Asymptotic F and t Tests in Cointegrating Regressions with Asymptotically Homogeneous Functions*

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Abstract

In this paper, we develop asymptotic F and t tests for nonlinear cointegrated regression, where the regressors are asymptotically homogeneous transformations of I(1) processes. These transformations encompass a broad class of functions, including distribution-like functions, logarithmic functions, and asymptotically polynomial functions. Our asymptotic F and t test theory covers both the case with exogenous regressors and the case with endogenous regressors. For the exogenous case, we construct a novel set of basis functions for series long-run variance estimation, effectively accounting for parameter estimation uncertainty. For the endogenous case, we extend the transformed-augmented OLS approach developed for linear cointegrated settings. Monte Carlo simulations show that our asymptotic F and t tests outperform competing tests, including the asymptotic chi-square test based on the fully modified OLS estimator and the non-standard fixed-b test based on the integrated modified OLS estimator. Furthermore, our theory extends to accommodate cases where the processes driving the regressors are nonstationary, fractionally integrated processes.

JEL Classification: C12, C13, C32

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1 Introduction

Cointegrated systems have been extensively studied in econometrics due to their importance in modeling long-term relationships among integrated variables. While much of the focus has been on linear cointegration, recent advancements have explored nonlinear dynamics. Nonlinear cointegration models are particularly useful for capturing nonlinear relationships between nonstationary economic variables (e.g., Park and Phillips (2001)). As might be expected, the asymptotic analysis of cointegrated regressions becomes more complex when nonlinear transformations of unit root I(1) processes are introduced into the system. Additionally, statistical inference faces challenges in accounting for nonparametric autocorrelation in the cointegration errors and their potential correlation with the regressors.

In this paper, we develop asymptotic F and t tests for a triangular cointegrated system, where the regressors are asymptotically homogeneous functions of I(1) processes. The class of asymptotically homogeneous functions, introduced by Park and Phillips (1999), encompasses a wide range of functions, including constant functions, distribution-like functions, logarithmic functions, and functions that asymptotically resemble polynomials. Empirical applications of asymptotically homogeneous cointegrated systems include smooth transition models (e.g., Saikkonen and Choi (2004)), the money demand function (e.g., Bae and De Jong (2007)), the carbon Kuznets curve (e.g., Chan and Wang (2015)), and the multifactor translog production function (e.g., Vogelsang and Wagner (2024)). Our asymptotic F and t test theory applies to this class of nonlinear cointegration models in both exogenous and endogenous settings.

We begin by examining cointegration under the assumption that the regressors are exogenous and establish the asymptotic mixed-normal distribution of the ordinary least squares (OLS) estimator. The conditional asymptotic variance depends on the long-run variance (LRV) of the cointegration errors, which may exhibit autocorrelation of unknown forms. It is standard practice to estimate the LRV using a kernel method and to construct test statistics based on the kernel LRV estimator. The conventional approach employs increasing-smoothing asymptotics, under which the distribution of the LRV estimator is approximated by a degenerate distribution concentrated at the true LRV. As a result, the Wald and t statistics are asymptotically chi-squared and normal, respectively. However, these asymptotic approximations fail to account for the estimation uncertainty of the LRV estimator and are prone to large size distortions in finite samples. See, for example, Kiefer and Vogelsang (2005), Sun et al. (2008), and Sun (2014).

In response, we propose using a series-based LRV estimator and approximating its distribution through fixed-smoothing asymptotics. This type of asymptotics is often referred to as fixed-K asymptotics in the literature, where K is the smoothing parameter held fixed in the limiting experiment. It has been utilized in prior studies, such as Phillips (2005), Müller (2007), and Sun (2013), albeit within frameworks involving only stationary processes. For linear cointegration regressions with I(1) regressors, Sun (2023) adopts a series-based method and shows that asymptotically pivotal inference is possible, although the mechanics are different from those in the stationary setting. However, for cointegration regressions with transformed I(1) regressors, as considered here, challenges emerge. Specifically, if we employ commonly used orthonormal basis functions in $L^2[0,1]$ to construct the series LRV estimator, as in the stationary framework and Sun (2023), the resulting test statistic will not be asymptotically pivotal.

To address this issue and develop asymptotically pivotal F and t test theory under fixed-smoothing asymptotics, we construct a novel set of basis functions tailored for LRV estimation in the cointegration framework. For any given set of basis functions in $L^2[0,1]$, we first project the corresponding basis vectors onto the orthogonal complement of the column space spanned by the regressors. We then normalize the projected basis vectors to obtain a new set of basis vectors, which are subsequently used in series LRV estimation. The use of these projected and normalized bases enables us to develop the asymptotic F and t test theory.

Our approach underscores the flexibility of the series-based method, as it allows us to design customized basis functions or basis vectors to suit specific needs for asymptotically pivotal inference. In contrast, a kernel-based method lacks such adaptability and requires the imposition of a restrictive assumption, the so-called full-design condition, to achieve an asymptotically pivotal limit under fixed-b asymptotics. This point was discovered and highlighted by Vogelsang and Wagner (2024).

We extend the asymptotic F and t testing framework to cases where the regressors are endogenous. In a linear cointegration setting, it is well known that the OLS estimator suffers from a second-order endogeneity bias, complicating asymptotically pivotal inference (e.g., Phillips and Hansen (1990)). This issue also appears in the polynomial cointegrated system as studied in Wagner and Hong (2016) and Chan and Wang (2015). Unlike the fully modified OLS method (e.g., Wagner (2015) and Wagner and Hong (2016)), we use an approach similar to those proposed by Hwang and Sun (2018), and Pellatt and Sun (2023), beginning by transforming the augmented homogeneous cointegrated regression using orthonormal basis functions.

Following Hwang and Sun (2018), we estimate the parameters by OLS based on a Transformed and Augmented (TA) regression model. We call the resulting estimator the TA-OLS estimator. Extending the results of Hwang and Sun (2018) for linear cointegration, we establish the asymptotic mixed-normality of the TA-OLS estimator. Importantly, this distribution is free from the second-order endogeneity bias that typically complicates inference in a cointegration model with endogenous regressors. Under fixed-smoothing asymptotics, we show that the test statistics based on the TA-OLS estimator are asymptotically F-distributed or t-distributed. As in the case of exogeneity, our procedure is particularly advantageous because critical values from the F and t distributions are readily available in any standard statistical software. This makes our method practically convenient, especially when compared to the integrated modified OLS (IM-OLS) method of Vogelsang and Wagner (2024), where the fixed-b limiting distribution is highly nonstandard, and critical values have to be simulated.

As a special case of the asymptotically homogeneous cointegrated setting, Vogelsang and Wagner (2014) study polynomial cointegrated regressions and impose the full-design condition to achieve asymptotically pivotal fixed-b inference. The full-design condition requires the inclusion of all possible transforms of the underlying I(1) processes, up to the highest order specified in the polynomial cointegration equation, which can be restrictive in practical applications. In contrast, our approach does not rely on the full-design condition, yet it still achieves standard F and t limits under fixed-K asymptotics. This flexibility is a key advantage of our method, as it does not restrict model specifications for the purpose of asymptotic development. This is especially appealing when certain higher-order terms are unnecessary in a polynomial cointegrated system.

In a Monte Carlo simulation study, we compare the finite-sample performance of our asymptotic F test in polynomial cointegration regressions to that of the existing tests based on the fully modified OLS (FM-OLS) and IM-OLS. In both exogenous and endogenous cases, the proposed F tests outperform competing tests. In particular, for the endogenous case, the F test is shown to be more accurate in size than FM-OLS with chi-square critical values and IM-OLS with non-standard critical values. The improved performance is especially evident when the degree of serial autocorrelation is strong. These findings are consistent with previous studies, such as Hwang and Sun (2018) and Pellatt and Sun (2023), which support the accuracy of the F test in linear cointegration regressions.

We use the TA-OLS method to estimate the carbon Kuznets curve (CKC) and employ the proposed F and t tests for inferences. The CKC examines how per capita CO_2 emissions in a country depend on its per capita GDP over time. Our method is particularly suited for this application, as a quadratic function of the logarithm of per capita GDP, which is widely regarded as a unit root process, is an integral part of the model. In addition, the logarithm of per capita GDP is likely to be endogenous. This setting aligns perfectly with the design of our method under endogeneity, which enables us to test the inverted U-shaped relationship between environmental pollution and economic activity with higher accuracy than existing methods.

As an additional contribution, we extend our method to a more general setting where the regressors are transformations of fractionally integrated processes. The generalization of I(1) components to nonstationary fractional processes has been explored in studies such as Robinson and Hualde (2003) and Hualde and Iacone (2019) in the context of linear cointegration. However, to the best of our knowledge, no existing studies have applied an asymptotically homogeneous transformation in this context. By replacing first differencing with fractional differencing in the TA-OLS regression, we show that the F and t test theory can be naturally extended to situations where the underlying driving process is a nonstationary fractional process. Moreover, we introduce a feasible TA-OLS estimation and testing procedure using a consistent estimator of the true fractional parameter, such as those proposed by Andrews and Sun (2004) and Shimotsu and Phillips (2005), among many others. We prove that the estimation error in the fractional parameter estimator does not affect our asymptotic theory, and the standard F test and t test remain asymptotically valid with an estimated fractional parameter. This appears to be the first study to establish the convenient asymptotic F test and t test in this generalized setting.

Our study contributes to the existing literature by introducing more accurate and reliable statistical tests, thereby broadening the scope of econometric analysis in contexts involving nonlinearity, nonstationarity, and cointegration. Building on foundational works for linear cointegrated systems, the literature on nonlinear cointegration has expanded to encompass both parametric and nonparametric approaches; see, for example, Chang et al. (2001), Saikkonen and Choi (2004),Wang and Phillips (2009), Chan and Wang (2015), and Dong et al. (2021). In particular, the asymptotically homogeneous cointegrated system considered in our paper includes finite-order polynomials as special cases. Consequently, our framework encompasses the cointegrating polynomial regressions of Wagner and Hong (2016) and Vogelsang and Wagner (2024). Transforming nonstationary variables using basis functions has also been adopted in recent cointegration literature (e.g., Müller and Watson (2013), Hwang and Sun (2018), and Sun et al. (2024)). Our approach and findings are expected to resonate with empirical researchers in macroeconomics and finance who frequently encounter nonlinear dynamic relationships among integrated variables.

Finally, we note that the TA-OLS method presented in this paper, along with that of Hwang and Sun (2018), is closely related to the high-dimensional trend instrumental variable (TIV) method, originally proposed by Phillips (2014), and further studied recently in Phillips and Kheifets (2024) and Sun et al. (2024). In the linear cointegration setting, Hwang and Sun (2018) show that under the large-K asymptotics, the TIV estimator and the TA-OLS estimator are asymptotically equivalent, and both are semi-parametrically efficient; see also Sun et al. (2024). While homogeneous cointegrated systems with more general nonstationary fractionally integrated processes have not yet been explored from the perspective of the TIV method, our fixed-K asymptotic framework, combined with easy-to-use F and t tests, is expected to provide more accurate approximations for the TIV method as well.

The rest of this paper is organized as follows. Section 2 introduces the asymptotically homogeneous cointegrated system and develops asymptotic F and t tests under exogeneity. Section 3 addresses the case with endogeneity and establishes the asymptotic properties of the TA-OLS estimator and the corresponding F and t tests. Sections 4 and 5 present Monte Carlo simulation results and an empirical application to real-world data, respectively. Section 6 extends the results in Section 3 to the case with nonstationary, fractionally integrated regressors. The final section concludes and outlines directions for future research.

We use the following notation throughout the paper: For any symmetric and positive definite matrix Ω, $\Omega^{1/2}$ is defined to be a symmetric and positive definite matrix such that $\Omega^{1/2}\Omega^{1/2} = \Omega$, unless stated otherwise. We define $\Omega^{-1/2} := \left(\Omega^{1/2}\right)^{-1}$ and let O be a matrix of zeros with dimensions that may change from each occurrence. For a matrix M with d_M rows, P_M is defined to be $M(M'M)^{-1}M'$ and Q_M is defined to be $I_{d_M} - P_M$. $\mathbb{N}$ represents the set of all positive integers (natural numbers), and $[a]$ denotes the greatest integer less than or equal to a (the floor function). The notation "$\Rightarrow$" indicates the weak convergence of a sequence of stochastic processes and random variables.

2 Cointegrated Homogeneous Regression under Exogeneity

We begin by describing the process X_t that drives the regressors in the cointegration model. We assume that $X_t = (X_{1,t}, X_{2,t}, \ldots, X_{d_x,t})'$ follows a d_x-dimensional unit root process (i.e.,

X_t is integrated of order one, I(1)):

$$X_t = X_{t-1} + u_{x,t}$$

for some stationary I(0) process $u_{x,t} \in \mathbb{R}^{d_x}$. Next, we consider the following cointegration model:

$$Y_t = Z_t' \beta_0 + u_{0,t}, \tag{1}$$

where $Y_t \in \mathbb{R}$, $u_{0,t} \in \mathbb{R}$ is a stationary I(0) process with zero mean, and each element of $Z_t = (Z_{1,t}, \ldots, Z_{d_z,t})'$ is driven by X_t according to:

$$Z_{i,t} = f_{i,0}(t) \cdot \prod_{j=1}^{d_x} f_{i,j}(X_{j,t}) \text{ for } i \in \{1, \ldots, d_z\},$$

with $f_{i,0}(\cdot)$ being a locally Riemann integrable function, and $f_{i,j}(\cdot)$ belonging to the class of asymptotically homogeneous functions as defined in Park and Phillips (1999). We formalize the requirements for $f_{i,0}(\cdot)$ and $f_{i,j}(\cdot)$ as an assumption below.

Assumption 1 *(i) $f_{i,0}(\cdot) : \mathbb{R} \to \mathbb{R}$ is locally Riemann integrable, and for some positive function $\kappa_{i0}(\cdot)$, the following*

$$f_{i,0}(t) = \kappa_{i0}(T) f_{i,0}\left(\frac{t}{T}\right)(1 + o(1))$$

holds uniformly over t, as $T \to \infty$.

