Delta X_t\gamma_0 + \Delta x_t\delta_0 + e_{0\cdot x,t}. \tag{36}$$

Under multicointegration, $e_{0\cdot x,t}$ is $I(0)$. If instead $e_{0\cdot x,t}$ is $I(1)$ and $\delta_0 = 0$, then the system becomes the conventional cointegration model:

$$y_t = x_t\beta_0 + \Delta x_t\gamma_0 + u_{0\cdot x,t}, \; u_{0\cdot x,t} \sim I(0).$$

A parameter of interest is β_0, which represents the long-run fraction of housing starts eventually completed.

[5] TAOLS also offers a practical advantage over TIV: the TA regression is classical, permitting standard asymptotic F and t tests. Note that the fTIV (fixed-K TIV) in Sun et al. (2025) is actually the TAOLS.

The data are obtained from the Federal Reserve Bank of St. Louis (FRED) as of June 1, 2025. Specifically, HOUST (x_t) denotes the total number of housing units started, while COMPUTSA (y_t) denotes the total number of housing units completed. Both series are measured in thousands of units and are seasonally adjusted. Our empirical analysis uses the full sample from January 1968 to June 2025, as well as five decade-based subsamples (the 1970s–2010s), all of which are plotted in Figure 1.

Using data from 1968 to 1994, Engsted and Haldrup (1999) find evidence that some housing starts were never completed. Phillips and Kheifets (2024) conduct the analysis using TIV separately by decade and report that the estimated cointegration coefficient between completions and starts is consistently below one (in the range of 0.95–0.97), suggesting that roughly 3–5% of housing starts are never completed.

Table 9 reports our results based on the TAOLS method. The tuning parameter K takes values $10, 20, 30$, and $T^{3.8/5}$ with the last one following Phillips and Kheifets (2024), while the adaptive t test uses $\kappa = 0.4, 0.5, 0.6$. We report the point estimates, two-sided 95% confidence intervals, and p-values for the one-sided test $\mathcal{H}_0 : \beta_0 = 1$ versus $\mathcal{H}_1 : \beta_0 < 1$. The point estimates broadly align with those in Phillips and Kheifets (2024) for comparable periods.

For the three TAOLS-based t tests, our main findings are as follows. First, the t-test based on $\mathbb{T}_c$ (assuming conventional cointegration) does not reject the null at 5% for 1980–1989 but rejects it for the full sample (1968–2025) and the early period 1968–1979. In later decades, inference is sensitive to the values of K.

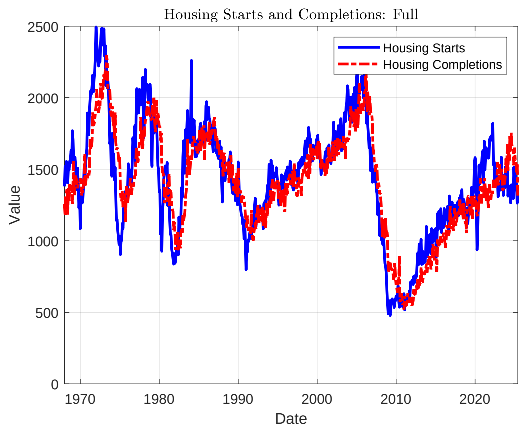
Second, the t-test based on $\mathbb{T}_m$ (assuming multicointegration) rejects the null at 1% across all sample periods. This provides strong evidence that the share of uncompleted housing units is statistically significant.

Third, the adaptive t-test $\mathbb{T}_a$ largely mirrors $\mathbb{T}_c$ because the estimated weight is small, suggesting weaker evidence for multicointegration than previously thought. Nevertheless, when κ is 0.4, the weighting effect becomes nontrivial. A key factor not accounted for is the presence of structural breaks, which can complicate the detection of singularity in the LRV of $(u_{0,t}, \Delta x_t)'$. We leave this issue for future research.

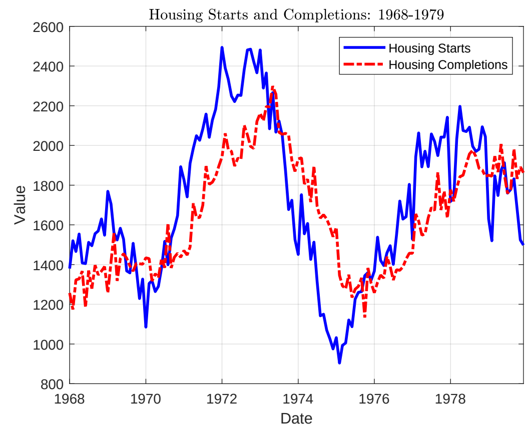
Finally, relative to the 'pvalue-M' results in Table 6 of Phillips and Kheifets (2024), our t-test based on $\mathbb{T}_m$ has uniformly smaller p-values, delivering stronger and more consistent evidence across all sample periods. The p-values in Phillips and Kheifets (2024) are based on a nonstandard approximation derived from the large-K and fixed-b asymptotics. This approximation may be less accurate than our t-approximation, as it does not account for the actual value of K in the limit and is also more cumbersome to implement.

Table 10 reports estimates for the feedback parameter γ_0, which is not studied in Phillips and Kheifets (2024). We test the null $\mathcal{H}_0 : \gamma_0 = 0$ against the alternative $\mathcal{H}_1 : \gamma_0 \neq 0$. The null is rejected at the 5% level (typically more strongly) in every sample period. This indicates that uncompleted housing projects may affect housing starts in the long run.

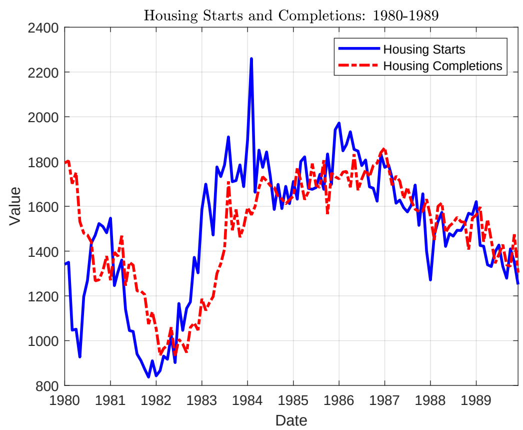
Overall, the evidence strongly supports $\beta_0 < 1$, implying a nontrivial inventory of uncompleted housing units, and overwhelmingly supports $\gamma_0 \neq 0$, consistent with endogeneity under conventional cointegration and with feedback effects under multicointegration.



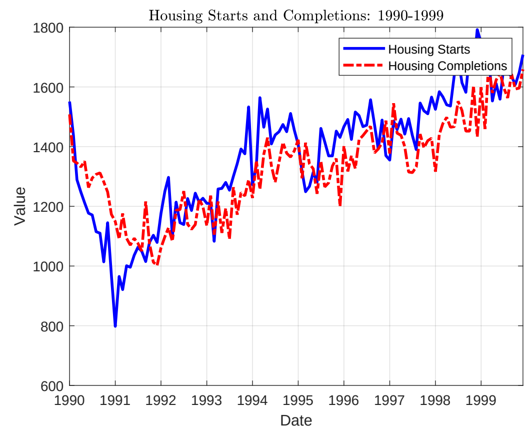
(a) Full sample period



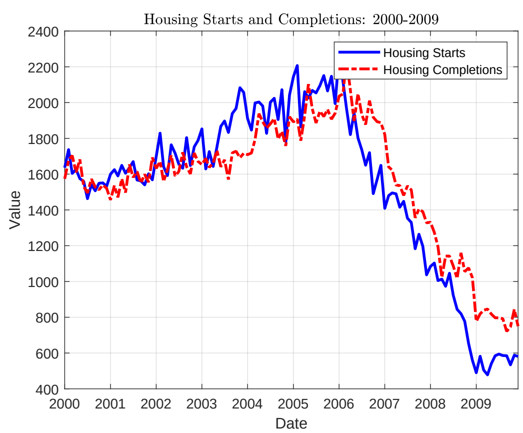
(b) 1970s



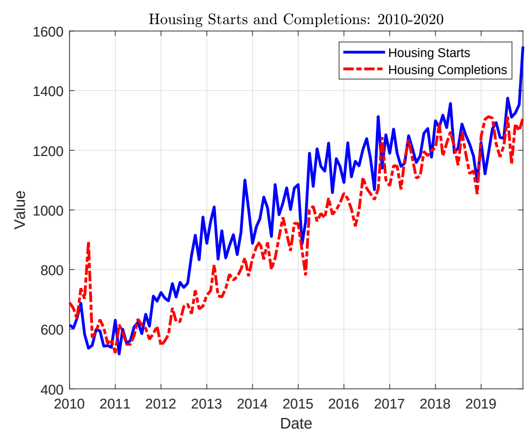
(c) 1980s



(d) 1990s



(e) 2000s



(f) 2010s

Figure 1: Housing starts (Starts) and completions (Completions) in the U.S. for the full sample period and successive decades: (a) full sample period, 1968-2025, (b) 1970s, (c) 1980s, (d) 1990s, (e) 2000s, and (f) 2010s

Table 9: TAOLS estimates, p-values, and two-sided 95% CIs for β_0 (the p-values are for testing $\mathcal{H}_0 : \beta_0 = 1$ against $\mathcal{H}_1 : \beta_0 < 1$)

K	$\tilde{\beta}_c$	$\tilde{\beta}_{cl}$	$\tilde{\beta}_{cu}$	p_c	$\hat{\beta}_m$	$\hat{\beta}_{ml}$	$\hat{\beta}_{mu}$	p_m	$p_a(.4)$	$p_a(.5)$	$p_a(.6)$
Period: 1968−2025											
10	0.974	0.961	0.987	0.001	0.976	0.973	0.979	0.000	0.000	0.000	0.000
20	0.970	0.943	0.996	0.013	0.977	0.975	0.979	0.000	0.012	0.013	0.013
30	0.966	0.940	0.992	0.007	0.977	0.975	0.979	0.000	0.006	0.007	0.007
143	0.964	0.947	0.980	0.000	0.977	0.976	0.978	0.000	0.000	0.000	0.000
Period: 1968−1979											
10	0.945	0.878	1.012	0.048	0.976	0.967	0.984	0.000	0.048	0.048	0.048
20	0.944	0.897	0.992	0.012	0.974	0.966	0.982	0.000	0.012	0.012	0.012
30	0.944	0.904	0.984	0.004	0.973	0.966	0.980	0.000	0.003	0.004	0.004
43	0.944	0.907	0.980	0.002	0.972	0.966	0.978	0.000	0.002	0.002	0.002
Period: 1980−1989											
10	0.979	0.920	1.037	0.218	0.976	0.967	0.985	0.000	0.215	0.218	0.218
20	0.980	0.930	1.029	0.203	0.975	0.969	0.982	0.000	0.185	0.201	0.203
30	0.979	0.937	1.020	0.150	0.974	0.969	0.980	0.000	0.134	0.149	0.150
38	0.979	0.941	1.017	0.133	0.974	0.969	0.979	0.000	0.113	0.131	0.133
Period: 1990−1999											
10	0.972	0.938	1.005	0.046	0.956	0.939	0.973	0.000	0.044	0.045	0.046
20	0.970	0.945	0.995	0.010	0.962	0.950	0.975	0.000	0.008	0.010	0.010
30	0.967	0.946	0.989	0.002	0.966	0.956	0.977	0.000	0.002	0.002	0.002
38	0.966	0.945	0.987	0.001	0.967	0.958	0.976	0.000	0.001	0.001	0.001
Period: 2000−2009											
10	0.973	0.929	1.017	0.101	0.974	0.970	0.979	0.000	0.017	0.065	0.096
20	0.976	0.941	1.012	0.088	0.972	0.967	0.977	0.000	0.013	0.054	0.082
30	0.979	0.949	1.009	0.083	0.971	0.967	0.976	0.000	0.004	0.035	0.071
38	0.981	0.952	1.010	0.098	0.971	0.967	0.975	0.000	0.006	0.045	0.086
Period: 2010−2019											
10	0.934	0.882	0.986	0.009	0.949	0.930	0.967	0.000	0.006	0.008	0.009
20	0.931	0.895	0.966	0.000	0.947	0.930	0.964	0.000	0.000	0.000	0.000
30	0.926	0.897	0.954	0.000	0.947	0.933	0.961	0.000	0.000	0.000	0.000
38	0.924	0.898	0.950	0.000	0.946	0.933	0.960	0.000	0.000	0.000	0.000
Period: 2010−2025											
10	0.961	0.891	1.031	0.122	0.965	0.943	0.987	0.003	0.103	0.118	0.121
20	0.957	0.906	1.007	0.045	0.958	0.942	0.973	0.000	0.037	0.044	0.045
30	0.950	0.908	0.992	0.010	0.955	0.943	0.968	0.000	0.007	0.010	0.010
53	0.947	0.915	0.979	0.001	0.952	0.943	0.962	0.000	0.001	0.001	0.001

Note: $[\tilde{\beta}_{cl}, \tilde{\beta}_{cu}]$ is the 95% two-sided CI based on $\tilde{\beta}_c$; $[\hat{\beta}_{ml}, \hat{\beta}_{mu}]$ is the 95% two-sided CI based on $\hat{\beta}_m$; p_c, p_m, and p_a are the p-values; $p_a\,(.4)$, $p_a\,(.5)$, and $p_a\,(.6)$ are the p-values when $\kappa = 0.4, 0.5$, and 0.6, respectively.

Table 10: TAOLS estimates, p-values, and two-sided 95% CIs for γ_0 (the p-values are for testing $\mathcal{H}_0 : \gamma_0 = 0$ against $\mathcal{H}_1 : \gamma_0 \neq 0$)

K	$\tilde{\gamma}_c$	$\tilde{\gamma}_{cl}$	$\tilde{\gamma}_{cu}$	p_c	$\hat{\gamma}_m$	$\hat{\gamma}_{ml}$	$\hat{\gamma}_{mu}$	p_m	$p_a(.4)$	$p_a(.5)$	$p_a(.6)$
Period: 1968−2025											
10	-8.541	-10.964	-6.119	0.000	-8.143	-10.717	-5.568	0.000	0.000	0.000	0.000
20	-6.499	-8.690	-4.308	0.000	-7.498	-9.052	-5.943	0.000	0.000	0.000	0.000
30	-5.386	-7.324	-3.448	0.000	-7.304	-8.637	-5.970	0.000	0.000	0.000	0.000
143	-2.815	-3.529	-2.100	0.000	-7.061	-7.745	-6.377	0.000	0.000	0.000	0.000
Period: 1968−1979											
10	-4.152	-7.332	-0.972	0.016	-5.244	-6.927	-3.562	0.000	0.016	0.016	0.016
20	-3.538	-5.549	-1.528	0.002	-5.252	-6.778	-3.725	0.000	0.001	0.002	0.002
30	-3.024	-4.504	-1.545	0.000	-5.224	-6.542	-3.906	0.000	0.000	0.000	0.000
43	-2.095	-3.201	-0.990	0.000	-5.151	-6.369	-3.933	0.000	0.000	0.000	0.000
Period: 1980−1989											
10	-2.624	-5.569	0.320	0.075	-5.149	-6.815	-3.482	0.000	0.072	0.075	0.075
20	-2.844	-4.789	-0.899	0.006	-4.962	-6.175	-3.750	0.000	0.005	0.006	0.006
30	-2.224	-3.499	-0.949	0.001	-4.827	-5.860	-3.795	0.000	0.001	0.001	0.001
38	-1.821	-2.857	-0.786	0.001	-4.822	-5.762	-3.883	0.000	0.001	0.001	0.001
Period: 1990−1999											
10	-3.431	-6.155	-0.707	0.019	-1.391	-5.410	2.627	0.458	0.019	0.019	0.019
20	-2.857	-4.644	-1.070	0.003	-2.749	-5.583	0.084	0.057	0.003	0.003	0.003
30	-2.071	-3.264	-0.878	0.001	-3.466	-5.852	-1.081	0.006	0.001	0.001	0.001
38	-1.293	-2.228	-0.358	0.008	-3.526	-5.716	-1.337	0.002	0.008	0.008	0.008
Period: 2000−2009											
10	-5.484	-8.291	-2.678	0.001	-6.231	-6.774	-5.687	0.000	0.000	0.000	0.001
20	-4.266	-6.151	-2.382	0.000	-6.189	-6.767	-5.611	0.000	0.000	0.000	0.000
30	-3.690	-5.183	-2.197	0.000	-6.161	-6.668	-5.654	0.000	0.000	0.000	0.000
38	-3.104	-4.425	-1.783	0.000	-6.148	-6.618	-5.678	0.000	0.000	0.000	0.000
Period: 2010−2019											
10	-1.756	-5.202	1.690	0.283	-7.881	-10.464	-5.297	0.000	0.080	0.194	0.270
20	-1.325	-3.448	0.799	0.208	-7.452	-9.829	-5.075	0.000	0.038	0.113	0.187
30	-0.692	-2.007	0.623	0.291	-7.364	-9.402	-5.327	0.000	0.032	0.135	0.257
38	-0.531	-1.546	0.484	0.297	-7.201	-9.096	-5.306	0.000	0.035	0.151	0.270
Period: 2010−2025											
10	-5.477	-13.879	2.924	0.177	-9.404	-13.763	-5.045	0.001	0.107	0.160	0.176
20	-4.052	-9.255	1.152	0.120	-8.337	-11.589	-5.085	0.000	0.083	0.114	0.120
30	-1.423	-3.817	0.970	0.234	-7.880	-10.524	-5.235	0.000	0.125	0.213	0.233
53	-0.567	-1.673	0.539	0.309	-7.294	-9.278	-5.311	0.000	0.236	0.304	0.309

Note: $[\tilde{\gamma}_{cl}, \tilde{\gamma}_{cu}]$ is the 95% two-sided CI based on $\tilde{\gamma}_c$; $[\hat{\gamma}_{ml}, \hat{\gamma}_{mu}]$ is the 95% two-sided CI based on $\hat{\gamma}_m$; p_c, p_m, and p_a are the p-values; $p_a(.4)$, $p_a(.5)$, and $p_a(.6)$ are the p-values when $\kappa = 0.4, 0.5$, and 0.6, respectively.

7 Conclusion

This paper studies the Transformed and Augmented Ordinary Least Squares method for multicointegration analysis. Its main advantage is that it transforms what appears to be a highly nonstandard and challenging inference problem—multicointegration among integrated time series—into a classical linear normal regression problem. Within this transformed setting, standard OLS theory applies directly, and conventional Wald and t statistics recover their familiar asymptotic F- and t-distributions. What previously required either restrictive parametric VAR assumptions or intricate $I(2)$ methodology is thereby reduced to a well-understood linear regression problem.

A key contribution is an adaptive testing procedure that combines the statistics from the conventional cointegration and multicointegration regimes through a data-driven weight. Monte Carlo simulations show that the resulting adaptive F test has accurate size and strong power without requiring the researcher to know in advance whether the data-generating process exhibits conventional cointegration or multicointegration, thereby providing a safeguard against model misspecification in applications where the underlying long-run structure is uncertain.

Together, the TAOLS framework and its adaptive extension provide applied researchers with a flexible, semiparametric, and easy-to-implement toolkit for inference in cointegrated and multicointegrated systems. The methodology offers a unified approach that is robust to both short-run dynamics and the form of the long-run cointegration structure. Directions for future research include extensions to multivariate systems with multiple cointegrating and multicointegrating relationships, as well as development of data-driven selection procedures for tuning parameters. Pursuing these directions would broaden the method's applicability and further strengthen the practical value of the TAOLS-based F- and t-tests for empirical work.