(ii) For $j \in \{1, \ldots, d_x\}$, the function $f_{i,j}(\cdot) : \mathbb{R} \to \mathbb{R}$ is asymptotically homogeneous in the sense that for any $\lambda > 0$,

$$f_{i,j}(\lambda x) = \kappa_{i,j}(\lambda) H_{i,j}(x) + \mathcal{R}_{i,j}(\lambda, x),$$

where $\kappa_{i,j}(\lambda) > 0$, $H_{i,j}(x)$ is locally Riemann integrable, and $\mathcal{R}_{i,j}(\lambda, x)$ satisfies one of the following two conditions:

(a) $|\mathcal{R}_{i,j}(\lambda, x)| \leq a_{i,j}(\lambda) \mathcal{P}_{i,j}(x)$, where $limsup_{\lambda \to \infty} a_{i,j}(\lambda) / \kappa_{i,j}(\lambda) = 0$ and $\mathcal{P}_{i,j}(x)$ is locally Riemann integrable;

(b) $|\mathcal{R}_{i,j}(\lambda, x)| \leq b_{i,j}(\lambda) \mathcal{Q}_{i,j}(\lambda x)$, where $limsup_{\lambda \to \infty} b_{i,j}(\lambda) / \kappa_{i,j}(\lambda) < \infty$ and $\mathcal{Q}_{i,j}(x)$ is bounded and vanishes at infinity (i.e., $\mathcal{Q}_{i,j}(x) \to 0$ as $|x| \to \infty$).

Assumption 1(i) ensures that, after appropriate normalization, we can convert the discrete time trend into a continuous time trend. Assumption 1(ii) states that $f_{i,j}(\lambda x)$ is approximately equal to $\kappa_{i,j}(\lambda) H_{i,j}(x)$, with a remainder that is asymptotically negligible. For example, each $f_{i,j}(\cdot)$ can be one of the following functions (with parameters that may vary for each $i \in \{1, \ldots, d_z\}$ and $j \in \{1, \ldots, d_x\}$):

(a) Homogeneous functions, such as x^θ with parameter $\theta > 0$, in which case, $\kappa(\lambda) = \lambda^\theta$ and $H(x) = x^\theta$; and $\text{sign}(x)$, in which case, $\kappa(\lambda) = 1$ and $H(x) = \text{sign}(x)$;

(b) Finite order polynomials, given by $x^k + a_1 x^{k-1} + \cdots + a_k$ with parameters $k \in \mathbb{N}$ and $(a_1, \ldots, a_k) \in \mathbb{R}^k$, in which case, $\kappa(\lambda) = \lambda^k$ and $H(x) = x^k$;

(c) The logarithm function $\log(|x|)$, in which case, $\kappa(\lambda) = \log(\lambda)$ and $H(x) = 1$;

(d) The cumulative distribution function of any random variable, in which case, $\kappa(\lambda) = 1$ and $H(x) = 1\{x \geq 0\}$.

Economic applications of polynomial transformations in Examples (a) and (b) can be found in Chan and Wang (2015) and Vogelsang and Wagner (2024) in the contexts of estimating carbon Kuznets curves and multifactor translog production functions. The logarithmic transformation in Example (c) is applied in Bae and De Jong (2007) for money demand analysis. The application of Example (d) includes smooth transition models, as considered in Saikkonen and Choi (2004). Remark 4.3 in Park and Phillips (1999) provides more detailed descriptions of the above asymptotically homogeneous functions. See also the discussions in the more recent monograph Wang (2015). Additionally, we note that an intercept can be included in our cointegration model given in (1) by setting $f_{1,0}, f_{1,1}, \ldots, f_{1,d_x}$ to be constant functions.

Remark 1 *Assumption 1 can be replaced by the following more general multivariate version: each $Z_{i,t}$ is an asymptotically homogeneous multivariate transform of the vector $(t', X_t')' \in \mathbb{R}^{d_x+1}$, that is, $Z_{i,t}$ takes the form of $f_i(t, X_t)$, where $f_i(\tau, \mathbf{x})$ is asymptotically homogeneous in the sense that for any $\Lambda_1 \in \mathbb{R}$ and $\mathbf{\Lambda}_2 \in \mathbb{R}^{1\times d_x}$,*

$$f_i(\Lambda_1 \tau, \mathbf{\Lambda}_2 \mathbf{x}) = \kappa_i(\Lambda_1, \mathbf{\Lambda}_2) H_i(\tau, \mathbf{x}) + \mathcal{R}_i(\Lambda_1, \mathbf{\Lambda}_2, \tau, \mathbf{x}),$$

where $\mathcal{R}_i(\Lambda_1, \mathbf{\Lambda}_2, \tau, \mathbf{x})$ satisfies either (a)

$$|\mathcal{R}_i(\Lambda_1, \mathbf{\Lambda}_2, \tau, \mathbf{x})| \leq a_i(\Lambda_1, \mathbf{\Lambda}_2) \mathcal{P}_i(\tau, \mathbf{x}),$$

where $lim_{\min(\Lambda_1, \|\mathbf{\Lambda}_2\|)\to 0} a_i(\Lambda_1, \mathbf{\Lambda}_2)/\kappa_i(\Lambda_1, \mathbf{\Lambda}_2) = 0$ and $\mathcal{P}_i(\tau, \mathbf{x})$ is locally Riemann integrable; or (b)

$$|\mathcal{R}_i(\Lambda_1, \mathbf{\Lambda}_2, \tau, \mathbf{x})| \leq b_i(\Lambda_1, \mathbf{\Lambda}_2) \mathcal{Q}_i(\Lambda_1 \tau, \mathbf{\Lambda}_2 \mathbf{x}),$$

where $lim_{\min(\Lambda_1, \|\mathbf{\Lambda}_2\|)\to 0} b_i(\Lambda_1, \mathbf{\Lambda}_2)/\kappa_i(\Lambda_1, \mathbf{\Lambda}_2) < \infty$ and $\mathcal{Q}_i(\tau, \mathbf{x})$ is bounded and vanishes at infinity (i.e., $\mathcal{Q}_i(\tau, \mathbf{x}) \to 0$ as $\min(|\tau|, \|\mathbf{x}\|) \to \infty$).

With some modifications of D_T and $Z(\cdot)$ defined later, our asymptotic theory still holds under the above assumption; but for simplicity, we maintain Assumption 1.

We turn to the assumption we impose on the initial value X_0 and the error processes. Denote the long-run variance of $u_t = (u_{0,t}', u_{x,t}')'$ and its submatrices by

$$\Omega = \sum_{j=-\infty}^{\infty} E u_t u_{t-j}' = \begin{pmatrix} \underset{1\times 1}{\Omega_{00}} & \underset{1\times d_x}{\Omega_{0x}} \\ \underset{d_x\times 1}{\Omega_{x0}} & \underset{d_x\times d_x}{\Omega_{xx}} \end{pmatrix}.$$

Assumption 2 *(i)* $X_0 = o_p(\sqrt{T})$.

(ii) The Functional Central Limit Theorem (FCLT) holds:

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[T\cdot]} u_t = \frac{1}{\sqrt{T}}\sum_{t=1}^{[T\cdot]} \begin{pmatrix} u_{0,t} \\ u_{x,t} \end{pmatrix} \Rightarrow B(\cdot) = \begin{pmatrix} B_0(\cdot) \\ B_x(\cdot) \end{pmatrix}, \tag{2}$$

where $B(\cdot) := (B_0'(\cdot), B_x'(\cdot))'$ *is a* (d_x+1)*-dimensional Brownian process with a positive definite variance* Ω.

(iii) $\{u_{0,t}\} := \{u_{0,t} : t \in \mathbb{N}\}$ *is independent of* $\{u_{x,t}\} := \{u_{x,t} : t \in \mathbb{N}\}$.

Assumption 2(i) ensures that the initial value of the I(1) process X_t will not affect our asymptotics. Assumption 2(ii) holds by the standard FCLT under well-known primitive conditions. The positive definiteness of Ω ensures that its submatrix Ω_{xx} is also positive definite, thereby implying that X_t is a full-rank I(1) process.

Assumption 2(iii) assumes that $\{u_{0,t}\}$ is independent of $\{u_{x,t}\}$, so that $\Omega_{0x} = 0$. We will address the case of dependence in the next section. Under Assumption 2(iii), we can represent $B(\cdot)$ as

$$B(\cdot) = \begin{pmatrix} \Omega_{00}^{1/2} & O \\ O & \Omega_{xx}^{1/2} \end{pmatrix} \begin{pmatrix} W_0(\cdot) \\ W_x(\cdot) \end{pmatrix},$$

where $W(\cdot) := (W_0'(\cdot), W_x'(\cdot))'$ is a (d_x+1)-dimensional standard Brownian. Note that $B_0(\cdot) = \Omega_{00}^{1/2} W(\cdot)$ is a one-dimensional process, which corresponds to the weak limit of the process $T^{-1/2}\sum_{t=1}^{[T\cdot]} u_{0,t}$.

When each element of Z_t is an I(1) variable, with no transformation applied to X_t, the model reduces to the triangular representation of a linear cointegration system as considered in Phillips (1991). When each element of Z_t takes the monomial form $t^{k_{i,0}}\prod_{j=1}^{d_x} X_{j,t}^{k_{i,j}}$, the model corresponds to the cointegrating polynomial regressions considered, for example, by Wagner (2015), Wagner and Hong (2016), Wagner et al. (2020), de Jong and Wagner (2022), Stypka et al. (2024), and Vogelsang and Wagner (2024). Our model is linear in parameters and, hence, easier to estimate than the nonlinear cointegration models that are nonlinear in parameters (cf. Wang (2015)).

Under Assumptions 1 and 2, we have, for $D_T = \mathrm{diag}((\nu_1(\sqrt{T}), \ldots, \nu_{d_z}(\sqrt{T}))')$ with $\nu_i(\lambda) := \kappa_{i,0}(\lambda^2) \cdot \prod_{j=1}^{d_x} \kappa_{i,j}(\lambda)$,

$$D_T^{-1} Z_{[T\tau]} \Rightarrow Z(\tau), \tag{3}$$

where $Z(\tau)$ is a vector of continuous-time processes defined on $\tau \in [0,1]$. Specifically, the conditions outlined in Assumptions 1 and 2, along with the continuous mapping theorem, allow us to establish the joint weak convergence of the vector-valued process $D_T^{-1} Z_{[T\tau]} =$

$D_T^{-1}(Z_{1,[T\tau]},\ldots,Z_{d_z,[T\tau]})'$. The marginal convergence for each component is given by

$$\begin{aligned}\frac{1}{\nu_i(\sqrt{T})}Z_{i,[T\tau]} &= \frac{1}{\nu_i(\sqrt{T})}\left(\kappa_{i,0}\left(T\right)H_{i,0}\left(\frac{[T\tau]}{T}\right)\cdot\prod_{j=1}^{d_x}\kappa_{i,j}(\sqrt{T})H_{i,j}\left(\frac{X_{j,[T\tau]}}{\sqrt{T}}\right)\right)+o_p(1)\\ &= H_{i,0}\left(\frac{[T\tau]}{T}\right)\cdot\left(\prod_{j=1}^{d_x}H_{i,j}\left(\frac{X_{j,[T\tau]}}{\sqrt{T}}\right)\right)+o_p(1)\\ &\Rightarrow H_{i,0}(\tau)\cdot\left(\prod_{j=1}^{d_x}H_{i,j}(B_{x,j}(\tau))\right):=H_i(\tau,B_x(\tau)),\end{aligned}$$

for $i\in\{1,\ldots,d_z\}$, where $B_{x,j}(\cdot)$ is the j-th component of the vector $B_x(\cdot)=(B_{x,1}(\cdot),\ldots,B_{x,d_x}(\cdot))'$.

When the i-th element $Z_{i,t}$ of Z_t takes the monomial form $t^{k_{i,0}}(\prod_{j=1}^{d_x}X_{j,t}^{k_{i,j}})$, we have

$$\begin{aligned}&f_{i,0}(t)=t^{k_{i,0}},\ H_{i,0}\left(t\right)=t^{k_{i,0}},\ \kappa_{i,0}\left(T\right)=T^{k_{i,0}};\\ &f_{i,j}(x)=x^{k_{i,j}},\ H_{i,j}\left(x\right)=x^{k_{i,j}},\ \kappa_{i,j}(T^{1/2})=T^{k_{i,j}/2},\ \text{for } j\in\{1,\ldots,d_x\},\end{aligned}$$

so $\nu_i(\sqrt{T})=T^{\ell_i}$ with $\ell_i=k_{i,0}+(k_{i,1}+\ldots+k_{i,d_x})/2$, and the weak limit of $\nu_i(\sqrt{T})^{-1}Z_{i,[T\tau]}$ can be written as

$$H_i(\tau,B_x(\tau))=\tau^{k_{i,0}}\left(\prod_{j=1}^{d_x}(B_{x,j}(\tau))^{k_{i,j}}\right).$$

Given the observations $\{X_t,Y_t\}_{t=1}^T$, we estimate β_0 by the OLS estimator:

$$\hat{\beta}_{\text{OLS}}=\left(\sum_{t=1}^{T}Z_tZ_t'\right)^{-1}\left(\sum_{t=1}^{T}Z_tY_t\right)=\left(Z'Z\right)^{-1}Z'Y,$$

where $Z=(Z_1,\ldots,Z_T)'$ and $Y=(Y_1,\ldots,Y_T)'$. Using the continuous mapping theorem and other standard arguments, we have:

$$\begin{aligned}\sqrt{T}D_T(\hat{\beta}_{\text{OLS}}-\beta_0) &= \left(\frac{1}{T}\sum_{t=1}^{T}\left(D_T^{-1}Z_t\right)\left(D_T^{-1}Z_t\right)'\right)^{-1}\left(\sum_{t=1}^{T}\left(D_T^{-1}Z_t\right)\frac{u_{0,t}}{\sqrt{T}}\right)\\ &\Rightarrow \Omega_{00}^{1/2}\left[\int_0^1 Z\left(\tau\right)Z\left(\tau\right)'d\tau\right]^{-1}\int_0^1 Z\left(\tau\right)dW_0\left(\tau\right),\end{aligned}$$

provided that $\int_0^1 Z\left(\tau\right)Z\left(\tau\right)'d\tau$ is positive definite almost surely.