Appendix of Proofs

Proof of Theorem 1. We prove Part (a) only, as Parts (b), (c), and (d) follow from Part (a) using a matrix inverse formula. Note that

$$D_T(\hat{\theta}_{\text{TAOLS}} - \theta_0) = \left(D_T^{-1} V'_{\ell,X,x,\Delta x} V_{\ell,X,x,\Delta x} D_T^{-1}\right)^{-1} D_T^{-1} V'_{\ell,X,x,\Delta x} V_{e\cdot x}.$$

Using the arguments in the main text, we have

$$V_{\ell,X,x,\Delta x}D_T^{-1} = (V_\ell, V_X, V_x, V_{\Delta x})\, D_T^{-1} = \left(T^{-1/2}V_\ell, T^{-2}V_X, T^{-1}V_x, V_{\Delta x}\right)$$
$$\Rightarrow (\xi_\ell, \xi_X, \xi_x, \eta_{\Delta x}) = \zeta.$$

Combining this with $V_{e\cdot x} \Rightarrow \Omega_{ee\cdot x}^{1/2}\eta_{e\cdot x}$, we then obtain: $D_T(\hat{\theta}_{\text{TAOLS}}-\theta_0) \Rightarrow \Omega_{ee\cdot x}^{1/2}\left(\zeta'\zeta\right)^{-1}\zeta'\eta_{e\cdot x}$.

Note that $\eta_{e\cdot x} = \int_0^1 \phi\,(r)\, dW_{e\cdot x}\,(r)$, and recall that ξ_ℓ is deterministic and $\eta_{e\cdot x}$ is independent of $(\xi_X, \xi_x, \eta_{\Delta x})$. Conditional on $(\xi_X, \xi_x, \eta_{\Delta x})$, $\eta_{e\cdot x}$ follows the normal distribution:

$$\mathcal{N}(0, \Omega_{ee\cdot x}\int_0^1 \boldsymbol{\phi}\,(r)\,\boldsymbol{\phi}'\,(r)\,dr) = \mathcal{N}(0, \Omega_{ee\cdot x}I_K),$$

since under Assumption 2(ii), we have $\int_0^1 \boldsymbol{\phi}\,(r)\,\boldsymbol{\phi}'\,(r)\,dr = I_K$. The limiting distribution of $D_T(\hat{\theta}_{\text{TAOLS}} - \theta_0)$ is thus mixed normal, with the conditional variance matrix given by $\Omega_{ee\cdot x}\left(\zeta'\zeta\right)^{-1}$. That is, $D_T(\hat{\theta}_{\text{TAOLS}} - \theta_0) \Rightarrow MN(0, \Omega_{ee\cdot x}\left(\zeta'\zeta\right)^{-1})$. ■

Proof of Theorem 2. We focus on establishing the asymptotic distribution of $\mathbb{W}_{m,\gamma}\,(K)$ only, as the other results can be proved using the same arguments. Since $\hat{V}_{e\cdot x} = Q_{[V_\ell,V_X,V_x,V_{\Delta x}]}V_{e\cdot x}$, we have

$$\hat{\Omega}_{ee\cdot x} = V_{e\cdot x}'Q_{[V_\ell,V_X,V_x,V_{\Delta x}]}V_{e\cdot x}/\,(K - 3d_x - 1)$$
$$\Rightarrow \Omega_{ee\cdot x}\eta_{e\cdot x}'Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}/\,(K - 3d_x - 1)\,,$$

where we have used $V_{e\cdot x} \Rightarrow \Omega_{ee\cdot x}^{1/2}\eta_{e\cdot x}$. Combining the above and Theorem 1(c), we obtain:

$$\mathbb{W}_{m,\gamma}\,(K) = \frac{[R\hat{\gamma}_{\text{ TAOLS}} - r_0]'\left[R\left(V_x'Q_{[V_\ell,V_X,V_{\Delta x}]}V_x\right)^{-1}R'\right]^{-1}[R\hat{\gamma}_{\text{TAOLS}} - r_0]\,/p}{V_{e\cdot x}'Q_{[V_\ell,V_X,V_x,V_{\Delta x}]}V_{e\cdot x}/\,(K - 3d_x - 1)}$$
$$= \frac{[R(\hat{\gamma}_{\text{TAOLS}} - \gamma_0)]'\left[R\left(V_x'Q_{[V_\ell,V_X,V_{\Delta x}]}V_x\right)^{-1}R'\right]^{-1}[R(\hat{\gamma}_{\text{TAOLS}} - \gamma_0)]/p}{V_{e\cdot x}'Q_{[V_\ell,V_X,V_x,V_{\Delta x}]}V_{e\cdot x}/\,(K - 3d_x - 1)}.$$

Note that

$$TR\,[\hat{\gamma}_{\text{TAOLS}} - \gamma_0] \Rightarrow \Omega_{ee\cdot x}^{1/2}R\left(\xi_x'Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\right)^{-1}\xi_x'Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\eta_{e\cdot x} := \Omega_{ee\cdot x}^{1/2}\tilde{R}'\eta_{e\cdot x}$$

for

$$\tilde{R} = \left\{R\left(\xi_x'Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\right)^{-1}\xi_x'Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\right\}'.$$

We then have

$$\mathbb{W}_{m,\gamma}(K) \Rightarrow \frac{\eta_{e\cdot x}'\tilde{R}\left[\tilde{R}'\tilde{R}\right]^{-1}\tilde{R}'\eta_{e\cdot x}/p}{\eta_{e\cdot x}'Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}/(K-3d_x-1)}$$
$$= \frac{\eta_{e\cdot x}'P_{\tilde{R}}\eta_{e\cdot x}/p}{\eta_{e\cdot x}'Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}/(K-3d_x-1)} = \frac{\left\|P_{\tilde{R}}\eta_{e\cdot x}\right\|^2/p}{\left\|Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}\right\|^2/(K-3d_x-1)}. \tag{37}$$

Under the assumption that $\left\{\phi_j(\cdot)\right\}_{j=1}^K$ are orthonormal, $\eta_{e\cdot x}$ follows the normal distribution $N(0, I_K)$. Note also that $\eta_{e\cdot x}$ is independent of $(\xi_X, \xi_x, \eta_{\Delta x})$. Hence, conditional on $(\xi_X, \xi_x, \eta_{\Delta x})$,

$$\left\|P_{\tilde{R}}\eta_{e\cdot x}\right\|^2 =^d \chi_p^2,$$
$$\left\|Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}\right\|^2 =^d \chi_{K-3d_x-1}^2,$$

where $=^d$ denotes distributional equivalence. The two chi-square variates above are conditionally independent, as they are based on two conditionally independent normal variables, namely $Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}$ and $\tilde{R}'\eta_{e\cdot x}$. The conditional independence between these two normal variables holds because, conditional on $(\xi_X, \xi_x, \eta_{\Delta x})$, we have

$$cov(Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}, \tilde{R}'\eta_{e\cdot x}) = E\left\{Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}\eta_{e\cdot x}'\tilde{R}\right\}$$
$$= Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\tilde{R} = Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\left[\xi_x'Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\right]^{-1}R'$$
$$= \left\{Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]} - P_{\left[Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\right]}\right\}Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\left[\xi_x'Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\right]^{-1}R'$$
$$= \left\{Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x - Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\right\}\left[\xi_x'Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\right]^{-1}R' = 0,$$

where we have used: $Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]} = Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]} - P_{\left[Q_{[\xi_\ell,\xi_X,\eta_{\Delta x}]}\xi_x\right]}$. Therefore, conditional on $(\xi_X, \xi_x, \eta_{\Delta x})$,

$$\frac{\left\|P_{\tilde{R}}\eta_{e\cdot x}\right\|^2/p}{\left\|Q_{[\xi_\ell,\xi_X,\xi_x,\eta_{\Delta x}]}\eta_{e\cdot x}\right\|^2/(K-3d_x-1)} =^d \frac{\chi_p^2/p}{\chi_{K-3d_x-1}^2/(K-3d_x-1)} =^d F_{p,K-3d_x-1}.$$

The conditional distribution does not depend on the conditioning variables $[\xi_X, \xi_x, \eta_{\Delta x}]$, and hence it is also the unconditional distribution. We have therefore shown that

$$\mathbb{W}_{m,\gamma}(K) \Rightarrow F_{p,K-3d_x-1}.$$

■

Proof of Theorem 3. Under Assumption 3(ii), we have

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\begin{pmatrix} u_{0\cdot x,t} \\ u_{x,t}\end{pmatrix} \Rightarrow \begin{pmatrix} B_{0\cdot x}(\cdot) \\ B_x(\cdot)\end{pmatrix} \equiv \begin{pmatrix} \Omega_{00\cdot x}^{1/2} W_{0\cdot x}(\cdot) \\ \Omega_{xx}^{1/2} W_x(\cdot)\end{pmatrix}. \tag{38}$$

Combining this with Assumptions 2 and 3(i), we obtain

$$T^{-1}V_x \Rightarrow \xi_x, V_{\Delta x} \Rightarrow \eta_{\Delta x}, V_{0\cdot x} \Rightarrow \Omega_{00\cdot x}^{1/2}\eta_{0\cdot x}.$$

Let $\tilde{D}_T = \text{diag}(TI_{d_x}, I_{d_x})$. It then follows that, when $(\xi_x, \eta_{\Delta x})'(\xi_x, \eta_{\Delta x})$ is invertible almost surely,

$$\begin{aligned} &\tilde{D}_T\left(\tilde{\beta}'_{\text{TAOLS}} - \beta'_0, \tilde{\gamma}'_{\text{TAOLS}} - \gamma'_0\right)' \\ &\Rightarrow \Omega_{00\cdot x}^{1/2}\left[(\xi_x, \eta_{\Delta x})'(\xi_x, \eta_{\Delta x})\right]^{-1}(\xi_x, \eta_{\Delta x})'\eta_{0\cdot x}, \end{aligned} \tag{39}$$

and

$$\tilde{\Omega}_{00\cdot x} = \frac{1}{K - 2d_x}V'_{0\cdot x}Q_{[V_x, V_{\Delta x}]}V_{0\cdot x} \Rightarrow \Omega_{00\cdot x}\frac{1}{K-2d_x}\eta'_{0\cdot x}Q_{[\xi_x,\eta_{\Delta x}]}\eta_{0\cdot x}. \tag{40}$$