Given some $p\times d_z$ matrix $R=[R\left(i,j\right)]$, where $R\left(i,j\right)$ represents the (i,j)-th component of R, and a $p\times 1$ vector r, we are interested in testing:

$$H_0: R\beta_0=r \text{ against } H_1: R\beta_0\neq r.$$

When $p = 1$, we may be interested in testing a one-sided alternative. For example, we may test

$$H_0 : R\beta_0 = r \text{ against } H_1 : R\beta_0 > r.$$

Different elements of $R\hat{\beta}$ may converge at different rates. We assume that there exists a $p \times p$ diagonal matrix $\tilde{D}_T$ such that $\lim_{T\to\infty} \tilde{D}_T R D_T^{-1} = A$ for a matrix $A \in \mathbb{R}^{p\times d_z}$ with full row rank p. Then, the rate of convergence of $R\hat{\beta}$ to $R\beta_0$ is given by $(\tilde{D}_T\sqrt{T})^{-1}$, as

$$\begin{aligned}
&\tilde{D}_T\sqrt{T}R\left(\hat{\beta}_{\text{OLS}} - \beta_0\right) \\
&= \left(\tilde{D}_T R D_T^{-1}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(D_T^{-1}Z_t\right)\left(D_T^{-1}Z_t\right)'\right)^{-1}\left(\sum_{t=1}^{T}\left(D_T^{-1}Z_t\right)\frac{u_{0,t}}{\sqrt{T}}\right) \\
&\Rightarrow \Omega_{00}^{1/2} A\left[\int_0^1 Z(\tau)Z(\tau)'\,d\tau\right]^{-1}\int_0^1 Z(\tau)\,dW_0(\tau) \\
&:= \Omega_{00}^{1/2}\int_0^1 Z^*(\tau)\,dW_0(\tau), \qquad (4)
\end{aligned}$$

where

$$Z^*(\tau) = A\left[\int_0^1 Z(\tau)Z(\tau)'\,d\tau\right]^{-1} Z(\tau).$$

The above asymptotic theory forms the basis for testing H_0 against H_1, but we still need to estimate the long-run variance Ω_{00}. Here we employ a series method for this estimation. For a set of K basis functions $\{\phi_i(\cdot)\}_{i=1}^K$ in $L_2[0,1]$, with K as the tuning parameter, the series long-run variance estimator, e.g., Phillips (2005), Müller (2007), and Sun (2013), takes the form of

$$\tilde{\Omega}_{00} = \frac{1}{KT}\sum_{i=1}^{K}(\boldsymbol{\phi}_i'\hat{u})^2 = \frac{1}{K}\sum_{i=1}^{K}\left[\frac{1}{\sqrt{T}}\phi_i\left(\frac{t}{T}\right)\hat{u}_{0t}\right]^2,$$

where $\hat{u}_0 = (\hat{u}_{0,t},\ldots,\hat{u}_{0,T})' = Y - Z\hat{\beta}_{\text{OLS}}$ is the vector of the OLS residuals and $\boldsymbol{\phi}_i = (\phi_i(1/T),\ldots,\phi_i(T/T))'$ is the basis vector corresponding to the basis function $\phi_i(\cdot)$.

For each $i \in \{1,\ldots,K\}$, we have, for $u_0 = (u_{0,1},\ldots,u_{0,T})'$:

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\boldsymbol{\phi}_i'\hat{u}_0 \\
&= \frac{1}{\sqrt{T}}\boldsymbol{\phi}_i' Q_Z u_0 = \frac{1}{\sqrt{T}}\left(\boldsymbol{\phi}_i'u_0 - \boldsymbol{\phi}_i'Z\left(Z'Z\right)^{-1}Z'u_0\right) \\
&\Rightarrow \Omega_{00}^{1/2}\int_0^1 \phi_i(\tau)\,dW_0(\tau) - \left(\int_0^1\phi_i(s)Z(s)'\,ds\right)\left(\int_0^1 Z(s)Z(s)'\,ds\right)^{-1}\int_0^1 Z(\tau)\,dW_0(\tau) \\
&= \Omega_{00}^{1/2}\int_0^1\tilde{\phi}_i(\tau)\,dW_0(\tau), \qquad (5)
\end{aligned}$$

where

$$\tilde{\phi}_i(\cdot) = \phi_i(\cdot) - \left(\int_0^1\phi_i(s)Z(s)'\,ds\right)\left(\int_0^1 Z(s)Z(s)'\,ds\right)^{-1}Z(\cdot). \qquad (6)$$

In the linear cointegration setting with $Z_t = (1, X_t')'$, Sun (2023) shows that $\tilde{\phi}_i(\cdot)$ in (5) does not depend on any nuisance parameters, such as Ω_{xx}. As a result, the long-run variance estimator $\tilde{\Omega}_{00}/\Omega_{00}$ weakly converges to a random variable that does not depend on any unknown parameters under fixed-K asymptotics, making the Wald inference based on $\tilde{\Omega}_{00}$ asymptotically pivotal. However, due to the non-standard limiting distribution, simulated critical values are necessary.

Although asymptotically pivotal inference applies in the linear cointegrated case, pivotal inference based on $\tilde{\Omega}_{00}$ cannot be directly extended to cointegrated nonlinear homogeneous regressions. This is because nuisance parameters that govern the limiting process $Z(\cdot)$ will retain their effect in $\tilde{\phi}_i(\cdot)$ in the nonlinear case, which implies that $\tilde{\Omega}_{00}/\Omega_{00}$ is not asymptotically pivotal. Consider, for example, $d_x = 2$ and $Z_t = \left(X_{1t}^2, X_{2t}^2\right)$. We have the following representation of $Z(\cdot)$, which is also considered in Vogelsang and Wagner (2024):

$$Z(\cdot) = \begin{pmatrix} B_{x,1}^2(\cdot) \\ B_{x,2}^2(\cdot) \end{pmatrix} = \underbrace{\begin{pmatrix} \varpi_{11}^2 & \varpi_{12}^2 & 2\varpi_{11}\varpi_{12} \\ 0 & \varpi_{22}^2 & 0 \end{pmatrix}}_{:=\Gamma} \underbrace{\begin{pmatrix} W_{x,1}^2(\cdot) \\ W_{x,2}^2(\cdot) \\ W_{x,1}(\cdot)W_{x,2}(\cdot) \end{pmatrix}}_{:=\mathcal{W}_x} = \Gamma\mathcal{W}_x(\cdot) \quad (7)$$

where ϖ_{11}, ϖ_{12}, and ϖ_{22} are defined according to

$$\Omega_{xx}^{1/2} = \begin{bmatrix} \varpi_{11} & \varpi_{12} \\ O & \varpi_{22} \end{bmatrix}$$

and $\mathcal{W}_x$ is free from any nuisance parameter. Let $\Gamma = \mathcal{U}\mathcal{D}\mathcal{V}'$, for $\mathcal{U} \in \mathbb{R}^{2\times 2}$ and $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2) \in \mathbb{R}^{3\times 2}$, be the singular value decomposition of Γ. Some simple algebra shows that

$$\tilde{\phi}_i(\cdot) = \phi_i(\cdot) - \left(\int_0^1 \phi_i(s)\left[\mathcal{V}'\mathcal{W}_x(s)\right] ds\right)\left(\int_0^1 \left[\mathcal{V}'\mathcal{W}_x(s)\right]\left[\mathcal{V}'\mathcal{W}_x(s)\right]' ds\right)^{-1}\left[\mathcal{V}'\mathcal{W}_x(\cdot)\right],$$

where

$$\mathcal{V}'\mathcal{W}_x(\cdot) = \begin{bmatrix} \mathcal{V}_1'\mathcal{W}_x(\cdot) \\ \mathcal{V}_2'\mathcal{W}_x(\cdot) \end{bmatrix}$$

and $\mathcal{V}_1 \in \mathbb{R}^3$ and $\mathcal{V}_2 \in \mathbb{R}^3$ are orthonormal column vectors. This representation shows that $\tilde{\phi}_i(\cdot)$ is a function of the orthogonal projection of $\mathcal{W}_x(\cdot)$ onto the proper subspace spanned by $\mathcal{V}_1$ and $\mathcal{V}_2$. The orientation of this subspace depends on $\mathcal{V}$. This is important because the distribution of $\mathcal{W}_x(\cdot)$ is not rotation-invariant, so the distribution of $\mathcal{V}'\mathcal{W}_x(\cdot)$ depends on $\mathcal{V}$, which in turn depends on Γ and Ω_{xx}. Since $\mathcal{V}$ is not a square matrix, there is no way for its effects on the second term in the definition of $\tilde{\phi}_i(\cdot)$ to cancel out. Therefore, the distribution of $\tilde{\phi}_i(\cdot)$ depends on Ω_{xx}, and as a result, $\tilde{\Omega}_{00}/\Omega_{00}$ is not asymptotically pivotal.

To enable asymptotically pivotal fixed-b asymptotic inference in polynomial cointegrated regressions, Vogelsang and Wagner (2024) impose the so-called full-design condition, which requires the inclusion of all monomials of $\{X_{jt}\}$ in Z_t. In the example above, this requires $Z_t = (X_{1t}^2, X_{2t}^2, X_{1t}X_{2t})$, so that both quadratic terms and the cross-product are included in the regression, and Γ, defined in (7), becomes a square matrix. The full-design condition guarantees a one-to-one mapping between $Z(\cdot)$ and the vector process $\mathcal{W}_x$ involving all monomials of $\{W_{x,j}(\cdot)\}$; see (7). This ensures that the effects of Γ (or $\mathcal{V}$) cancel out and that $\tilde{\phi}_i(\cdot)$ does not depend on nuisance parameters. However, the full-design condition can be restrictive when a more parsimonious specification is desired, or when some monomials of $\{X_{jt}\}$ have no effect on Y_t. Our aim is to develop an asymptotically pivotal inference method that does not impose the restrictive full-design condition while achieving standard F and t limits under fixed-K asymptotics.

The idea is to transform any candidate basis functions in an initial step in order to "preempt" the estimation error so that the long-run variance estimator is invariant to the use of the estimated $\hat{u}_0$ or the true u_0. This, combined with an orthonormalization step, yields a new set of basis vectors that can be used to construct a new LRV estimator $\hat{\Omega}_{00}$, ensuring the asymptotic pivotality $\hat{\Omega}_{00}/\Omega_{00}$.

We now describe the details. For a given set of basis functions $\{\phi_i(\cdot)\}_{i=1}^K$ and the corresponding basis vectors $\{\boldsymbol{\phi}_i\}_{i=1}^K$, we let $\Phi = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_K) \in \mathbb{R}^{T\times K}$ be the matrix that concatenates $\{\boldsymbol{\phi}_i\}_{i=1}^K$. Projecting each column of Φ onto the orthogonal complement of the space spanned by the columns of Z yields the new matrix $\tilde{\Phi} = Q_Z\Phi := (\tilde{\boldsymbol{\phi}}_1, \ldots, \tilde{\boldsymbol{\phi}}_K)$. Normalizing $\tilde{\Phi}$ yields

$$\tilde{\Phi}^* = \tilde{\Phi}(\tilde{\Phi}'\tilde{\Phi})^{-1/2} = (\tilde{\boldsymbol{\phi}}_1, \ldots, \tilde{\boldsymbol{\phi}}_K)(\tilde{\Phi}'\tilde{\Phi})^{-1/2}.$$

This step may also be referred to as an orthonormalization step, as $\tilde{\Phi}^{*\prime}\tilde{\Phi}^* = I_K$ and so $\tilde{\Phi}^*$ is an orthogonal matrix. Let $\tilde{\boldsymbol{\phi}}_i^*$ be the i-th column of $\tilde{\Phi}^*$; then we can write $\tilde{\Phi}^* = (\tilde{\boldsymbol{\phi}}_1^*, \ldots, \tilde{\boldsymbol{\phi}}_K^*) = Q_Z\Phi\left(\Phi' Q_Z \Phi\right)^{-1/2}$. We then replace the original basis vectors $\{\boldsymbol{\phi}_i\}_{i=1}^K$ used in $\tilde{\Omega}_{00}$ with the transformed basis vectors $\{\tilde{\boldsymbol{\phi}}_i^*\}_{i=1}^K$. This yields the following estimator for Ω_{00} :

$$\hat{\Omega}_{00} = \frac{1}{K}\sum_{i=1}^K \left[(\tilde{\boldsymbol{\phi}}_i^*)'\hat{u}_0\right]^2. \tag{8}$$

In a matrix form, $\hat{\Omega}_{00}$ can be equivalently expressed as

$$\hat{\Omega}_{00} = \frac{1}{K}\left(\hat{u}_0' Q_Z \Phi\right)\left(\Phi' Q_Z \Phi\right)^{-1}\left(\Phi' Q_Z \hat{u}_0\right).$$

In essence, we have transformed the original basis matrix Φ into the new basis matrix $Q_Z\Phi\left(\Phi' Q_Z \Phi\right)^{-1/2}$ and used the transformed basis vectors in constructing the long-run variance estimator.

Based on $\hat{\Omega}_{00}$, we construct the Wald statistic:

$$F_T = \frac{1}{p\hat{\Omega}_{00}}\left(R\hat{\beta}_{\text{OLS}} - r\right)'\left[R\left(Z'Z\right)^{-1}R'\right]^{-1}\left(R\hat{\beta}_{\text{OLS}} - r\right). \tag{9}$$

When $p = 1$, we construct the t-statistic:

$$t_T = \frac{R\hat{\beta}_{\text{OLS}} - r}{\sqrt{\hat{\Omega}_{00} R\left(Z'Z\right)^{-1}R'}}.$$

To derive the asymptotic distributions of F_T and t_T, we impose the following assumptions on the stochastic process $Z(\cdot)$ in (3), the basis functions $\{\phi_i(\cdot)\}_{i=1}^K$, and the corresponding transformed (random) basis functions $\{\tilde{\phi}_i(\cdot)\}_{i=1}^K$.