Furthermore, using the same arguments as in Hwang and Sun (2018), we can show that, when $K > 2d_x$,

$$\mathbb{W}_{c,\beta}(K) \Rightarrow F_{p,K-2d_x} \text{ and } \mathbb{T}_{c,\beta}(K) \Rightarrow t_{K-2d_x}.$$

Similar arguments can be used to show that $W_{c,\gamma}(K) \Rightarrow F_{p,K-2d_x}$ and $T_{c,\gamma}(K) \Rightarrow t_{K-2d_x}$. ■

Proof of Proposition 4. Define $\hat{\vartheta}_{\text{TAOLS}} = (\hat{\beta}'_{\text{TAOLS}}, \hat{\gamma}'_{\text{TAOLS}})'$, $\vartheta_0 = (\beta'_0, \gamma'_0)'$, and $\mathcal{D}_T = \text{diag}(T^2 I_{d_x}, TI_{d_x})$. Then

$$V_{X,x}\mathcal{D}_T^{-1} \Rightarrow (\xi_X, \xi_x) := \xi_{X,x},$$

and

$$\begin{aligned} \mathcal{D}_T(\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) &= \left[(V_{X,x}\mathcal{D}_T^{-1})'Q_{V_\ell, V_{\Delta x}}V_{X,x}\mathcal{D}_T^{-1}\right]^{-1}(V_{X,x}\mathcal{D}_T^{-1})'Q_{V_\ell}V_{\tilde{Z}} \\ &\Rightarrow \Omega_{ee\cdot x}^{1/2}\left(\xi'_{X,x}Q_{[\xi_\ell,\eta_{\Delta x}]}\xi_{X,x}\right)^{-1}\xi_{X,x}Q_{[\xi_\ell,\eta_{\Delta x}]}\eta_{e\cdot x}. \end{aligned} \tag{41}$$

Next, for $\tilde{K} = K - 3d_x - 1$ and $\tilde{x}_t = (x'_t, \Delta x'_t)'$, we have

$$\begin{aligned} &\hat{\Omega}_{zz} \\ &= \frac{1}{\tilde{K}}\sum_{i=1}^{K}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\Delta e_{0,t} - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\tilde{x}'_t(\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0)\right]^2 \\ &= \frac{1}{\tilde{K}}\sum_{i=1}^{K}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\Delta e_{0,t} - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\left(\mathcal{D}_T^{-1}\tilde{x}_t\right)'\mathcal{D}_T(\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0)\right]^2. \end{aligned}$$

For each $i=1,\ldots,K$, we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\left(\mathcal{D}_T^{-1}\tilde{x}_t\right)$$
$$=\left(\begin{array}{c}\frac{1}{T}\times\frac{1}{T}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\left(\frac{x_t}{\sqrt{T}}\right)\\ \frac{1}{T}\times\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}u_{x,t}\right)\end{array}\right)=O_p\left(\frac{1}{T}\right).$$

Combining this with (41), we have

$$\hat{\Omega}_{zz}=\frac{1}{\tilde{K}}\sum_{i=1}^{K}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\Delta e_{0,t}+O_p\left(\frac{1}{T}\right)\right]^2.$$

Under Assumptions 1 and 2, we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\Delta e_{0,t}$$
$$=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)e_{0,t}-\frac{1}{\sqrt{T}}\sum_{t=0}^{T-1}\phi_i\left(\frac{t+1}{T}\right)e_{0,t}$$
$$=\frac{1}{\sqrt{T}}\phi_i\left(1\right)e_{0,T}-\frac{1}{\sqrt{T}}\phi_i\left(\frac{1}{T}\right)e_{0,0}-\frac{1}{\sqrt{T}}\sum_{t=1}^{T-1}e_{0,t}\left[\phi_i\left(\frac{t+1}{T}\right)-\phi_i\left(\frac{t}{T}\right)\right]$$
$$=\frac{1}{\sqrt{T}}\phi_i\left(1\right)e_{0,T}-\frac{1}{\sqrt{T}}\phi_i\left(\frac{1}{T}\right)e_{0,0}-\frac{1}{T}\frac{1}{\sqrt{T}}\sum_{t=1}^{T-1}e_{0,t}\dot{\phi}_i\left(\frac{s\left(t\right)}{T}\right),$$

where $s\left(t\right)\in\left[t,t+1\right]$ and $\dot{\phi}_i\left(r\right)=d\phi_i\left(r\right)/dr$. In view of

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T-1}e_{0,t}\dot{\phi}_i\left(\frac{s\left(t\right)}{T}\right)\Rightarrow\int_0^1\dot{\phi}_i\left(r\right)dB_e\left(r\right),$$

we have

$$\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\Delta e_{0,t}=\phi_i\left(1\right)e_{0,T}-\phi_i\left(\frac{1}{T}\right)e_{0,0}+o_p(1). \tag{42}$$

Therefore,

$$T\hat{\Omega}_{zz}=\frac{1}{\tilde{K}}\sum_{i=1}^{K}\left[\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right)\Delta e_{0,t}+o_p\left(1\right)\right]^2$$
$$=\frac{1}{\tilde{K}}\sum_{i=1}^{K}\left[\phi_i\left(1\right)e_{0,T}-\phi_i\left(0\right)e_{0,0}+o_p\left(1\right)\right]^2$$
$$\Rightarrow\frac{1}{\tilde{K}}\left\|\boldsymbol{\phi}\left(1\right)e_{0,\infty}-\boldsymbol{\phi}\left(0\right)e_{0,0}\right\|^2=\frac{1}{\tilde{K}}\left\|\boldsymbol{\varrho}\right\|^2.$$

It remains to prove the limit of $\hat{\Sigma}_{zz}$. Let $\tilde{T} = T - 3d_x - 1$. Then

$$\begin{aligned}
\hat{\Sigma}_{zz} &= \frac{1}{\tilde{T}} \sum_{t=1}^{T} (\hat{z}_t)^2 \\
&= \frac{1}{\tilde{T}} \sum_{t=1}^{T} \left[\Delta e_{0,t} - \tilde{x}_t' \left(\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0 \right) \right]^2 \\
&= \frac{1}{\tilde{T}} \sum_{t=1}^{T} (\Delta e_{0,t})^2 - \frac{2}{\tilde{T}} \left(\sum_{t=1}^{T} \Delta e_0 \tilde{x}_t' \right) \mathcal{D}_T^{-1} \left[\mathcal{D}_T (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) \right] \\
&\quad + \left[\mathcal{D}_T (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) \right]' \mathcal{D}_T^{-1} \left(\frac{1}{\tilde{T}} \sum_{t=1}^{T} \tilde{x}_t \tilde{x}_t' \right) \mathcal{D}_T^{-1} \left[\mathcal{D}_T (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) \right].
\end{aligned}$$

Since $D_T(\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) = O_p(1)$,

$$\mathcal{D}_T^{-1} \left(\frac{1}{T} \sum_{t=1}^{T} \tilde{x}_t \tilde{x}_t' \right) \mathcal{D}_T^{-1} = \frac{1}{T} \begin{pmatrix} \frac{1}{T^4} \sum_{t=1}^{T} x_t x_t' & \frac{1}{T^3} \sum_{t=1}^{T} x_t \Delta x_t' \\ \frac{1}{T^3} \sum_{t=1}^{T} \Delta x_t x_t' & \frac{1}{T^2} \sum_{t=1}^{T} \Delta x_t \Delta x_t' \end{pmatrix} = o_p(1),$$

and

$$\frac{1}{T} \left(\sum_{t=1}^{T} \Delta e_{0,t} \tilde{x}_t' \right) \mathcal{D}_T^{-1} = \frac{1}{T} \left[\begin{array}{cc} \frac{1}{T^2} \sum_{t=1}^{T} \Delta e_{0,t} x_t', & \frac{1}{T} \sum_{t=1}^{T} \Delta e_{0,t} \Delta x_t' \end{array} \right] = o_p(1),$$

we have

$$\hat{\Sigma}_{zz} = \frac{1}{\tilde{T}} \sum_{t=1}^{T} (\Delta e_{0,t})^2 + o_p(1) \to^p \Sigma_{\Delta e, \Delta e},$$

as desired. ■

Proof of Proposition 5. As in the proof of Proposition 4, define $\vartheta_0 = (\beta_0', \gamma_0')'$, and $\mathcal{D}_T = \text{diag}(T^2 I_{d_x}, T I_{d_x})$. We still have $V_{X,x} \mathcal{D}_T^{-1} \Rightarrow (\xi_X, \xi_x) := \xi_{X,x}$, but now $\tilde{Z}_t = U_{0 \cdot x, t}$ and so

$$\frac{1}{T} V_{\tilde{Z}} = \frac{1}{T} \sum_{t=1}^{T} \phi\left(\frac{t}{T}\right) \frac{U_{0 \cdot x, t}}{\sqrt{T}} \Rightarrow \Omega_{00 \cdot x}^{1/2} \int_0^1 \phi(r) W_{0 \cdot x}(r) \, dr := \Omega_{00 \cdot x}^{1/2} \xi_{0 \cdot x}.$$

Hence,

$$\begin{aligned}
\frac{\mathcal{D}_T}{T} (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) &= \left((V_{X,x} \mathcal{D}_T^{-1})' Q_{[V_\ell, V_{\Delta x}]} V_{X,x} \mathcal{D}_T^{-1} \right)^{-1} (V_{X,x} \mathcal{D}_T^{-1})' Q_{[V_\ell, V_{\Delta x}]} \frac{1}{T} V_{\tilde{Z}} \\
&\Rightarrow \Omega_{00 \cdot x}^{1/2} \left[\xi_{X,x}' Q_{[\xi_\ell, \eta_{\Delta x}]} \xi_{X,x} \right]^{-1} \xi_{X,x}' Q_{[\xi_\ell, \eta_{\Delta x}]} \xi_{0 \cdot x} := \begin{pmatrix} \beta_{c,I}^\infty \\ \gamma_{c,I}^\infty \end{pmatrix}.
\end{aligned}$$

Compared to (41), this shows that, under conventional cointegration, $\hat{\vartheta}_{\text{TAOLS}}$ does not converge as fast as it does under multicointegration.

Next, recalling that $\tilde{x}_t = (x_t', \Delta x_t')'$ and $\tilde{K} = K - 3d_x - 1$, we write $\hat{\Omega}_{zz}$ as

$$\hat{\Omega}_{zz} = \frac{1}{\tilde{K}} \sum_{i=1}^{K} \left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi_i \left(\frac{t}{T} \right) u_{0\cdot x,t} - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi_i \left(\frac{t}{T} \right) \left(T\mathcal{D}_T^{-1} \tilde{x}_t \right)' \frac{\mathcal{D}_T}{T} (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) \right]^2 .$$

For each $i = 1, \dots, K$, we have

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi_i \left(\frac{t}{T} \right) u_{0\cdot x,t} \Rightarrow \Omega_{00\cdot x}^{1/2} \int_0^1 \phi_i (r)\, dW_{0\cdot x} (r) := \Omega_{00\cdot x}^{1/2} \eta_{0\cdot x,i}.$$

In addition,

$$\begin{aligned} &\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi_i \left(\frac{t}{T} \right) T\mathcal{D}_T^{-1} \tilde{x}_t \\ &= \begin{pmatrix} \frac{1}{T} \sum_{t=1}^{T} \phi_i \left(\frac{t}{T} \right) \left(\frac{x_t}{\sqrt{T}} \right) \\ \sum_{t=1}^{T} \phi_i \left(\frac{t}{T} \right) \left(\frac{1}{\sqrt{T}} u_{x,t} \right) \end{pmatrix} \Rightarrow \begin{pmatrix} \int_0^1 \phi_i (r)\, B_x (r)\, dr \\ \int_0^1 \phi_i (r)\, dB_x (r) \end{pmatrix} := \begin{pmatrix} \xi_{x,i} \\ \eta_{\Delta x,i} \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \hat{\Omega}_{zz} &\Rightarrow \frac{1}{\tilde{K}} \sum_{i=1}^{K} \left[\Omega_{00\cdot x}^{1/2} \eta_{0\cdot x,i} - \Omega_{00\cdot x}^{1/2} \begin{pmatrix} \xi_{x,i} \\ \eta_{\Delta x,i} \end{pmatrix}' \left[\xi_{X,x}' Q_{[\xi_\ell, \eta_{\Delta x}]} \xi_{X,x} \right]^{-1} \xi_{X,x}' Q_{[\xi_\ell, \eta_{\Delta x}]} \xi_{0\cdot x} \right]^2 \\ &= \frac{1}{\tilde{K}} \Omega_{00\cdot x} \left\| \eta_{0\cdot x} - (\xi_x, \eta_{\Delta x}) \left[\xi_{X,x}' Q_{[\xi_\ell, \eta_{\Delta x}]} \xi_{X,x} \right]^{-1} \xi_{X,x}' Q_{[\xi_\ell, \eta_{\Delta x}]} \xi_{0\cdot x} \right\|^2 . \end{aligned}$$

Next, letting $\tilde{T} = T - 3d_x - 1$ and noting $z_t = u_{0\cdot x,t}$, we have

$$\begin{aligned} \hat{\Sigma}_{zz} &= \frac{1}{\tilde{T}} \sum_{t=1}^{T} (\hat{z}_t)^2 \\ &= \frac{1}{\tilde{T}} \sum_{t=1}^{T} \left[u_{0\cdot x,t} - \tilde{x}_t' (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) \right]^2 \\ &= \frac{1}{\tilde{T}} \sum_{t=1}^{T} u_{0\cdot x,t}^2 - \frac{2}{\tilde{T}} \left(\sum_{t=1}^{T} u_{0\cdot x,t} \tilde{x}_t' \right) T\mathcal{D}_T^{-1} \frac{\mathcal{D}_T}{T} (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) \\ &+ \left[\frac{\mathcal{D}_T}{T} (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) \right]' T\mathcal{D}_T^{-1} \left(\frac{1}{\tilde{T}} \sum_{t=1}^{T} \tilde{x}_t \tilde{x}_t' \right) T\mathcal{D}_T^{-1} \left[\frac{\mathcal{D}_T}{T} (\hat{\vartheta}_{\text{TAOLS}} - \vartheta_0) \right]. \end{aligned}$$

Since

$$
\begin{aligned}
& T\mathcal{D}_T^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_t\tilde{x}_t'\right)T\mathcal{D}_T^{-1} \\
&= \begin{pmatrix} \frac{1}{T^2}\frac{1}{T}\sum_{t=1}^{T}x_t x_t' & \frac{1}{T}\frac{1}{T}\sum_{t=1}^{T}x_t\Delta x_t' \\ \frac{1}{T}\frac{1}{T}\sum_{t=1}^{T}\Delta x_t x_t' & \frac{1}{T}\sum_{t=1}^{T}\Delta x_t\Delta x_t' \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{T}\frac{1}{T}\sum_{t=1}^{T}\frac{x_t}{\sqrt{T}}\frac{x_t'}{\sqrt{T}} & \frac{1}{T}\sum_{t=1}^{T}\frac{x_t}{\sqrt{T}}\frac{\Delta x_t'}{\sqrt{T}} \\ \frac{1}{T}\sum_{t=1}^{T}\frac{\Delta x_t}{\sqrt{T}}\frac{x_t'}{\sqrt{T}} & \frac{1}{T}\sum_{t=1}^{T}\Delta x_t\Delta x_t' \end{pmatrix} \to^p \begin{pmatrix} O & O \\ O & \Sigma_{u_x,u_x} \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{T}\left(\sum_{t=1}^{T}u_{0\cdot x,t}\tilde{x}_t'\right)T\mathcal{D}_T^{-1} \\
&= \frac{1}{T}\left[\begin{array}{cc} \frac{1}{T}\sum_{t=1}^{T}u_{0\cdot x,t}x_t', & \sum_{t=1}^{T}u_{0\cdot x,t}\Delta x_t' \end{array}\right] \\
&= \left[\begin{array}{cc} \frac{1}{T}\sum_{t=1}^{T}\frac{u_{0\cdot x,t}}{\sqrt{T}}\frac{x_t'}{\sqrt{T}}, & \frac{1}{T}\sum_{t=1}^{T}u_{0\cdot x,t}\Delta x_t' \end{array}\right] \to_p \left[\begin{array}{cc} 0, & \Sigma_{0\cdot x,u_x} \end{array}\right],
\end{aligned}
$$

we have

$$
\begin{aligned}
\hat{\Sigma}_{zz} &= \frac{1}{\tilde{T}}\sum_{t=1}^{T}u_{0\cdot x,t}^2 + 2\Sigma_{0\cdot x,u_x}\left(\tilde{\gamma}_{\text{TAOLS}}-\gamma_0\right) \\
&\quad + \left(\tilde{\gamma}_{\text{TAOLS}}-\gamma_0\right)'\Sigma_{u_x,u_x}\left[\left(\tilde{\gamma}_{\text{TAOLS}}-\gamma_0\right)\right] + o_p(1) \qquad (43) \\
&\quad \Rightarrow \Sigma_{0\cdot x,0\cdot x} - 2\Sigma_{0\cdot x,u_x}\gamma_{c,I}^{\infty} + \left(\gamma_{c,I}^{\infty}\right)'\Sigma_{u_x,u_x}\gamma_{c,I}^{\infty} \\
&\quad = \begin{pmatrix} -\gamma_{c,I}^{\infty} \\ 1 \end{pmatrix}'\begin{pmatrix} \Sigma_{u_x,u_x} & \Sigma_{0\cdot x,u_x} \\ \Sigma_{0\cdot x,u_x}' & \Sigma_{0\cdot x,0\cdot x} \end{pmatrix}\begin{pmatrix} -\gamma_{c,I}^{\infty} \\ 1 \end{pmatrix} > 0.
\end{aligned}
$$

Finally, we prove the stochastic boundedness of the test statistics. We prove only $\mathbb{W}_{m,\beta}(K) = O_p(1)$ as other cases follow from the same argument. Under conventional cointegration, we have

$$Y_t = \alpha_0 + X_t'\beta_0 + x_t'\gamma_0 + \Delta x_t'\delta_0 + U_{0\cdot x,t}.$$

Based on this, we can represent $\hat{\beta}_{\text{TAOLS}} - \beta_0$ as

$$\hat{\beta}_{\text{TAOLS}} - \beta_0 = \left(V_X'Q_{[V_\ell,V_x,V_{\Delta x}]}V_X\right)^{-1}V_X'Q_{[V_\ell,V_x,V_{\Delta x}]}V_{U\cdot x},$$

where $V_{U\cdot x} = (V_{U\cdot x,1},\ldots,V_{U\cdot x,K})'$ and

$$V_{U\cdot x} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\boldsymbol{\phi}\left(\frac{t}{T}\right)U_{0\cdot x,t}.$$

Observing that

$$\frac{1}{T}V_{U\cdot x}=\frac{1}{T}\sum_{t=1}^{T}\frac{U_{0\cdot x,t}}{\sqrt{T}}\phi\left(\frac{t}{T}\right)\Rightarrow\Omega_{00\cdot x}^{1/2}\int_{0}^{1}\phi\left(r\right)W_{0\cdot x}\left(r\right)dr:=\Omega_{00\cdot x}^{1/2}\xi_{0\cdot x},$$

where $W_{0\cdot x}\left(r\right)$ is defined in (38), we have, under conventional cointegration,

$$\begin{aligned}\frac{1}{T^{2}}\hat{\Omega}_{ee\cdot x}&=\frac{1}{K}\left(\frac{1}{T}V_{U\cdot x}^{\prime}Q_{[V_{\ell},V_{X},V_{x},V_{\Delta x}]}\frac{1}{T}V_{U\cdot x}\right)\\&\Rightarrow\frac{1}{K}\Omega_{00\cdot x}\xi_{0\cdot x}^{\prime}Q_{[\xi_{\ell},\xi_{X},\xi_{x},\eta_{\Delta x}]}\xi_{0\cdot x}.\end{aligned}$$