Assumption 3 *The stochastic process $Z(\cdot)$ satisfies the condition that $\int_0^1 Z(\tau) Z(\tau)' d\tau$ is positive definite almost surely.*

Assumption 4 *(i) For each $i \in \{1, \ldots, K\}$, $\phi_i(\cdot)$ is continuously differentiable. (ii) For $\mathcal{S}_{\tilde{\Phi}} \in \mathbb{R}^{K \times K}$ with the (i,j)-th element given by $\mathcal{S}_{\tilde{\Phi}}(i,j) := \int_0^1 \tilde{\phi}_i(\tau) \tilde{\phi}_j(\tau) d\tau$, $\mathcal{S}_{\tilde{\Phi}}$ is of the full rank K almost surely.*

Theorem 1 *Let Assumptions 1–4 hold. Assume further that $\lim_{T\to\infty} \tilde{D}_T R D_T^{-1}$ is of full row rank p. Then, for a fixed K as $T \to \infty$, we have that*

$$F_T \Rightarrow F_{p,K} \text{ and } t_T \Rightarrow t_K \text{ under } H_0 : R\beta_0 = r,$$

where $F_{p,K}$ is the standard F distribution with degrees of freedom (p, K), and t_K is the standard t distribution with degrees of freedom K.

Remark 2 *Define $[\tilde{\phi}_1^*(\cdot), \ldots, \tilde{\phi}_K^*(\cdot)] = [\tilde{\phi}_1(\cdot), \ldots, \tilde{\phi}_K(\cdot)]\mathcal{S}_{\tilde{\Phi}}^{-1/2}$. Then, $\hat{\Omega}_{00}$ is a series LRV estimator using the new set of basis functions $\{\tilde{\phi}_i^*(\cdot)\}_{i=1}^K$. As in Sun (2023), each $\tilde{\phi}_i^*(\cdot)$ is a random function, as it depends on the trajectory of $Z(\cdot)$, which is a random element. Note that $\hat{\Omega}_{00} = (\hat{u}_0'\Phi)(\Phi' Q_Z \Phi)^{-1}(\Phi'\hat{u}_0)/K$ and $\tilde{\Omega}_{00} = (\hat{u}_0'\Phi)(\Phi'\hat{u}_0)/K$ have similar forms and can both be written as $(\hat{u}_0'\Phi)\, G\, (\Phi'\hat{u}_0)/K$ for a matrix G, which is equal to either $(\Phi' Q_Z \Phi)^{-1}$ or I_K. Therefore, we may also interpret $\hat{\Omega}_{00}$ as a weighted version of $\tilde{\Omega}_{00}$ using $(\Phi' Q_Z \Phi)^{-1}$ as the weighting matrix. The series method to LRV estimation offers flexibility in crafting the basis functions or reweighting them when necessary. The kernel-based method does not have such flexibility, and the full-design condition has to be imposed in order to obtain an asymptotically pivotal limit under the fixed-b asymptotics; see Vogelsang and Wagner (2024) for more details.*

3 Cointegrated Homogeneous Regression under Endogeneity

We consider the same model as before:

$$\begin{aligned} Y_t &= \alpha_0 + Z_t'\beta_0 + u_{0,t}, \\ X_t &= X_{t-1} + u_{x,t}. \end{aligned} \tag{10}$$

In the above, we explicitly include an intercept in the model because X_t is now allowed to be endogenous, and thus it cannot accommodate a deterministic constant regressor. We allow $\{u_{0,t}\}$ and $\{u_{x,t}\}$ to be arbitrarily correlated, in which case their long-run covariance Ω_{0x} can be a non-zero vector. With a slight abuse of notation, we now let Z_t denote the vector of nonconstant regressors, and let d_z be the number of elements in Z_t, that is, the number of non-constant regressors in (10).

We still maintain Assumption 2(ii) but now

$$\Omega^{1/2} = \begin{bmatrix} \Omega_{00\cdot x}^{1/2} & \Omega_{0x}\Omega_{xx}^{-1/2} \\ O & \Omega_{xx}^{1/2} \end{bmatrix},$$

where $\Omega_{00\cdot x} := \Omega_{00} - \Omega_{0x}\Omega_{xx}^{-1}\Omega_{x0}$, and $B(\cdot)$ can be represented by

$$B(\cdot) = \Omega^{1/2}W(\cdot) = \begin{pmatrix} \Omega_{00\cdot x}^{1/2}W_0(\cdot) + \Omega_{0x}\Omega_{xx}^{-1/2}W_x(\cdot) \\ \Omega_{xx}^{1/2}W_x(\cdot) \end{pmatrix}$$

with $W_0(\cdot)$ being independent of $W_x(\cdot)$. As in the previous section, we assume that $D_T^{-1}Z_{[T\tau]} \Rightarrow Z(\tau)$, where $Z(\tau)$ satisfies Assumption 3.

Given that Ω_{0x} may not be zero, there may be a long-run correlation between $\{u_{0,t}\}$ and $\{u_{x,t}\}$. To remove this potential correlation, we perform a long-run projection. Define the long-run projection coefficient $\gamma_0 = \Omega_{xx}^{-1}\Omega_{x0}$ and let

$$u_{0\cdot x,t} = u_{0,t} - u_{x,t}'\gamma_0.$$

We can then represent Y_t as

$$Y_t = \alpha_0 + Z_t'\beta_0 + \Delta X_t'\gamma_0 + u_{0\cdot x,t}, \tag{11}$$

where the long-run variance of $u_{0\cdot x,t}$ is $\Omega_{00\cdot x}$, and the long-run covariance between $\{u_{0\cdot x,t}\}$ and $\{u_{x,t}\}$ is 0.

It is well known in the time series econometrics literature that the OLS estimator based on (11) still exhibits a second-order endogeneity bias, which hinders asymptotically pivotal inference. To remove the bias, Phillips and Hansen (1990) introduce the fully modified method for linear cointegration regressions with I(1) regressors. When the regressor is a quadratic transform of an I(1) process, Liang et al. (2016) provide an explicit expression for the second-order bias and use a fully modified approach to conduct inference. Here we take a different approach and use an estimator that is free of second-order bias.

As in Hwang and Sun (2018), Sun (2023), and Phillips and Kheifets (2024), our estimation method begins by using some basis functions to transform (11). Let $\{\phi_i(\cdot)\}_{i=1}^{\infty}$ be a set of complete basis functions in $L^2[0,1]$. For each $i \in \{1,\dots,K\}$, we define

$$\begin{aligned} V_{\alpha,i} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_i\left(\frac{t}{T}\right), \\ V_{Y,i} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}Y_t\phi_i\left(\frac{t}{T}\right),\ V_{Z,i} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}Z_t\phi_i\left(\frac{t}{T}\right), \\ V_{\Delta x,i} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\Delta X_t\phi_i\left(\frac{t}{T}\right),\ V_{0\cdot x,i} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{0\cdot x,t}\phi_i\left(\frac{t}{T}\right). \end{aligned} \tag{12}$$

We then obtain

$$V_{Y,i} = V_{\alpha,i}\alpha_0 + V'_{Z,i}\beta_0 + V'_{\Delta x,i}\gamma_0 + V_{0\cdot x,i} \tag{13}$$

for $i \in \{1, \ldots, K\}$. This can be viewed as a cross-sectional regression with K observations. We assume that K is fixed but sufficiently large so that the number of observations is larger than the number of regressors ($K > d_z + d_x$).

We make the following assumptions on the basis functions, which are identical to the corresponding assumptions in Hwang and Sun (2018).

Assumption 5 *(i) For every $i \in \{1, \ldots, K\}$, $\phi_i(\cdot)$ is continuously differentiable. (ii) for every $i \in \{1, \ldots, K\}$, $\phi_i(\cdot)$ satisfies $\int_0^1 \phi_i(\tau)\, d\tau = 0$. (iii) the functions $\{\phi_i(\cdot)\}_{i=1}^K$ are orthonormal in $L^2[0,1]$.*

Under Assumption 5(i, ii),

$$V_{\alpha,i} = \sqrt{T}\int_0^1 \phi_i(\tau)d\tau + \sqrt{T}O(1/T) = O(1/\sqrt{T}) = o(1).$$

Thus, for $i \in \{1, \ldots, K\}$,

$$V_{Y,i} = V'_{Z,i}\beta_0 + V'_{\Delta x,i}\gamma_0 + V^{\alpha}_{0\cdot x,i}, \tag{14}$$

where

$$V^{\alpha}_{0\cdot x,i} = V_{0\cdot x,i} + V_{\alpha,i}\alpha_0 = V_{0\cdot x,i} + o(1).$$

Our estimation and inference will be based on equation (14), treating $V^{\alpha}_{0\cdot x,i}$ as the regression error. Following Hwang and Sun (2018), we refer to (14) as the Transformed and Augmented (TA) regression, and the associated OLS estimator as the Transformed and Augmented OLS (TA-OLS) estimator, denoted by $(\hat{\beta}_{\text{TAOLS}}, \hat{\gamma}_{\text{TAOLS}})$.

Define $V_Y = (V_{Y,1}, \ldots, V_{Y,K})' \in \mathbb{R}^{K\times 1}$, $V_Z = (V_{Z,1}, \ldots, V_{Z,K})' \in \mathbb{R}^{K\times d_z}$, and similarly define $V_{\Delta x} \in \mathbb{R}^{K\times d_x}$ and $V^{\alpha}_{0\cdot x} \in \mathbb{R}^{K\times 1}$. Then, we have

$$V_Y = V_Z\beta_0 + V_{\Delta x}\gamma_0 + V^{\alpha}_{0\cdot x}.$$

Clearly, the TA-OLS estimator $\hat{\beta}_{\text{TAOLS}}$ of β_0 satisfies

$$\hat{\beta}_{\text{TAOLS}} - \beta_0 = \left(V'_Z Q_{V_{\Delta x}} V_Z\right)^{-1} V'_Z Q_{V_{\Delta x}} V^{\alpha}_{0\cdot x}.$$

Define

$$\xi_i \equiv \int_0^1 \phi_i(\tau)\, Z(\tau)d\tau,\ \eta_i \equiv \int_0^1 \phi_i(\tau)\, dB_x(\tau),\ \nu_i \equiv \int_0^1 \phi_i(\tau)\, dW_0(\tau),$$

and denote $\xi \equiv (\xi_1, \xi_2, \ldots, \xi_K)' \in \mathbb{R}^{K\times d_z}$, with $\eta \in \mathbb{R}^{K\times d_x}$ and $\nu \in \mathbb{R}^{K\times 1}$ defined similarly.

Under Assumptions 1, 2(i, ii), and 5(i, ii), we can use summation by parts, the continuous mapping theorem, and integration by parts to obtain

$$T^{-1/2}D_T^{-1}V_Z \Rightarrow \xi,\ V_{\Delta x} \Rightarrow \eta, \text{ and } V^{\alpha}_{0\cdot x} \Rightarrow \nu$$

holds jointly, and this implies that

$$\sqrt{T}D_T(\hat{\beta}_{\text{TAOLS}} - \beta_0) \Rightarrow \Omega_{00\cdot x}^{1/2} \left(\xi' Q_\eta \xi\right)^{-1} \xi' Q_\eta \nu. \tag{15}$$

It is important to note that ν, which is a functional of $W_0(\cdot)$, is independent of ξ and η, which are functionals of $W_x(\cdot)$. Additionally, the full-rank condition imposed on Ω_{xx} and Assumption 3, together with $K > d_z + d_x$, ensure that $\xi' Q_\eta \xi$ is positive definite almost surely. As a result, conditional on (ξ, η), the limiting distribution in (15) is normal with mean zero, and the limiting distribution is mixed normal. There is no second-order endogeneity bias in the TA-OLS estimator.

To make inferences on $R\beta_0$, we estimate $\Omega_{00\cdot x}$ by

$$\hat{\Omega}_{00\cdot x} = \frac{1}{K} \left\| V_Y - V_Z' \hat{\beta}_{\text{TAOLS}} - V_{\Delta x}' \hat{\gamma}_{\text{TAOLS}} \right\|^2$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^K$. Based on the above estimator, we construct the Wald statistic:

$$F(\hat{\beta}_{\text{TAOLS}}) = \frac{1}{p\hat{\Omega}_{00\cdot x}} \left[R\hat{\beta}_{\text{TAOLS}} - r\right]' \left[R\left(V_Z' Q_{V_{\Delta x}} V_Z\right)^{-1} R'\right]^{-1} \left[R\hat{\beta}_{\text{TAOLS}} - r\right], \tag{16}$$

and the t-statistic when $p = 1$:

$$t(\hat{\beta}_{\text{TAOLS}}) = \frac{R\hat{\beta}_{\text{TAOLS}} - r}{\sqrt{\hat{\Omega}_{00\cdot x} R\left(V_Z' Q_{V_{\Delta x}} V_Z\right)^{-1} R'}}. \tag{17}$$

As in the previous section, we assume that there exists a diagonal matrix $\tilde{D}_T \in \mathbb{R}^{p\times p}$ such that $\lim_{T\to\infty} \tilde{D}_T R D_T^{-1} = A$ for a matrix $A \in \mathbb{R}^{p\times d_z}$ with full row rank p.

Theorem 2 *Let Assumptions 1, 2(i,ii), 3, and 5 hold. Assume further that* $\lim_{T\to\infty} \tilde{D}_T R D_T^{-1} = A$ *is of full row rank* p. *Then under the fixed-*K *asymptotics where* K *is held fixed as* $T \to \infty$, *we have the following:*

(i) $\sqrt{T}D_T(\hat{\beta}_{\text{TAOLS}} - \beta_0) \Rightarrow \Omega_{00\cdot x}^{1/2} (\xi' Q_\eta \xi)^{-1} \xi' Q_\eta \nu$, *where* ν *is independent of* (ξ, η), *and* $\nu \sim N(0, I_K)$.