In addition,

$$\begin{aligned}&TH\left(\hat{\beta}_{\mathrm{TAOLS}}-\beta_{0}\right)\\&=H\left(\frac{1}{T^{2}}V_{X}^{\prime}Q_{[V_{\ell},V_{x},V_{\Delta x}]}\frac{1}{T^{2}}V_{X}\right)^{-1}\frac{1}{T^{2}}V_{X}^{\prime}Q_{[V_{\ell},V_{x},V_{\Delta x}]}\frac{1}{T}V_{U\cdot x}\\&\Rightarrow\Omega_{00\cdot x}^{1/2}H\left(\xi_{X}^{\prime}Q_{[\xi_{\ell},\xi_{x},\eta_{\Delta x}]}\xi_{X}\right)^{-1}\xi_{X}^{\prime}Q_{[\xi_{\ell},\xi_{x},\eta_{\Delta x}]}\xi_{0\cdot x}\\&:=\Omega_{00\cdot x}^{1/2}\tilde{H}_{m}^{\prime}\xi_{0\cdot x},\end{aligned}$$

where $\tilde{H}_{m}^{\prime}=H\left(\xi_{X}^{\prime}Q_{[\xi_{\ell},\xi_{x},\eta_{\Delta x}]}\xi_{X}\right)^{-1}\xi_{X}^{\prime}Q_{[\xi_{\ell},\xi_{x},\eta_{\Delta x}]}$. Therefore, under the null hypothesis $H\beta_{0}=h_{0}$, we have

$$\begin{aligned}\mathbb{W}_{m,\beta}\left(K\right)&=\frac{K-3d_{x}-1}{p}\frac{\left[H(\hat{\beta}_{\mathrm{TAOLS}}-\beta_{0})\right]^{\prime}\left[H\left(V_{X}^{\prime}Q_{[V_{\ell},V_{x},V_{\Delta x}]}V_{X}\right)^{-1}H^{\prime}\right]^{-1}H(\hat{\beta}_{\mathrm{TAOLS}}-\beta_{0})}{V_{U\cdot x}^{\prime}Q_{[V_{\ell},V_{X},V_{x},V_{\Delta x}]}V_{U\cdot x}}\\&\Rightarrow\frac{K-3d_{x}-1}{p}\frac{\xi_{0\cdot x}^{\prime}\tilde{H}_{m}\left[H\left(\xi_{X}^{\prime}Q_{[\xi_{\ell},\xi_{x},\eta_{\Delta x}]}\xi_{X}\right)^{-1}H^{\prime}\right]^{-1}\tilde{H}_{m}^{\prime}\xi_{0\cdot x}}{\xi_{0\cdot x}^{\prime}Q_{[\xi_{\ell},\xi_{X},\xi_{x},\eta_{\Delta x}]}\xi_{0\cdot x}}\\&=\frac{K-3d_{x}-1}{p}\frac{\xi_{0\cdot x}^{\prime}\tilde{H}_{m}\left[\tilde{H}_{m}^{\prime}\tilde{H}_{m}\right]^{-1}\tilde{H}_{m}^{\prime}\xi_{0\cdot x}}{\xi_{0\cdot x}^{\prime}Q_{[\xi_{\ell},\xi_{X},\xi_{x},\eta_{\Delta x}]}\xi_{0\cdot x}}\\&=\frac{K-3d_{x}-1}{p}\frac{\left\|P_{\tilde{H}_{m}}\xi_{0\cdot x}\right\|^{2}}{\left\|Q_{[\xi_{\ell},\xi_{X},\xi_{x},\eta_{\Delta x}]}\xi_{0\cdot x}\right\|^{2}}.\end{aligned}$$

Combining this with the assumption that $\xi_{0\cdot x}^{\prime}Q_{[\xi_{\ell},\xi_{X},\xi_{x},\eta_{\Delta x}]}\xi_{0\cdot x}>0$ almost surely, we complete the proof for $\mathbb{W}_{m,\beta}(K)=O_{p}\left(1\right)$. ■

Proof of Proposition 6 . When $\Omega_{00\cdot x}=0$, we have $z_{t}=\Delta e_{0,t}$. Then, under Assumptions 1 and 2, we have, using the same argument as in (42),

$$\begin{aligned}\sqrt{T}V_{z,i}&=e_{0,T}\phi_{i}\left(1\right)-e_{0,0}\phi_{i}\left(0\right)+O_{p}(1/\sqrt{T})\\&\Rightarrow e_{0,\infty}\phi_{i}\left(1\right)-e_{0,0}\phi_{i}\left(0\right).\end{aligned}$$

So

$$\sqrt{T}V_z \Rightarrow \boldsymbol{\phi}(1)\, e_{0,\infty} - \boldsymbol{\phi}(0)\, e_{0,0} = \boldsymbol{\varrho}.$$

Let $\tilde{D}_T = \operatorname{diag}(TI_{d_x}, I_{d_x})$. Using the above and

$$V_{x,\Delta x}\tilde{D}_T^{-1} \Rightarrow (\xi_x, \eta_{\Delta x}),$$

we obtain

$$\begin{aligned}\sqrt{T}\tilde{D}_T\left(\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0\right) &= \left(\tilde{D}_T^{-1}V'_{x,\Delta x}V_{x,\Delta x}\tilde{D}_T^{-1}\right)^{-1}\tilde{D}_T^{-1}V'_{x,\Delta x}\sqrt{T}V_z \\ &\Rightarrow \left[(\xi_x, \eta_{\Delta x})'(\xi_x, \eta_{\Delta x})\right]^{-1}(\xi_x, \eta_{\Delta x})'\,\boldsymbol{\varrho}.\end{aligned}$$

Comparing this with (39), we can see that $\tilde{\vartheta}_{\text{TAOLS}}$ now converges at a faster rate. In particular,

$$T^{3/2}\left(\tilde{\beta}_{\text{TAOLS}} - \beta_0\right) = \left(\frac{V'_x}{T}Q_{V_{\Delta x}}\frac{V_x}{T}\right)^{-1}\frac{V'_x}{T}Q_{V_{\Delta x}}\sqrt{T}V_z \Rightarrow \left(\xi'_x Q_{[\eta_{\Delta x}]}\xi_x\right)^{-1}\xi'_x Q_{[\eta_{\Delta x}]}\boldsymbol{\varrho},$$

$$T^{1/2}\left(\tilde{\gamma}_{\text{TAOLS}} - \gamma_0\right) = \left(V'_{\Delta x}Q_{V_x}V_{\Delta x}\right)^{-1}V'_{\Delta x}Q_{V_x}\sqrt{T}V_z \Rightarrow \left(\eta'_{\Delta x}Q_{[\xi_x]}\eta_{\Delta x}\right)^{-1}\eta'_{\Delta x}Q_{[\xi_x]}\boldsymbol{\varrho}.$$

In view of $\sqrt{T}V_z \Rightarrow \boldsymbol{\varrho}$ and $V_{x,\Delta x}\tilde{D}_T^{-1} \Rightarrow (\xi_x, \eta_{\Delta x})$, we have

$$T\tilde{\Omega}_{zz} = \frac{1}{K - 2d_x}(\sqrt{T}V_z)'Q_{V_{x,\Delta x}}\sqrt{T}V_z \Rightarrow \frac{1}{K - 2d_x}\ \boldsymbol{\varrho}'Q_{[\xi_x, \eta_{\Delta x}]}\ \boldsymbol{\varrho}.$$

Using the above limit of $T^{3/2}\left(\tilde{\beta}_{\text{TAOLS}} - \beta_0\right)$ and letting

$$\tilde{H}'_c = H\left(\xi'_x Q_{[\eta_{\Delta x}]}\xi_x\right)^{-1}\xi'_x Q_{[\eta_{\Delta x}]},$$

we get

$$\begin{aligned}&\mathbb{W}_{c,\beta}(K) \\ &= \frac{\left(H\tilde{\beta}_{\text{TAOLS}} - h_0\right)'\left[H\left(V'_x Q_{V_{\Delta x}}V_x\right)^{-1}H'\right]^{-1}\left[H\tilde{\beta}_{\text{TAOLS}} - h_0\right]}{p\tilde{\Omega}_{00\cdot x}} \\ &= \frac{V'_z Q_{V_{\Delta x}}V_x\left(V'_x Q_{V_{\Delta x}}V_x\right)^{-1}H'\left[H\left(V'_x Q_{V_{\Delta x}}V_x\right)^{-1}H'\right]^{-1}H\left(V'_x Q_{V_{\Delta x}}V_x\right)^{-1}V'_x Q_{V_{\Delta x}}V_z}{pV'_z Q_{[V_x, V_{\Delta x}]}V_z/(K - 2d_x)} \\ &= \frac{\left(\sqrt{T}V_z\right)'Q_{V_{\Delta x}}V_x\left(V'_x Q_{V_{\Delta x}}V_x\right)^{-1}H'[H\left(V'_x Q_{V_{\Delta x}}V_x\right)^{-1}H']^{-1}H\left(V'_x Q_{V_{\Delta x}}V_x\right)^{-1}V'_x Q_{V_{\Delta x}}\left(\sqrt{T}V_z\right)}{p\left(\sqrt{T}V_z\right)'Q_{[V_x, V_{\Delta x}]}\left(\sqrt{T}V_z\right)/(K - 2d_x)} \\ &\Rightarrow \frac{\boldsymbol{\varrho}'\tilde{H}_c\left[\tilde{H}'_c H_c\right]^{-1}\tilde{H}'_c\boldsymbol{\varrho}}{\boldsymbol{\varrho}'Q_{[\xi_x, \eta_{\Delta x}]}\boldsymbol{\varrho}}\frac{(K - 2d_x)}{p}.\end{aligned}$$

Hence, under the assumption that $\boldsymbol{\varrho}' Q_{[\xi_x,\eta_{\Delta x}]}\boldsymbol{\varrho} > 0$ almost surely, $\mathbb{W}_{c,\beta}(K) = O_p(1)$. Similarly, we can show that $\mathbb{T}_{c,\beta}(K)$, $\mathbb{W}_{c,\gamma}(K)$ and $\mathbb{T}_{c,\gamma}(K)$ are all of order $O_p(1)$; details are omitted.

Finally, we consider $\tilde{\Sigma}_{zz}$:

$$\begin{aligned}
\tilde{\Sigma}_{zz} &= \frac{1}{T-2d_x}\sum_{t=1}^{T}(\tilde{z}_t)^2 \\
&= \frac{1}{T-2d_x}\sum_{t=1}^{T}\left[\Delta e_{0,t} - \tilde{x}_t'\left(\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0\right)\right]^2 \\
&= \frac{1}{T-2d_x}\sum_{t=1}^{T}(\Delta e_{0,t})^2 - \frac{2}{T-2d_x}\left(\sum_{t=1}^{T}\Delta e_{0,t}\tilde{x}_t'\right)(\sqrt{T}\tilde{D}_T)^{-1}\left[\sqrt{T}\tilde{D}_T\left(\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0\right)\right] \\
&+ \left[\sqrt{T}\tilde{D}_T\left(\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0\right)\right]'(\sqrt{T}\tilde{D}_T)^{-1}\left(\frac{1}{T-2d_x}\sum_{t=1}^{T}\tilde{x}_t\tilde{x}_t'\right)(\sqrt{T}\tilde{D}_T)^{-1}\left[\sqrt{T}\tilde{D}_T\left(\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0\right)\right].
\end{aligned}$$

Since $\sqrt{T}\tilde{D}_T(\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0) = O_p(1)$,

$$\begin{aligned}
&(\sqrt{T}\tilde{D}_T)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_t\tilde{x}_t'\right)(\sqrt{T}\tilde{D}_T)^{-1} \\
&= \begin{pmatrix} \frac{1}{T^2}\frac{1}{T^2}\sum_{t=1}^{T}x_t x_t', & \frac{1}{T^{5/2}}\frac{1}{T}\sum_{t=1}^{T}x_t\Delta x_t' \\ \frac{1}{T^{5/2}}\frac{1}{T}\sum_{t=1}^{T}\Delta x_t x_t', & \frac{1}{T}\frac{1}{T}\sum_{t=1}^{T}\Delta x_t\Delta x_t' \end{pmatrix} = o_p(1),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{T}\left(\sum_{t=1}^{T}\Delta e_{0,t}\tilde{x}_t'\right)(\sqrt{T}\tilde{D}_T)^{-1} \\
&= \left[\begin{array}{cc} \frac{1}{T^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\Delta e_{0,t}x_t', & \frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\Delta e_{0,t}\Delta x_t' \end{array}\right] = o_p(1),
\end{aligned}$$

we have

$$\tilde{\Sigma}_{zz} = \frac{1}{T-2d_x}\sum_{t=1}^{T}(\Delta e_{0,t})^2 + o_p(1) \to^p \Sigma_{\Delta e,\Delta e}.$$

■

Proof of Proposition 7.

The convergence of $\tilde{\Omega}_{zz}$ is the same as (40), and so it suffices to prove the limit of $\tilde{\Sigma}_{zz}$.

Under conventional cointegration, we have $z_t = u_{0\cdot x,t}$. Hence,

$$\begin{aligned}
\tilde{\Sigma}_{zz} &= \frac{1}{\tilde{T}} \sum_{t=1}^{T} \left[u_{0\cdot x,t} - \tilde{x}_t' \left(\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0 \right) \right]^2 \\
&= \frac{1}{\tilde{T}} \sum_{t=1}^{T} (u_{0\cdot x,t})^2 - \frac{2}{\tilde{T}} \left[\left(\sum_{t=1}^{T} u_{0\cdot x,t} \tilde{x}_t' \right) \tilde{D}_T^{-1} \right] \tilde{D}_T (\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0) \\
&+ \left[\tilde{D}_T \left(\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0 \right) \right]' \left[\tilde{D}_T^{-1} \frac{1}{\tilde{T}} \sum_{t=1}^{T} \tilde{x}_t \tilde{x}_t' \tilde{D}_T^{-1} \right] \tilde{D}_T (\tilde{\vartheta}_{\text{TAOLS}} - \vartheta_0). \qquad (44)
\end{aligned}$$

Note that

$$\tilde{D}_T^{-1} \frac{1}{T} \sum_{t=1}^{T} \tilde{x}_t \tilde{x}_t' \tilde{D}_T^{-1} = \begin{pmatrix} \frac{1}{T} \frac{1}{T^2} \sum_{t=1}^{T} x_t x_t' & \frac{1}{T} \frac{1}{T} \sum_{t=1}^{T} x_t \Delta x_t' \\ \frac{1}{T} \frac{1}{T} \sum_{t=1}^{T} \Delta x_t x_t' & \frac{1}{T} \sum_{t=1}^{T} \Delta x_t \Delta x_t' \end{pmatrix} \to^p \begin{pmatrix} O_{d_x \times d_x}, & O_{d_x \times d_x} \\ O_{d_x \times d_x}, & \Sigma_{u_x, u_x} \end{pmatrix},$$

$$\frac{1}{T} \left(\sum_{t=1}^{T} u_{0\cdot x,t} \tilde{x}_t' \right) \tilde{D}_T^{-1} = \begin{pmatrix} \frac{1}{T} \frac{1}{T} \sum_{t=1}^{T} u_{0\cdot x,t} x_t', & \frac{1}{T} \sum_{t=1}^{T} u_{0\cdot x,t} \Delta x_t' \end{pmatrix} \to^p \left[O_{1 \times d_x}, \Sigma_{0\cdot x, u_x} \right].$$

We have

$$\begin{aligned}
\tilde{\Sigma}_{zz} &= \Sigma_{0\cdot x, 0\cdot x} + o_p(1) - 2\Sigma_{0\cdot x, u_x} \left(\tilde{\gamma}_{\text{TAOLS}} - \gamma_0 \right) (1 + o_p(1)) \\
&+ \left[\left(\tilde{\gamma}_{\text{TAOLS}} - \gamma_0 \right)' \Sigma_{u_x, u_x} \left(\tilde{\gamma}_{\text{TAOLS}} - \gamma_0 \right) \right] (1 + o_p(1)).
\end{aligned}$$

Using $\tilde{\gamma}_{\text{TAOLS}} - \gamma_0 \Rightarrow \Omega_{00\cdot x}^{1/2} \left(\eta_{\Delta x}' Q_{\xi_x} \eta_{\Delta x} \right)^{-1} \eta_{\Delta x}' Q_{\xi_x} \eta_{0\cdot x} := \gamma_{c,L}^{\infty}$, we obtain:

$$\begin{aligned}
\tilde{\Sigma}_{zz} &\Rightarrow \Sigma_{00\cdot x} + \left(\gamma_{c,L}^{\infty} \right)' \Sigma_{u_x, u_x} \gamma_{c,L}^{\infty} - 2\Sigma_{0\cdot x, u_x} \gamma_{c,L}^{\infty} \\
&= \begin{pmatrix} -\gamma_{c,L}^{\infty} \\ 1 \end{pmatrix}' \begin{pmatrix} \Sigma_{u_x, u_x} & \Sigma_{0\cdot x, u_x} \\ \Sigma_{0\cdot x, u_x}' & \Sigma_{0\cdot x, 0\cdot x} \end{pmatrix} \begin{pmatrix} -\gamma_{c,L}^{\infty} \\ 1 \end{pmatrix} := \Sigma_{c,L}^{\infty}.
\end{aligned}$$

Given that the variance matrix is positive definite, $\Sigma_{c,L}^{\infty} > 0$ almost surely. ■

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Supplemental Appendix: Asymptotic F and t Tests in the Presence of Deterministic Trends

The supplemental appendix extends the asymptotic F and t tests and the adaptive procedure to allow for deterministic trends in the cointegration system.

S.1 Multicointegration

In this subsection, we show how deterministic trends can be incorporated into the multicointegration model while preserving the applicability of the asymptotic F and t test theory.

Suppose the latent $I(1)$ process $\{(y_t^*, x_t^*)\}$ satisfies a multicointegration model without deterministic trends, as described in the main text:

$$\begin{aligned} y_t^* &= (x_t^*)'\beta_0 + u_{0,t}, \\ x_t^* &= x_{t-1}^* + u_{x,t}, \\ U_{0,t} &= \alpha_0^* + (x_t^*)'\gamma_0 + e_{0,t}, \end{aligned}$$

where $U_{0,t} = \sum_{\tau=1}^{t} u_{0,\tau}$ and $\alpha_0^* = -e_{0,0} - (x_0^*)'\gamma_0$. We now model the observed process $\{(y_t, x_t)\}$ as the latent $I(1)$ process augmented by deterministic linear trends:

$$\begin{aligned} y_t &= c_{y,0} + c_{y,1}t + y_t^*, \\ x_t &= c_{x,0} + c_{x,1}t + x_t^*, \end{aligned} \tag{S.1}$$

where $c_{y,0}, c_{y,1} \in \mathbb{R}$ and $c_{x,0}, c_{x,1} \in \mathbb{R}^{d_x}$ are constants. Then, a simple algebraic manipulation gives

$$y_t = c_0 + c_1 t + x_t'\beta_0 + u_{0,t}, \tag{S.2}$$

for $c_0 = c_{y,0} - c_{x,0}'\beta_0$ and $c_1 = c_{y,1} - c_{x,1}'\beta_0$. Thus, when either x_t or y_t has a linear trend, the model in (1) should also include a linear trend term.