(ii) Under the null hypothesis of $H_0 : R\beta_0 = r$, *we have that*

$$F^*(\hat{\beta}_{\text{TAOLS}}) := \frac{K - d_z - d_x}{K} F(\hat{\beta}_{\text{TAOLS}}) \Rightarrow F_{p,K-d_z-d_x};$$
$$t^*(\hat{\beta}_{\text{TAOLS}}) := \sqrt{\frac{K - d_z - d_x}{K}} t(\hat{\beta}_{\text{TAOLS}}) \Rightarrow t_{K-d_z-d_x} \text{ for } p = 1.$$

4 Monte Carlo Simulations

We consider the following data generation process (DGP):

$$Y_t = Z_t'\beta_0 + u_{0,t} \text{ for } t \in \{1, \ldots, T\}$$

with

$$\text{(i): } Z_t = (1, X_{1t}, X_{2t}, X_{1t}^2, X_{2t}^2, X_{1t}X_{2t})'; \tag{18}$$

$$\text{(ii): } Z_t = (1, X_{1t}, X_{2t}, X_{1t}^2, X_{2t}^2)', \tag{19}$$

where $X_t = (X_{1t}, X_{2t})'$ satisfies $X_t = X_{t-1} + u_{x,t}$ and $X_0 = 0$. The I(0) component of the cointegration model, i.e., $u_t = \left(u_{0,t}, u_{x,t}'\right)'$, is generated as follows:

$$u_t = \Theta u_{t-1} + \epsilon_t, \tag{20}$$

where

$$\epsilon_t = \begin{pmatrix} \epsilon_{0,t} \\ \epsilon_{x,t} \end{pmatrix} \sim \text{i.i.d } N\left(0, \Sigma\right), \ \Theta = \rho \cdot I_{d_x+1},\ \Sigma = J_{d_x+1,d_x+1} \cdot \varphi + I_{d_x+1} \cdot (1-\varphi),$$

$d_x = 2$, and $J_{p,q}$ is the $p \times q$ matrix of ones. The parameter ρ controls the persistence of individual components in $u_t \in \mathbb{R}^3$ while the parameter φ, which is equal to the pairwise correlation coefficient between the elements of u_t for the above model, characterizes the degree of endogeneity. We set the values of $\rho \in \{0.05, 0.25, 0.50, 0.75, 0.90\}$.

For the true coefficients β_0 for (i) and (ii), we set them as $(0, 10, 10, 0, 0, 0, 0)'$ and $(0, 10, 10, 0, 0)'$, respectively, so the (unknown) true cointegrating relationship is assumed to be linear. Without loss of generality, we set the intercept parameter to be zero. Considering quadratic and interactive specifications for (i) and (ii) in (18) and (19), we test whether the elements of β_0 associated with the nonlinear regressors are jointly zero. The corresponding null hypotheses are formulated as $H_0 : R\beta_0 = r$, where

$$\text{(i): } R = (O_{3\times 3},\ I_{3\times 3}) \in \mathbb{R}^{3\times 6} \text{ with } r = (0,0,0)'; \tag{21}$$

$$\text{(ii): } R = (O_{2\times 3},\ I_{2\times 2}) \in \mathbb{R}^{2\times 5} \text{ with } r = (0,0)'. \tag{22}$$

In the following two subsections, we examine cointegrated regressions with exogenous and endogenous regressors and evaluate the finite-sample performance of the procedures developed in Sections 2 and 3. We also compare the finite-sample performance of our methods with several existing methods in the literature, using a nominal significance level of 5%.

4.1 Cointegrated homogeneous regression with exogenous regressors

This subsection considers the case where the parameter φ equals 0, indicating that there is no endogeneity in the cointegrated homogeneous regression. Based on the OLS estimator $\hat{\beta}_{\text{OLS}}$, the first group of tests, referred to as "OLS-HAC", employs the following Wald statistic:

$$F_{\text{HAC},T} = \frac{1}{p\hat{\Omega}_{\text{HAC},00}} \left(R\hat{\beta}_{\text{OLS}} - r\right)' \left[R\left(Z'Z\right)^{-1} R'\right]^{-1} \left(R\hat{\beta}_{\text{OLS}} - r\right),$$

where $\hat{\Omega}_{\text{HAC},00}$ is the standard kernel estimator for the LRV $\Omega_{00} := \sum_{j=-\infty}^{\infty} E[u_{0,t}u_{0,t-j}]$, using either the Bartlett or Quadratic Spectral (QS) kernel. The subscript 'HAC' on

$\hat{\Omega}_{\mathrm{HAC},00}$ signifies that it is a Heteroskedasticity and Autocorrelation Consistent (HAC) LRV estimator, which converges to the true LRV Ω_{00} under the conventional increasing-smoothing asymptotics. The critical values for the "OLS-HAC" test are from the chi-squared distributions χ_p^2/p with $p = 3$ and $p = 2$ for (i) and (ii), respectively.

The second test, referred to as "OLS-HAR", is the asymptotic F test proposed in Section 2 for an exogenous cointegrated homogenous regression. The method replaces $\hat{\Omega}_{\mathrm{HAC},00}$ by $\hat{\Omega}_{00}$, as given in (8), which is constructed based on the transformed Fourier basis functions. The Fourier basis functions are given by $\{\sqrt{2}\sin(2\pi jr), \sqrt{2}\cos(2\pi jr)\}_{j=1}^{K/2}$, assuming K is even. In the literature, an LRV estimator like $\hat{\Omega}_{00}$ is often referred to as a Heteroskedasticity and Autocorrelation Robust (HAR) LRV estimator when it converges to a random variable in distribution, as is the case under the fixed-smoothing asymptotics. For notational symmetry, we will rewrite $\hat{\Omega}_{00}$ as $\hat{\Omega}_{\mathrm{HAR},00}$ from now on.

For both OLS-HAC and OLS-HAR, smoothing parameters are required. The smoothing parameter for $\hat{\Omega}_{\mathrm{HAC},00}$ is the kernel bandwidth, while the smoothing parameter for $\hat{\Omega}_{\mathrm{HAR},00}$ is the number of (transformed) basis functions. We use data-driven smoothing parameters for $\hat{\Omega}_{\mathrm{HAC},00}$ and $\hat{\Omega}_{\mathrm{HAR},00}$ that minimize the asymptotic mean squared errors (AMSE), as developed by Andrews (1991) and Phillips (2005), respectively. In both cases, we apply a parametric plug-in approach using AR(1) to compute the unknown parameters. See Section 4 of Sun (2023) for the formulas and implementation details.

Table 1 reports the empirical rejection probabilities of OLS-HAC and OLS-HAR under the null hypotheses in (21) and (22), using data-dependent smoothing parameter choices. To conserve space, for OLS-HAC, we report only the results using the QS kernel. In all our Monte Carlo simulations, we consider two sample sizes, $T \in \{100, 200\}$, with $10,000$ simulation replications. The results demonstrate that the empirical size distortion of OLS-HAC with chi-square critical values can be substantially higher than the nominal size level, namely 0.05. The over-rejection of OLS-HAC is due to neglecting the estimation error in the LRV estimator $\hat{\Omega}_{\mathrm{HAC},00}$. In contrast, except for cases where the degree of temporal dependence is high (i.e., when $\rho > 0.75$), our proposed OLS-HAR test, which uses standard F critical values, has significantly lower size distortion, with the null rejection probabilities much closer to the nominal level of 0.05. The results in Table 1 are also consistent with the numerical findings in Sun (2023), which demonstrate the accuracy of the asymptotic F test in a different setting. Our results confirm that the asymptotic F test in linear cointegration regression can be successfully extended to cases where regressors are polynomial functions of I(1) variables. We also note that our asymptotic F test exhibits appealing finite-sample performance for testing both (21) and (22), as it allows for flexible polynomial functions without requiring a full-design condition.

4.2 Cointegrated homogeneous regression with endogenous regressors

This subsection considers the case of an endogenous cointegration regression with $\varphi = 0.75$. We compare the asymptotic F test based on the TA-OLS estimator, which we propose in Section 3, against existing approaches, including the integrated modified OLS (IM-OLS) and fully-modified OLS (FM-OLS) methods. We first describe the FM-OLS of Phillips and

Hansen (1990), which has been extended to cointegrated polynomial regression in Wagner and Hong (2016). Define the one-sided long-run variance:

$$\Delta = \sum_{j=0}^{\infty} E[u_{t-j}u_t'] = \begin{pmatrix} \Delta_{00} & \Delta_{0x} \\ \Delta_{x0} & \Delta_{xx} \end{pmatrix}.$$

The estimator for Δ takes the following form:

$$\hat{\Delta} := \begin{pmatrix} \hat{\Delta}_{00} & \hat{\Delta}_{0x} \\ \hat{\Delta}_{x0} & \hat{\Delta}_{xx} \end{pmatrix} = \frac{1}{T}\sum_{s=1}^{T}\sum_{t=s}^{T} k\left(\frac{|s-t|}{B_T}\right)\hat{u}_s\hat{u}_t', \tag{23}$$

where $\hat{u}_t = \left(\hat{u}_{0,t}, u_{x,t}'\right)'$ and $\hat{u}_{0,t} = Y_t - Z_t'\hat{\beta}_{\text{OLS}}$. In the above, $k(\cdot)$ is a kernel function, such as the Bartlett and QS kernels, and B_T is the bandwidth parameter. We partition $\hat{\Delta}_{x0}$ and $\hat{\Delta}_{xx}$ as $\hat{\Delta}_{x0} = (\hat{\Delta}_{x_1,0}, \hat{\Delta}_{x_2,0})'$ and $\hat{\Delta}_{xx} = [(\hat{\Delta}_{x,x_1}, \hat{\Delta}_{x,x_2})'$. Also, we define $\hat{\Delta}_{x0}^{+} = (\hat{\Delta}_{x_1,0}^{+}, \hat{\Delta}_{x_2,0}^{+})'$ as

$$\hat{\Delta}_{x0}^{+} = \hat{\Delta}_{x0} - \hat{\Delta}_{xx}\hat{\Omega}_{xx}^{-1}\hat{\Omega}_{x0} = \begin{pmatrix} \hat{\Delta}_{x_1,0} - \hat{\Delta}_{x,x_1}'\hat{\Omega}_{xx}^{-1}\hat{\Omega}_{x0} \\ \hat{\Delta}_{x_2,0} - \hat{\Delta}_{x,x_2}'\hat{\Omega}_{xx}^{-1}\hat{\Omega}_{x0} \end{pmatrix},$$

where

$$\hat{\Omega} = \begin{pmatrix} \hat{\Omega}_{00} & \hat{\Omega}_{0x} \\ \hat{\Omega}_{x0} & \hat{\Omega}_{xx} \end{pmatrix} := \frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T} k\left(\frac{|s-t|}{B_T}\right)\hat{u}_s\hat{u}_t'$$

is the kernel LRV estimator.

In the context of our DGPs, the FM-OLS method in Wagner and Hong (2016) covers only case (ii) given in (19), where only the quadratic terms of X_{1t} and X_{2t}, but not their interaction, are included in Z_t. In this case, the FM-OLS estimator is

$$\hat{\beta}_{\text{FM-OLS}} = (Z'Z)^{-1}(Z'Y^{+} - A^{*}),$$

with $Y^{+} = (Y_1^{+}, \ldots, Y_T^{+})'$ and $Y_t^{+} = Y_t - \Delta X_t'\hat{\Omega}_{xx}^{-1}\hat{\Omega}_{x0}$. The additive correction factor A^{*} is given by

$$A^{*} = \begin{bmatrix} 0 \\ M^{*} \end{bmatrix} \in \mathbb{R}^5 \text{ and } M^{*} = \begin{bmatrix} T\hat{\Delta}_{x_1,0}^{+} \\ T\hat{\Delta}_{x_2,0}^{+} \\ \left(2\sum_{t=1}^{T} X_{1t}\right)\cdot\hat{\Delta}_{x_1,0}^{+} \\ \left(2\sum_{t=1}^{T} X_{2t}\right)\cdot\hat{\Delta}_{x_2,0}^{+} \end{bmatrix} \in \mathbb{R}^4,$$

The corresponding FM-OLS Wald statistic is defined as

$$W_{\text{FM-OLS}} := \left(R\hat{\beta}_{\text{FM-OLS}} - r\right)' \left[\hat{\sigma}_{0\cdot x}^2 R(Z'Z)^{-1}R'\right]^{-1} \left(R\hat{\beta}_{\text{FM-OLS}} - r\right),$$

where $\hat{\sigma}_{0\cdot x}^2 := \hat{\Omega}_{00} - \hat{\Omega}_{0x}\hat{\Omega}_{xx}\hat{\Omega}_{x0}$. Wagner and Hong (2016) show that $W_{\text{FM-OLS}}$ is asymptotically chi-squared distributed with p degrees of freedom. A key assumption behind this chi-squared limiting result, similar to Phillips and Hansen (1990), is that the amount of smoothing increases to infinity as the sample size grows (i.e., B_T grows with the sample size but at a slower rate). Consequently, $\hat{\Omega}$, $\hat{\Delta}$, and $\hat{\sigma}^2_{0\cdot x}$ converge to their respective true values Ω, Δ, and $\sigma^2_{0\cdot x}$, regardless of the choice of the kernel function or bandwidth B_T used in the nonparametric estimators.