Using the same derivations as in Section 2, we obtain the following for the latent $I(2)$ processes:

$$Y_t^* = \alpha_0^* + (X_t^*)'\beta_0 + (x_t^*)'\gamma_0 + (\Delta x_t^*)'\delta_0 + e_{0\cdot x,t}, \tag{S.3}$$

where $Y_t^* = \sum_{\tau=1}^{t} y_\tau^*$ and $X_t^* = \sum_{\tau=1}^{t} x_\tau^*$. Substituting:

$$Y_t = c_{y,0}t + c_{y,1}\frac{(t+1)\,t}{2} + Y_t^* \text{ and } X_t = c_{x,0}t + c_{x,1}\frac{(t+1)\,t}{2} + X_t^*$$

into the latent multicointegration equation yields:

$$\begin{aligned} Y_t &= \alpha_0 + \alpha_1 t + \alpha_2 t^2 + X_t'\beta_0 + x_t'\gamma_0 + \Delta x_t'\delta_0 + e_{0\cdot x,t} \\ &= \ell_t'\rho_0 + X_t'\beta_0 + x_t'\gamma_0 + \Delta x_t'\delta_0 + e_{0\cdot x,t}, \end{aligned} \tag{S.4}$$

where

$$\alpha_0 = \alpha_0^* - c_{x,0}'\gamma_0 - c_{x,1}'\delta_0,$$

$$\alpha_1 = c_{y,0} + \frac{1}{2}c_{y,1} - \left(c_{x,0} + \frac{1}{2}c_{x,1}\right)'\beta_0 - c_{x,1}'\gamma_0,$$

$$\alpha_2 = \frac{1}{2}c_{y,1} - \frac{1}{2}c_{x,1}'\beta_0,$$

$\ell_t = \left(1, t, t^2\right)'$, and $\rho_0 = (\alpha_0, \alpha_1, \alpha_2)'$. The model extends (8) by incorporating linear and quadratic trends; the only change is replacing $\ell_t = 1$ with $\ell_t = \left(1, t, t^2\right)'$.

Define

$$V_{\ell,i} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\ell_t\phi_i\left(\frac{t}{T}\right).$$

The TA regression equation corresponding to (S.4) is

$$V_{Y,i} = V_{\ell,i}'\rho_0 + V_{X,i}'\beta_0 + V_{x,i}'\gamma_0 + V_{\Delta x,i}'\delta_0 + V_{e\cdot x,i}. \tag{S.5}$$

Estimating this regression by OLS yields the TAOLS estimator $\hat{\theta}_{\text{TAOLS}}$ of the full parameter vector $\theta_0 = (\rho_0', \beta_0', \gamma_0', \delta_0')'$. Our primary interest lies in inference on $\left(\beta_0', \gamma_0', \delta_0'\right)'$, which captures the cointegrating relationships and adjustment dynamics. By the Frisch–Waugh–Lovell theorem, we have

$$(\hat{\beta}_{\text{TAOLS}}', \hat{\gamma}_{\text{TAOLS}}', \hat{\delta}_{\text{TAOLS}}')' = \left(V_{X,x,\Delta x}'Q_{V_\ell}V_{X,x,\Delta x}\right)^{-1}V_{X,x,\Delta x}'Q_{V_\ell}V_Y,$$

where $V_{X,x,\Delta x} = (V_X, V_x, V_{\Delta x})$.

Since

$$(X_t', x_t', \Delta x_t') = \left(X_t^{*\prime}, x_t^{*\prime}, \Delta x_t^{*\prime}\right) + \ell_t'\psi_x \text{ and } Y_t = Y_t^* + \ell_t'\psi_y,$$

with

$$\psi_x = \begin{pmatrix} \mathbf{0}_{1\times d_x} & c_{x,0}' & c_{x,1}' \\ c_{x,0}' + \frac{1}{2}c_{x,1}' & c_{x,1}' & \mathbf{0}_{1\times d_x} \\ \frac{1}{2}c_{x,1}' & \mathbf{0}_{1\times d_x} & \mathbf{0}_{1\times d_x} \end{pmatrix}_{3\times 3d_x} \text{ and } \psi_y = \begin{pmatrix} 0 \\ c_{y,0} + \frac{1}{2}c_{y,1} \\ \frac{1}{2}c_{y,1} \end{pmatrix},$$

we have

$$V_{X,x,\Delta x} = V_{X^*,x^*,\Delta x^*} + V_\ell\psi_x \text{ and } V_Y = V_{Y^*} + V_\ell\psi_y.$$

Consequently, the TAOLS estimator simplifies to:

$$\left(\hat{\beta}_{\text{TAOLS}}', \hat{\gamma}_{\text{TAOLS}}', \hat{\delta}_{\text{TAOLS}}'\right)' = \left(V_{X^*,x^*,\Delta x^*}'Q_{V_\ell}V_{X^*,x^*,\Delta x^*}\right)^{-1}V_{X^*,x^*,\Delta x^*}'Q_{V_\ell}V_{Y^*}.$$

Thus, the asymptotic properties of the above estimator coincide with those of the infeasible OLS estimator $(\hat{\beta}^{*\prime}_{\text{TAOLS}}, \hat{\gamma}^{*\prime}_{\text{TAOLS}}, \hat{\delta}^{*\prime}_{\text{TAOLS}})'$ based on the regression model involving only the latent processes:

$$V_{Y^*,i} = V'_{\ell,i}\rho_0 + V'_{X^*,i}\beta_0 + V'_{x^*,i}\gamma_0 + V'_{\Delta x^*,i}\delta_0 + V_{e\cdot x,i}. \tag{S.6}$$

The above model mirrors the earlier TAOLS setup in (10) with the only difference being the new definition of $V_{\ell,i}$, which arises from replacing $\ell_t = 1$ with $\ell_t = \left(1, t, t^2\right)'$. Instead of using the earlier result that $T^{-1/2}V_\ell \Rightarrow \int_0^1 \ell(r)\boldsymbol{\phi}(r)\,dr = \xi_\ell$ for $\ell_t = 1$ and $\ell(r) = 1$, we now use

$$D_{\ell,T}^{-1}V_\ell \Rightarrow \int_0^1 \boldsymbol{\phi}(r)\,\ell(r)'dr := \xi_\ell, \text{ for } \ell(r) = \left(1, r, r^2\right)',$$

where $D_{\ell,T} = \text{diag}\left(T^{1/2}, T^{3/2}, T^{5/2}\right)$. With this adjustment to accommodate deterministic trends, the asymptotic F and t test theory for testing β_0, γ_0, and δ_0 remains applicable to the regression model in (S.6), and therefore also to the regression model in (S.5).

More specifically, if $K > 3d_x + 3$ and the null hypothesis is $H\beta_0 = h_0$ or $R\gamma_0 = r_0$, then, for $\diamondsuit \in \{\beta, \gamma\}$,

$$\mathbb{W}_{m,\diamondsuit}(K) \Rightarrow F_{p,K-3d_x-3} \text{ and } \mathbb{T}_{m,\diamondsuit}(K) \Rightarrow t_{K-3d_x-3},$$

where $\mathbb{W}_{m,\diamondsuit}(K)$ and $\mathbb{T}_{m,\diamondsuit}(K)$ are the standard Wald and t statistics, respectively, based on the model in (S.5) and constructed in the usual way.

Intuitively, compared to the original model without deterministic trends, two additional coefficients (α_1 and α_2) corresponding to linear and quadratic trends have to be estimated. This increases the model complexity and reduces the available degrees of freedom by two. The degrees of freedom in (16) are adjusted accordingly: $\hat{\Omega}_{ee\cdot x} = ||\hat{V}_{e\cdot x}||^2/(K - 3d_x - 3)$, and the asymptotic F- and t-distributions are adjusted similarly.

S.2 Nonsingular Cointegration and Adaptive Tests

If there is no second layer of cointegration, we have the model in (S.2) with a nonsingular LRV for $\left(u_{0,t}, u'_{x,t}\right)'$. Augmenting the model with Δx_t leads to

$$y_t = c_0 + c_1 t + x'_t\beta_0 + \Delta x'_t\gamma_0 + u_{0\cdot x,t},$$

where $\{u_{0\cdot x,t}\}$ has a positive long-run variance. The asymptotic F and t theory of Hwang and Sun (2018) extends naturally:

$$\mathbb{W}_{c,\diamondsuit}(K) \Rightarrow F_{p,K-2d_x-2} \text{ and } \mathbb{T}_{c,\diamondsuit}(K) \Rightarrow t_{K-2d_x-2}.$$

Comparing this result with Theorem 3, only a degree-of-freedom adjustment is needed.

If the cointegration regime is unknown, the two regimes can be represented jointly as:

$$y_t = c_0 + c_1 t + x_t' \beta_0 + \Delta x_t' \gamma_0 + \underbrace{1 \{\Omega_{00\cdot x} > 0\} u_{0\cdot x,t} + 1 \{\Omega_{00\cdot x} = 0\} \Delta e_{0,t}}_{:= z_t} \tag{S.7}$$

or

$$Y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + X_t' \beta_0 + x_t' \gamma_0 + \Delta x_t' \delta_0 + \underbrace{1 \{\Omega_{00\cdot x} > 0\} U_{0\cdot x,t} + 1 \{\Omega_{00\cdot x} = 0\} e_{0\cdot x,t}}_{:= \tilde{Z}_t}, \tag{S.8}$$

where

$$\delta_0 = 0 \times 1 \{\Omega_{00\cdot x} > 0\} + \Omega_{xx}^{-1} \Omega_{xe} \times 1 \{\Omega_{00\cdot x} = 0\}.$$

The adaptive test in Section 4.3 applies with minor modifications. For example, for a given K, we now define $K_m = K + 3d_x + 3$ and $K_c = K + 2d_x + 2$ so that $K_m = K_c + d_x + 1$. This aligns the effective degrees of freedom across regimes, ensuring the data-driven weighting procedure remains asymptotically valid.

The above discussion assumes that x_t and y_t contain linear trends so that their integrated versions X_t and Y_t contain quadratic trends. Higher-order polynomial trends can be similarly accommodated. On the other hand, we can reduce the linear trends to constants by setting $c_{x,1} = 0$ and $c_{y,1} = 0$ in (S.1). In this case, the observed y_t and x_t have no linear trends but may have nonzero means. We then have

$$y_t = c_0 + x_t' \beta_0 + u_{0,t},$$

and

$$Y_t = \alpha_0 + \alpha_1 t + X_t' \beta_0 + x_t' \gamma_0 + \Delta x_t' \delta_0 + e_{0\cdot x,t}.$$

The asymptotic F- and t-test theory, including the adaptive version, remains valid after appropriate degrees-of-freedom adjustments. This highlights the flexibility of TAOLS in accommodating polynomial trends while preserving the applicability of asymptotic F and t tests.