To describe the IM-OLS method in Vogelsang and Wagner (2024), we let $S^O_t := \sum_{j=1}^t O_j$, which represents the integration O_j up to period t, and $S^O := [S^O_1, \ldots, S^O_T]'$. Also, denote $\tilde{S}^Z_t := [(S^Z_t)', X'_t]'$ and $\tilde{S}^Z = [\tilde{S}^Z_1, \ldots, \tilde{S}^Z_T]' = [S^Z, X]$. The IM-OLS estimator of $\theta_0 = (\beta'_0, \gamma'_0)'$ is then defined as

$$\hat{\theta}_{\text{IM-OLS}} = [(\tilde{S}^Z)'\tilde{S}^Z]^{-1}(\tilde{S}^Z)'S^Y.$$

Let $\tilde{R} = [R, O] \in \mathbb{R}^{p\times(d_z+d_x)}$. The IM-OLS Wald statistic is then formulated as

$$W_{\text{IM-OLS}} := \left(\tilde{R}\hat{\theta}_{\text{IM-OLS}} - r\right)' \left[\tilde{R}\hat{V}_{\text{IM,M}}\tilde{R}'\right]^{-1} \left(\tilde{R}\hat{\theta}_{\text{IM-OLS}} - r\right),$$

where

$$\hat{V}_{\text{IM,M}} = \hat{\sigma}^2_{0\cdot x,\text{M}} \cdot [(\tilde{S}^Z)'\tilde{S}^Z]^{-1}C'C[(\tilde{S}^Z)'\tilde{S}^Z]^{-1}$$

with $C := [c_1, \ldots, c_T]'$, $c_t := S^{\tilde{S}^Z}_T - S^{\tilde{S}^Z}_{t-1}$, $S^{\tilde{S}^Z}_t := \sum_{j=1}^t \tilde{S}^Z_j$ for $t \in \{1, \ldots, T\}$ and $S^{\tilde{S}^Z}_0 = 0$. The estimator $\hat{\sigma}^2_{0\cdot x,\text{M}}$ is constructed as

$$\hat{\sigma}^2_{0\cdot x,\text{M}} = \frac{1}{T}\sum_{\tau=2}^T\sum_{t=2}^T k\left(\frac{|\tau - t|}{B_T}\right)\Delta\hat{S}^u_{\tau,\text{M}}\Delta\hat{S}^u_{t,\text{M}},$$

where $B_T = bT$ for some $b \in (0, 1]$, $\Delta\hat{S}^u_{t,\text{M}} = \hat{S}^u_{t,\text{M}} - \hat{S}^u_{t-1,\text{M}}$, and $\hat{S}^u_{t,\text{M}}$ is a modification of the IM-OLS residual $\hat{S}^u_t := S^Y_t - \hat{\theta}'_{\text{IM-OLS}}\tilde{S}^Z_t$. The modification is required to obtain the asymptotically pivotal fixed-b limiting distribution of $W_{\text{IM-OLS}}$. See Vogelsang and Wagner (2024) for the detailed formulation of the modified residual $\hat{S}^u_{t,\text{M}}$. The non-standard fixed-b critical values are simulated by extending the algorithm in Hwang and Vogelsang (2024) to the cointegrated polynomial regression setting. It is noteworthy that the fixed-b inference using IM-OLS imposes a full-design condition to establish the asymptotically pivotal fixed-b limit of $W_{\text{IM-OLS}}$. Thus, the application of $W_{\text{IM-OLS}}$ rules out case (ii) in (19).

In both FM-OLS and IM-OLS, we use the data-driven values of B_T, adopting the formulas from Andrews (1991). Similar to the exogenous case, the unknown parameters in the data-driven formulas are estimated using the plug-in method, with VAR(1) serving as the approximating model for $\{\hat{u}_t = (\hat{u}_{0t}, \Delta x'_t)'\}$. For the TA-OLS method, we apply the same approximating model for $\{\hat{u}_t\}$ and implement the data-driven smoothing parameter K.

Table 2 reports the empirical rejection probabilities for IM-OLS, FM-OLS, and TA-OLS under the null. It is clear that for all values of ρ and sample sizes $T \in \{100, 200\}$, our proposed F test based on TA-OLS outperforms the tests based on FM-OLS and IM-OLS, particularly when ρ is greater than 0.50. These findings are consistent with previous studies, such as Pellatt and Sun (2023) and Hwang and Vogelsang (2024), which support the accuracy of the F test based on the TA-OLS estimator in linear cointegration regressions.

In fact, the results in Table 2 indicate that combining a data-driven smoothing parameter with F critical values enhances the size accuracy of the TA-OLS method for cointegrated homogeneous regressions. Practically, our TA-OLS framework with endogenous I(1) variables does not require assumptions about how to specify the regressor vector Z_t, which contrasts with the existing FM-OLS and IM-OLS methods.

5 Empirical Application to Carbon Kuznets Curve

In this section, we apply the TA-OLS method developed in 3 to real data and compare it with existing methods for a quadratic cointegration system under endogeneity.

We examine the long-run relationship between each country's per capita CO_2 emissions and per capita GDP. The carbon Kuznets curve (CKC) hypothesis suggests an inverted U-shaped relationship between these two variables, observed across time for each country (e.g., Holtz-Eakin and Selden (1995)). To test this hypothesis, we consider the model:

$$\log(e_t) = \alpha_0 + \beta_{0,1} \log(x_t) + \beta_{0,2}(\log(x_t))^2 + u_{0,t} \tag{24}$$

for $t \in \{1, \ldots, T\}$, where e_t and x_t denote the per capita CO_2 emissions and the per capita GDP in period t, respectively, and $u_{0,t}$ is an I(0) error term, which is potentially correlated with the (log) per capita GDP. The specification in (24) follows the same quadratic formulation as in Chan and Wang (2015), which uses a least squares (LS) method. While the LS approach in Chan and Wang (2015) becomes the nonlinear least squares when the cointegrated system is nonlinear in parameters, we note that their LS approach applied to (24) is equivalent to the OLS, as the model is linear in parameters, with $\log(x_t)$ and $(\log(x_t))^2$ as the regressors. Chan and Wang (2015) simulate the non-standard limiting distribution of the OLS estimator based on consistent estimators of the LRV and half-LRV of the I(0) components of the cointegrated system.

Since the logarithm of per capita GDP is widely known to exhibit nonstationary behavior over time, the issue addressed in this paper becomes relevant when applying a quadratic transformation to the logarithm of per capita GDP. Wagner (2015) and Chan and Wang (2015) provide detailed statistical evidence on the nonstationary I(1) properties of $\log(e_t)$ and $\log(x_t)$ in (24). These papers also investigate the CKC hypothesis, using time-domain methods such as FM-OLS and LS. Our TA-OLS method, combined with Fourier basis functions for a fixed-K specification, estimates the cointegrating vector by focusing on only K low-frequency components of the underlying time series. In this sense, our approach closely aligns with the empirical questions posited by the CKC hypothesis regarding the nonlinear long-run relationship between the two I(1) variables.

Following Chan and Wang (2015), we analyze 13 early industrialized countries over the post World War II period from 1951 to 2008. The CO_2 emission data are sourced from the Carbon Dioxide Information Analysis Center (Boden et al. (2010))[1]. For per capita GDP, we use data from Maddison (2003), adjusted to 1990 Geary-Khamis dollars.[2] The list of

[1] https://www.osti.gov/biblio/1389324

[2] The data are available from Angus Maddison's archived website:
https://www.rug.nl/ggdc/historicaldevelopment/maddison/

country names and their codes is summarized in Table 3. Figure 1 plots the logarithm of per capita CO_2 against the logarithm of per capita GDP for selected countries: France, Germany, UK, and USA.

In our application of the TA-OLS method, the choice of K is necessary. Given the limited time span of 58 years of data, we use a fixed $K \in \{8, 12\}$ instead of a data-driven approach. These choices align with recent HAR literature using low-frequency transformation techniques, which recommend selecting K values that reflect business cycle frequencies. Hwang and Sun (2018) also show favorable finite-sample performance with these K choices in linear cointegration regressions. To save space, we report only the results with $K = 12$, as the results with $K = 8$ yield similar qualitative and quantitative implications.

In addition to the TA-OLS method, we present estimation results for the parameters $(\beta_{0,1}, \beta_{0,2})'$ and their asymptotic confidence intervals using the LS from Chan and Wang (2015) and the FM-OLS from Wagner (2015) and Wagner and Hong (2016). The FM-OLS uses the QS kernel with AMSE-based data-driven bandwidth selection method in Andrews (1991).

The results are presented in Figures 2 and 3 and Tables 4–7. The figures and tables show that all three methods – TA-OLS, FM-OLS, and LS – indicate the presence of a significant nonlinear inverted U-shaped cointegration relationship between log(per capita CO_2 emissions) and log(per capita GDP) for most countries. These findings are supported by a highly significant negative coefficient for the quadratic term ($\beta_{0,2}$). Therefore, our empirical analysis confirms Chan and Wang (2015)'s finding, which provides strong evidence of inverted U-shaped patterns for the selected 13 countries in the post World War II period, from 1951 to 2008.

While the coefficient estimates from all three methods indicate statistically significant non-zero coefficients, the results in Table 4 show that in most countries, the implied turning points, calculated as $\exp(-\hat{\beta}_{0,1}/(2\hat{\beta}_{0,2}))$, based on TA-OLS estimates are lower than those obtained using the LS and FM-OLS approaches. Specifically, with the exception of Belgium, France, and Italy, the turning points calculated using TA-OLS estimates are lower than those from FM-OLS estimates, ranging from \$48 to \$2,668, and lower than those from LS estimates, ranging from \$60 to \$6,970.

Our results in Tables 5–7 and Figures 2 and 3 also show that the confidence intervals for the linear cointegration coefficient from the FM-OLS and LS methods are substantially narrower than those from the TA-OLS method for all countries. These findings can be attributed to the fact that both the LS method from Chan and Wang (2015) and the FM-OLS method assume the consistency of nonparametric estimators of the nuisance parameters, such as the LRV and half LRV. Consequently, their confidence intervals may fail to account for significant variability in parameter estimates in finite samples. In contrast, our TA-OLS procedure, along with the corresponding confidence intervals derived from fixed-K asymptotics, indicates that estimation uncertainty can be substantial in the CKC analysis when long-run endogeneity and serial dependence of an unknown form are present.

6 Extension to Fractionally Integrated Regressors

This section explores how our TA-OLS method can be extended to a more general homogeneous cointegration setting, where the regressors are driven by nonstationary, fractionally integrated processes.

Let L be the lag operator and define $\Delta = (1 - L)$. We assume that each component $X_{i,t}$ of X_t now follows a fractional process of the form:

$$\Delta^{\delta_0} X_{i,t} = u_{x,i,t}, \tag{25}$$

for $i \in \{1, \ldots, d_x\}$, where $\delta_0 \geq 1/2$, $u_{x,i,t} = 0$ for all $t \leq 0$, and for $t > 0$, $u_{x,i,t}$ is stationary with zero mean and continuous and positive spectrum $f_{u_{x,i}}(\lambda) : [-\pi, \pi] \to \mathbb{R}$. This formulation corresponds to a Type II fractional process, commonly used to model a nonstationary fractional process in econometrics (see Marinucci and Robinson (1999)). Expanding the binomial Δ^{δ_0} in (25) yields the form:

$$\sum_{j=0}^{t} a_j(\delta_0) X_{i,t-j} = u_{x,i,t},$$

where $a_j(\delta)$ is given by

$$a_j(\delta) = \frac{\Gamma(j - \delta)}{\Gamma(-\delta)\Gamma(1 + j)},$$

and $\Gamma(\cdot)$ is the gamma function: $\Gamma(m) = \int_0^\infty h^{m-1} e^{-h} dh$ such that $\Gamma(m) = \infty$ for $m = 0, -1, -2, \ldots,$ and $\Gamma(0)/\Gamma(0) = 1$. Equivalently, we define $X_{i,t}$ as

$$X_{i,t} = \Delta^{-\delta_0} u_{x,i,t} = \sum_{j=0}^{t} a_j(-\delta_0) u_{x,i,t-j}. \tag{26}$$

For expositional simplicity, we have assumed above that all elements of X_t share the same fractional parameter δ_0, but our theory can be generalized to accommodate different fractional parameters for each component of X_t.

When $\delta_0 = 1$, we have $a_j(-\delta_0) = 1$ for all j, and (26) simplifies to $X_{i,t} = \sum_{j=0}^{t} u_{x,i,t-j}$, so that $X_{i,t} = X_{i,t-1} + u_{x,i,t}$ with the initial value $X_{i,0} = 0$. In this case, $X_{i,t}$ reduces to the I(1) process considered in previous sections. Similarly, when δ_0 is equal to any other positive integer, $X_{i,t}$ follows the commonly defined I(δ_0) process with $X_{i,0} = 0$. Therefore, within the framework of Type II fractional processes, we have a unified definition of an I(δ_0) process for any $\delta_0 \in \mathbb{R}^+$, but our focus of interest is on the case where X_t is nonstationary, so we restrict $\delta_0 \geq 1/2$.

Taking an asymptotically homogenous transform of $X_t = (X_{1,t}, \ldots, X_{d_x,t})'$ yields Z_t. We consider the model:

$$Y_t = \alpha_0 + Z_t'\beta_0 + u_{0,t}, \tag{27}$$

for $t \in \{1, \ldots, T\}$, where $u_{0,t}$ is a stationary I(0) process with zero mean and a continuous, positive spectrum. Robinson and Hualde (2003) and Hualde and Iacone (2019) consider a similar model but with Z_t replaced by X_t, so no asymptotically homogenous transformation is applied. On the other hand, they allow $u_{0,t}$ to be fractionally integrated with an order lower than that of X_t. Our asymptotic F theory will not directly apply in such a DGP for $u_{0,t}$, and we leave the extension to future research.

Within the framework of Type II fractional processes, the limit distributions involve Type II fractional Brownian motion defined as follows:

$$W_x(\tau;\delta) = \frac{1}{\Gamma(\delta)} \int_0^\tau (\tau - s)^{\delta-1} dW_x(s) \text{ with } W_x(0;\delta) = 0,$$

where $W_x(\cdot)$ is the standard d_x-dimensional Brownian motion. Applying the invariance principle for fractional processes, which is established, for example, in Wu and Shao (2006), we have

$$\frac{X_{[T\tau]}}{T^{\delta_0 - 1/2}} \Rightarrow B_x(\tau;\delta_0) := \Omega_{xx}^{1/2} W_x(\tau;\delta_0), \tag{28}$$

where $B_x(\tau;\delta_0) = (B_{x,1}(\tau;\delta_0), \ldots, B_{x,d_x}(\tau;\delta_0))'$ is a d_x-dimensional fractional Brownian motion. This, together with the continuous mapping theorem, implies that for

$$\nu_{i,T}(\delta_0) := \kappa_{i,0}(T) \prod_{j=1}^{d_x} \kappa_{i,j}(T^{\delta_0 - 1/2}),$$

we have

$$\begin{aligned} \frac{1}{\nu_{i,T}(\delta_0)} Z_{i,[T\tau]} &= H_{i,0}\left(\frac{[T\tau]}{T}\right) \cdot \left(\prod_{j=1}^{d_x} H_{i,j}\left(\frac{X_{j,[T\tau]}}{T^{\delta_0-1/2}}\right)\right) + o_p(1) \\ &\Rightarrow H_{i,0}(\tau) \cdot \left(\prod_{j=1}^{d_x} H_{i,j}(B_{x,j}(\tau;\delta_0))\right) := H_i(\tau, B_x(\tau;\delta_0)), \end{aligned}$$

which holds jointly for $i \in \{1, \ldots, d_z\}$. This indicates that our previous definition of D_T can be suitably modified to $D_T = \text{diag}((\nu_{1,T}(\delta_0), \ldots, \nu_{d_z,T}(\delta_0))')$ so that

$$D_T^{-1} Z_{[T\tau]} \Rightarrow Z(\tau) \tag{29}$$

holds, where $Z(\cdot) = [H_1(\cdot, B_x(\cdot;\delta_0)), \ldots, H_{d_z}(\cdot, B_x(\cdot;\delta_0))]'$ is a full-rank vector process.

Under the high-level conditions in (28) and (29), our TA-OLS framework can be naturally extended as

$$V_{Y,i} = V_{\alpha,i}\alpha_0 + V_{Z,i}'\beta_0 + V_{\Delta^{\delta_0}x,i}'\gamma_0 + V_{0\cdot x,i}, \tag{30}$$

where the augmented term $V_{\Delta^{\delta_0}x,i}$ is defined as

$$V_{\Delta^{\delta_0}x,i} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[\Delta^{\delta_0} X_t\right] \phi_i\left(\frac{t}{T}\right)$$

for $i \in \{1, \ldots, K\}$.

Denote $(\tilde{\beta}_{\text{TAOLS}}, \tilde{\gamma}_{\text{TAOLS}})$ as the (infeasible) TA-OLS estimator from (30), assuming that δ_0 is known. Note that $V_{\Delta^{\delta_0}x}$ with $\delta_0 = 1$ is equal to $V_{\Delta x}$ in the I(1) regressor case, and $(\tilde{\beta}_{\text{TAOLS}}, \tilde{\gamma}_{\text{TAOLS}})$ is equal to the feasible estimator $(\hat{\beta}_{\text{TAOLS}}, \hat{\gamma}_{\text{TAOLS}})$ for the I(1) cointegrated system in Section 3. Invoking summation by parts, the continuous mapping theorem, and integration by parts, we can obtain

$$T^{-1/2} D_T^{-1} V_Z \Rightarrow \xi, \; V_{\Delta^{\delta_0}x} \Rightarrow \eta, \text{ and } V_{0\cdot x}^{\alpha} \Rightarrow \nu \tag{31}$$

holds jointly, where η, ν, and ξ are the same as defined in Section 3, but the stochastic process $Z(\cdot)$ in ξ is now defined using a functional of $B_x(\cdot;\delta_0)$. A careful inspection of our proof for Theorem 2 indicates that the asymptotic F and t limit results for the TA-OLS test statistics do not depend on the order of fractional integration $\delta_0 \in (1/2, \infty)$. Therefore, the TA-OLS estimation and inference framework can be extended to a more general fractionally integrated system as in (27).

For the practical implementation of the TA-OLS method, knowledge of the fractional parameter δ_0 is required to construct $V_{\Delta^{\delta_0}x}$. After obtaining a consistent estimator $\hat{\delta}$ of δ_0, we can implement the feasible TA-OLS, $(\hat{\beta}_{\text{TAOLS}}, \hat{\gamma}_{\text{TAOLS}})$, using $V_{\Delta^{\hat{\delta}}x}$ in place of $V_{\Delta^{\delta_0}x}$ with

$$V_{\Delta^{\hat{\delta}}x,i} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[\Delta^{\hat{\delta}} X_t\right] \phi_i\left(\frac{t}{T}\right).$$

To establish the asymptotic equivalence between the feasible $\hat{\beta}_{\text{TAOLS}}$ and its infeasible counterpart $\tilde{\beta}_{\text{TAOLS}}$, we impose the following conditions, which are analogous to Assumptions 3(i, ii) of Robinson and Hualde (2003).

Assumption 6 *$|\hat{\delta} - \delta_0| \leq \mathcal{C}$ for some $\mathcal{C} < \infty$, and $\hat{\delta} - \delta_0 = O_p(T^{-\psi})$ for some $\psi \in (0, 1/2)$.*

The conditions in Assumption 6 are also imposed in Robinson and Hualde (2003) and Hualde and Iacone (2019), which consider fractional cointegration with unknown integration orders. These conditions can be guaranteed if the parameter space for δ_0 is compact, which is commonly assumed when studying nonlinear estimators, such as the local Whittle estimator.

Theorem 3 *Let Assumptions 1, 2(i, ii) for the cointegrated system in (25) and (27), and Assumptions 3, 5, and 6 hold. Assume further that $\lim_{T\to\infty} \tilde{D}_T R D_T^{-1} = A$ is of full row rank p. Then, under the fixed-K asymptotics where K is held fixed as $T \to \infty$, we have the following:*

(i) $\sqrt{T} D_T(\hat{\beta}_{\text{TAOLS}} - \beta_0) = \sqrt{T} D_T(\tilde{\beta}_{\text{TAOLS}} - \beta_0) + o_p(1)$ so that

$$\sqrt{T} D_T(\hat{\beta}_{\text{TAOLS}} - \beta_0) \Rightarrow \Omega_{00\cdot x}^{1/2} \left(\xi' Q_\eta \xi\right)^{-1} \xi' Q_\eta \nu,$$

where ν is independent of (ξ, η), and $\nu \sim N(0, I_K)$.

(ii) Under the null hypothesis of $H_0 : R\beta_0 = r$, we have that

$$F^*(\hat{\beta}_{\text{TAOLS}}) := \frac{K - d_z - d_x}{K} F(\hat{\beta}_{\text{TAOLS}}) \Rightarrow F_{p,K-d_z-d_x};$$

$$t^*(\hat{\beta}_{\text{TAOLS}}) := \sqrt{\frac{K - d_z - d_x}{K}}\, t(\hat{\beta}_{\text{TAOLS}}) \Rightarrow t_{K-d_z-d_x} \text{ for } p = 1.$$

The result in Theorem 3 parallels those in Theorem 2, showing that the TA-OLS estimator, based on a $T^{-\psi}$-consistent fractional parameter estimator, retains an asymptotically mixed normal limit centered at the true parameter. Additionally, it shows that the scaled Wald and t statistics are asymptotically F and t distributed. Therefore, our asymptotic F and t testing theory holds for asymptotically homogenous cointegration regressions when the underlying driving process is a nonstationary fractional process. This appears to be the first time that the convenient F and t asymptotic theory is developed in this setting.

7 Conclusion

The paper develops the asymptotic F and t test theory for cointegrated regressions, where each regressor is an asymptotically homogeneous transformation of an I(1) process. Unlike the fixed-b framework of Vogelsang and Wagner (2024) which considers the cointegrated polynomial regressions, the asymptotic theory developed here does not require the full-design assumption. Our theory has been extended to accommodate cases where the regressors are driven by nonstationary, fractionally integrated processes of order δ_0 for any $\delta_0 \geq 1/2$, while the regression error is I(0).

Further extensions are possible. In the exogenous case, we only require that each of the suitably standardized regressors converges to a continuous-time process (i.e., $D_T^{-1} Z_{[T\cdot]} \Rightarrow Z(\cdot)$ for some D_T) and the (scaled) partial sum of the error process converges to a Brownian motion. In the endogenous case, we further require that the endogeneity can be eliminated by a long-run projection. The asymptotic F and t theory holds as long as these general requirements are met. Another interesting extension is to consider fractional cointegration when the regressor error is also fractionally integrated but of a lower order than the processes driving the regressors. This would involve employing a consistent estimator for the integration order of the regression error and transforming the original regression into one with an I(0) regression error. This extension requires further technical development, which we leave for future research.

8 Appendix of Proofs, Tables, and Figures

Proof of Theorem 1. We prove only the case for F_T, as the proof for t_T is similar. We write

$$\hat{\Omega}_{00} = \frac{1}{K} \hat{u}_0' \left(\tilde{\boldsymbol{\phi}}_1, \ldots, \tilde{\boldsymbol{\phi}}_K \right) (\tilde{\Phi}'\tilde{\Phi})^{-1} \left(\tilde{\boldsymbol{\phi}}_1, \ldots, \tilde{\boldsymbol{\phi}}_K \right)' \hat{u}_0.$$

Noting the invariance property: $\tilde{\boldsymbol{\phi}}_i' \hat{u}_0 = \tilde{\boldsymbol{\phi}}_i' Q_Z u_0 = \tilde{\boldsymbol{\phi}}_i' u_0$ (since $\hat{u}_0 = Q_Z u_0$ and $Q_Z \tilde{\boldsymbol{\phi}}_i = \tilde{\boldsymbol{\phi}}_i$), we have, under Assumptions 1–4(i):

$$\frac{1}{\sqrt{T}} \tilde{\boldsymbol{\phi}}_i' \hat{u}_0 = \frac{1}{\sqrt{T}} \tilde{\boldsymbol{\phi}}_i' u_0 \Rightarrow \Omega_{00}^{1/2} \int_0^1 \tilde{\phi}_i(\tau)\, dW_0(\tau), \tag{32}$$

jointly over $i \in \{1, \ldots, K\}$, where $\tilde{\phi}_i(\tau)$ is defined in (6). This can be proved directly or using the fact that $\tilde{\boldsymbol{\phi}}_i' \hat{u}_0 = (Q_Z \boldsymbol{\phi}_i)' \hat{u}_0 = (\boldsymbol{\phi}_i)' Q_Z \hat{u}_0 = (\boldsymbol{\phi}_i)' \hat{u}_0$, showing that the above result is identical to (5). Furthermore, by the continuous mapping theorem, the (i, j)-th element of $\tilde{\Phi}' \tilde{\Phi} / T$ satisfies

$$
\begin{aligned}
&\frac{1}{T} \boldsymbol{\phi}_i' Q_Z \boldsymbol{\phi}_j \\
&= \frac{1}{T} \boldsymbol{\phi}_i' \boldsymbol{\phi}_j - \frac{1}{T} \boldsymbol{\phi}_i' Z (Z'Z)^{-1} Z' \boldsymbol{\phi}_j \\
&\Rightarrow \int_0^1 \phi_i(\tau) \phi_j(\tau) \, d\tau - \left(\int_0^1 \phi_i(\tau) Z(\tau)' \, d\tau \right) \left(\int_0^1 Z(\tau) Z(\tau)' \, d\tau \right)^{-1} \left(\int_0^1 \phi_j(\tau) Z(\tau) \, d\tau \right) \\
&= \int_0^1 \tilde{\phi}_i(\tau) \tilde{\phi}_j(\tau) \, d\tau = \mathcal{S}_{\tilde{\Phi}}(i, j),
\end{aligned}
$$

where the convergence holds jointly for all pairs (i, j). This implies that

$$
\frac{\tilde{\Phi}' \tilde{\Phi}}{T} \Rightarrow \mathcal{S}_{\tilde{\Phi}}, \tag{33}
$$

where $\mathcal{S}_{\tilde{\Phi}}$ is invertible almost surely under Assumption 4(ii). Let

$$
\tilde{\boldsymbol{\phi}}^*(\tau) := \left[\tilde{\phi}_1(\tau), \cdots, \tilde{\phi}_K(\tau) \right] \mathcal{S}_{\tilde{\Phi}}^{-1/2},
$$

which is a (row) vector of functions. By construction,

$$
\int_0^1 \tilde{\boldsymbol{\phi}}^*(\tau)' \tilde{\boldsymbol{\phi}}^*(\tau) \, d\tau = \mathcal{S}_{\tilde{\Phi}}^{-1/2} \left[\underbrace{\int_0^1 \tilde{\phi}_i(\tau) \tilde{\phi}_j(\tau) d\tau}_{= \mathcal{S}_{\tilde{\Phi}}(i,j)} \right]_{1 \le i, j \le K} \mathcal{S}_{\tilde{\Phi}}^{-1/2} = I_K. \tag{34}
$$

The joint weak convergence results in (32) and (33) lead to

$$
\begin{aligned}
&\frac{1}{\sqrt{T}} \left[\tilde{\boldsymbol{\phi}}_1' \hat{u}_0, \ldots, \tilde{\boldsymbol{\phi}}_K' \hat{u}_0 \right] \cdot \left(\frac{1}{T} \tilde{\Phi}' \tilde{\Phi} \right)^{-1/2} \\
&\Rightarrow \Omega_{00}^{1/2} \left[\int_0^1 \tilde{\phi}_1(\tau) \, dW_0(\tau), \ldots, \int_0^1 \tilde{\phi}_K(\tau) \, dW_0(\tau) \right] \mathcal{S}_{\tilde{\Phi}}^{-1/2} \\
&= \Omega_{00}^{1/2} \left[\int_0^1 \tilde{\phi}_1^*(\tau) \, dW_0(\tau), \ldots, \int_0^1 \tilde{\phi}_K^*(\tau) \, dW_0(\tau) \right].
\end{aligned}
$$

By the continuous mapping theorem, we then have that

$$
\begin{aligned}
\hat{\Omega}_{00} &= \frac{1}{K} \hat{u}_0' \left(\tilde{\boldsymbol{\phi}}_1, \ldots, \tilde{\boldsymbol{\phi}}_K \right) (\tilde{\Phi}' \tilde{\Phi})^{-1} \left(\tilde{\boldsymbol{\phi}}_1, \ldots, \tilde{\boldsymbol{\phi}}_K \right)' \hat{u}_0 \\
&\Rightarrow \frac{1}{K} \Omega_{00} \sum_{i=1}^K \left(\int_0^1 \tilde{\phi}_i^*(\tau) \, dW_0(\tau) \right)^2.
\end{aligned}
$$

Combining the above asymptotic results, we have

$$F_T \Rightarrow \frac{\left\{\int_0^1 Z^*(\tau)\, dW_0(\tau)\right\}' \left\{\int_0^1 Z^*(\tau)\, Z^*(\tau)'\, d\tau\right\}^{-1} \left\{\int_0^1 Z^*(\tau)\, dW_0(\tau)\right\}/p}{\sum_{i=1}^{K}\left[\int_0^1 \tilde{\phi}_i^*(\tau)\, dW_0(\tau)\right]^2/K}$$

$$:= \frac{\zeta_0'\zeta_0/p}{\sum_{i=1}^{K}\zeta_i^2/K}, \tag{35}$$

where

$$\zeta_0 = \left\{\int_0^1 Z^*(\tau)\, Z^*(\tau)'\, d\tau\right\}^{-1/2} \int_0^1 Z^*(\tau)\, dW_0(\tau),$$

and

$$\zeta := (\zeta_1, \zeta_2, \ldots, \zeta_K)' = \int_0^1 \tilde{\boldsymbol{\phi}}^*(\tau)'\, dW_0(\tau).$$

Under Assumption 4(ii), $\zeta \sim N(0, I_K)$ and $\zeta_0 \sim N(0, I_p)$ conditional on $Z(\cdot)$. Also, conditional on $Z(\cdot)$, ζ is independent of ζ_0, as both are conditionally normal and their conditional covariance is zero, since $\int_0^1 \tilde{\phi}_i(\tau)\, Z^*(\tau)\, d\tau = 0$ for all $i \in \{1, \ldots, K\}$. Thus, conditional on $Z(\cdot)$,

$$\frac{\zeta_0'\zeta_0/p}{\zeta'\zeta/K} \sim F_{p,K}.$$

Given that the conditional distribution $F_{p,K}$ does not depend on the conditioning variable (i.e., $Z(\cdot)$), $\frac{\zeta_0'\zeta_0/p}{\zeta'\zeta/K} \sim F_{p,K}$ unconditionally. Therefore, $F_T \Rightarrow F_{p,K}$. ■

Proof of Theorem 2. Since the result in Part (i) has been derived in Section 3, we only prove the results in Part (ii). To prove Part (ii), note that under the $H_0: R\beta_0 = r$,

$$\sqrt{T}\tilde{D}_T\left(R\hat{\beta}_{\text{TAOLS}} - r\right) = (\tilde{D}_T R D_T^{-1})\sqrt{T} D_T\left[\hat{\beta}_{\text{TAOLS}} - \beta_0\right] \Rightarrow A\Omega_{00\cdot x}^{1/2}\left(\xi' Q_\eta \xi\right)^{-1}\xi' Q_\eta \nu.$$

Noting that

$$\hat{\Omega}_{00\cdot x} = \frac{1}{K}\left\|Q_{V_Z, V_{\Delta x}} V_{0\cdot x}^{\alpha}\right\|^2 = \frac{1}{K}\left\|\left(Q_{V_{\Delta x}} - P_{Q_{V_{\Delta x}} V_Z}\right) V_{0\cdot x}^{\alpha}\right\|^2,$$

we have:

$$F(\hat{\beta}_{\text{TAOLS}}) \Rightarrow \frac{\left[A\left(\xi' Q_\eta \xi\right)^{-1}\xi' Q_\eta \nu\right]' \left[A\left(\xi' Q_\eta \xi\right)^{-1} A'\right]^{-1} \left[A\left(\xi' Q_\eta \xi\right)^{-1}\xi' Q_\eta \nu\right]/p}{\left\|\left(Q_\eta - P_{Q_\eta \xi}\right)\nu\right\|^2/K}$$

$$= \frac{\left\|\left[A\left(\xi' Q_\eta \xi\right)^{-1} A\right]^{-1/2} A\left(\xi' Q_\eta \xi\right)^{-1}\xi' Q_\eta \nu\right\|^2/p}{\left\|\left(Q_\eta - P_{Q_\eta \xi}\right)\nu\right\|^2/K},$$

where

$$\left[A\left(\xi'Q_\eta\xi\right)^{-1}A\right]^{-1/2}A\left(\xi'Q_\eta\xi\right)^{-1}\xi'Q_\eta\nu\sim\chi_p^2,$$
$$\left\|\left(Q_\eta-P_{Q_\eta\xi}\right)\nu\right\|^2\sim\chi^2_{K-d_z-d_x}.$$

Conditional on (ξ,η), both $\left(Q_\eta-P_{Q_\eta\xi}\right)\nu$ and $\xi'Q_\eta\nu$ are normal and their conditional covariance is

$$cov(\left(Q_\eta-P_{Q_\eta\xi}\right)\nu,\xi'Q_\eta\nu)=\left(Q_\eta-P_{Q_\eta\xi}\right)Q_\eta\xi=Q_\eta\xi-Q_\eta\xi=0.$$

Hence, the limiting distribution of $F^*(\hat\beta_{\text{TAOLS}})$ equals that of a ratio of two independent (scaled) chi-square random variables, namely χ_p^2/p and $\chi^2_{K-d_z-d_x}/\left(K-d_z-d_x\right)$, and is therefore the $F_{p,K-d_z-d_x}$ distribution. The case for $p=1$ can be shown similarly, and the details are omitted. ■

Proof of Theorem 3. We prove only Part (i) since the proof for Part (ii) is essentially the same as that of Theorem 2, where the stochastic process $Z(\cdot)$ and (η,ξ) are now defined as functionals of $B_x(\cdot;\delta_0)$. We first show that

$$V_{\Delta^{\hat\delta}x,i}-V_{\Delta^{\delta_0}x,i}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\Delta^{\hat\delta}X_t-\Delta^{\delta_0}X_t\right]\phi_i\left(\frac{t}{T}\right)=o_p(1)$$

for all $i\in\{1,2,\ldots,K\}$ for a finite K. Using $a_0(\cdot)\equiv1$ and $u_{x,0}=0$, we can write that, for some $\tilde\delta$ satisfying $|\tilde\delta-\delta_0|\le|\hat\delta-\delta_0|$,

$$\begin{aligned}\Delta^{\hat\delta}X_t-\Delta^{\delta_0}X_t&=\Delta^{(\hat\delta-\delta_0)}u_{x,t}-u_{x,t}\\&=\sum_{j=0}^{t}a_j(\hat\delta-\delta_0)u_{x,t-j}-u_{x,t}=\sum_{j=1}^{t-1}a_j(\hat\delta-\delta_0)u_{x,t-j}\\&=\sum_{j=1}^{t-1}\left\{\sum_{m=1}^{M-1}\frac{(\hat\delta-\delta_0)^m}{m!}a_j^{(m)}(0)u_{x,t-j}+\frac{(\hat\delta-\delta_0)^M}{M!}a_j^{(M)}(\tilde\delta-\delta_0)u_{x,t-j}\right\}\\&=\sum_{m=1}^{M-1}\frac{(\hat\delta-\delta_0)^m}{m!}g^{(m)}(u_{x,t},0)+\frac{(\hat\delta-\delta_0)^M}{M!}g^{(M)}(u_{x,t},\tilde\delta-\delta_0),\end{aligned}\tag{36}$$

for any $M\ge2$, where

$$g^{(m)}(u_{x,t},b)=\sum_{j=1}^{t-1}a_j^{(m)}(b)u_{x,t-j}\text{ with }a_j^{(m)}(b)=\frac{d^ma_j\left(b\right)}{db^m}.$$

Therefore, we have $V_{\Delta^{\hat\delta}x,i}-V_{\Delta^{\delta_0}x,i}=A_T+B_T$, where

$$A_T=\sum_{m=1}^{M-1}\frac{(\hat\delta-\delta_0)^m}{m!}A_{mT},\;\;A_{mT}:=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sum_{j=1}^{t-1}a_j^{(m)}(0)u_{x,t-j}\phi_i\left(\frac{t}{T}\right),$$

$$B_T=\frac{(\hat\delta-\delta_0)^M}{M!}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}g^{(M)}(u_{x,t},\tilde\delta-\delta_0)\phi_i\left(\frac{t}{T}\right).$$

For notational simplicity, we have suppressed the dependence of A_T, A_{mT}, and B_T on i, the index for the basis functions.

To proceed, we note that $\hat{\delta} - \delta_0 = O_p(T^{-\psi})$ from Assumption 6. Then, we can apply the result (C.13) of Lemma C.4 in Robinson and Hualde (2003) and obtain that

$$g^{(M)}(u_{x,t}, \tilde{\delta} - \delta_0) = O_p(t^{1/2}),$$

which holds uniformly in $t \in \{1, \ldots, T\}$. For a large enough M, it follows that

$$\begin{aligned}
\|B_T\| &= O_p\left(T^{-\psi M}\right) \times \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \sqrt{t} \left|\phi_i\left(\frac{t}{T}\right)\right| \\
&= O_p\left(T^{-\psi M+1}\right) \times \frac{1}{T} \sum_{t=1}^{T} \sqrt{\frac{t}{T}} \left|\phi_i\left(\frac{t}{T}\right)\right| \\
&= O_p\left(T^{-\psi M+1}\right) \left(\int_0^1 \tau \left|\phi_i(\tau)\right| d\tau + o(1)\right) = o_p(1),
\end{aligned}$$

where the final equality follows from Assumption 5.

To prove that $\|A_T\| = o_p(1)$ for any $M \geq 2$, we first establish a bound for A_{mT} for any $m \in \{1, \ldots M-1\}$. We consider the scalar case with $d_x = 1$ only, as the vector case can be reduced to the scalar case by considering A_{mT} element by element. We have

$$\begin{aligned}
A_{mT} &= \frac{1}{\sqrt{T}} \sum_{s=1}^{T-1} \sum_{t=s+1}^{T} a_{t-s}^{(m)}(0) u_{x,s} \phi_i\left(\frac{t}{T}\right) \\
&= \frac{1}{\sqrt{T}} \sum_{s=1}^{T-1} \underbrace{\left[\sum_{\tau=1}^{T-s} a_\tau^{(m)}(0) \phi_i\left(\frac{\tau+s}{T}\right)\right]}_{:=h_{i,s}} u_{x,s} := \frac{1}{\sqrt{T}} \sum_{s=1}^{T-1} h_{i,s} u_{x,s}.
\end{aligned}$$

Using the continuity and positivity of the spectral density $f_{u_x}(\cdot)$ of $u_{x,t}$, we can obtain that for some positive constant $\mathcal{C}_f$:

$$\begin{aligned}
E\left(A_{mT}^2\right) &= \frac{1}{T} \sum_{s_1=1}^{T-1} \sum_{s_2=1}^{T-1} h_{i,s_1} h_{i,s_2} E[u_{x,s_1} u_{x,s_2}] = \frac{1}{T} \sum_{s_1=1}^{T-1} \sum_{s_2=1}^{T-1} h_{i,s_1} h_{i,s_2} \int_{-\pi}^{\pi} f_{u_x}(\lambda) e^{i(s_1-s_2)\lambda} d\lambda \\
&= \frac{1}{T} \int_{-\pi}^{\pi} f(\lambda) \left|\sum_{s=1}^{T-1} h_{i,s} e^{is\lambda}\right|^2 d\lambda \\
&\leq \frac{\mathcal{C}_f}{T} \int_{-\pi}^{\pi} \left|\sum_{s=1}^{T-1} h_{i,s} e^{is\lambda}\right|^2 d\lambda = O\left(\frac{1}{T}\right) \cdot \sum_{s=1}^{T-1} h_{i,s}^2.
\end{aligned}$$

Next, it follows from Lemma D.4 of Robinson and Hualde (2003) that

$$\begin{aligned}
a_\tau^{(m)}(0) &= 0 \text{ for } \tau < m; \\
|a_\tau^{(m)}(0)| &\leq \frac{\mathcal{C}_m (\log(\tau+1))^{m-1}}{\tau - m + 1} \text{ for } \tau \geq m,
\end{aligned}$$

where $\mathcal{C}_m < \infty$ depends only on m. Using these results and Assumption 5, we obtain that, for some positive constant $\tilde{\mathcal{C}}_m$ depending only on m,

$$\begin{aligned}\sum_{s=1}^{T-1} h_{i,s}^2 &= \sum_{s=1}^{T-1}\left[\sum_{\tau=1}^{T-s} a_\tau^{(m)}(0)\phi_i\left(\frac{\tau+s}{T}\right)\right]^2 \\ &= \sum_{s=1}^{T-1}\left[\sum_{\tau=m}^{T-s} a_\tau^{(m)}(0)\phi_i\left(\frac{\tau+s}{T}\right)\right]^2 \leq \tilde{\mathcal{C}}_m \sum_{s=1}^{T-1}\left[\sum_{\tau=m}^{T-s}\frac{[\log(\tau+1)]^{m-1}}{\tau-m+1}\right]^2 \\ &\leq \tilde{\mathcal{C}}_m \sum_{s=1}^{T-1}[\log(T)]^{2m} = O(T(\log(T))^{2m}).\end{aligned}$$

It then follows that $E[A_{mT}^2] = O([\log(T)]^{2m})$, and consequently $A_{mT} = O_p\left([\log(T)]^m\right)$. As a result, for any $M \geq 2$:

$$|A_T| \leq \sum_{m=1}^{M-1}\frac{|\hat{\delta}-\delta_0|^m}{m!}|A_{mT}| = O_p\left(T^{-\psi}[\log(T)]^m\right) = o_p(1).$$

We have therefore proved that

$$\left\|V_{\Delta^{\hat{\delta}}x,i} - V_{\Delta^{\delta_0}x,i}\right\| \leq \|A_T\| + \|B_T\| = o_p(1)$$

for all $i \in \{1,\ldots,K\}$. This, together with (31) and Slutsky's theorem, implies that

$$T^{-1/2}D_T^{-1}V_Z \Rightarrow \xi,\ V_{\Delta^{\hat{\delta}}x} \Rightarrow \eta,\ \text{and}\ V_{0\cdot x}^{\alpha} \Rightarrow \nu$$

hold jointly. Part (i) then follows straightforwardly by the continuous mapping theorem and Slutsky's theorem. ■

Table 1: Empirical Rejection Probabilities of the OLS-HAC-χ^2 Test and OLS-HAR-F Test in Exogenous Cointegrated Homogeneous Regression

$T = 100$				
	Case (i)		Case (ii)	
ρ	OLS-HAC	OLS-HAR	OLS-HAC	OLS-HAR
0.05	0.0873	0.0511	0.0769	0.0539
0.25	0.1189	0.0671	0.1036	0.0672
0.50	0.1866	0.0908	0.1482	0.0823
0.75	0.3236	0.1332	0.2371	0.1016
0.90	0.5386	0.2122	0.4214	0.1563
$T = 200$				
	Case (i)		Case (ii)	
ρ	OLS-HAC	OLS-HAR	OLS-HAC	OLS-HAR
0.05	0.0659	0.0473	0.0636	0.0473
0.25	0.0971	0.0611	0.0836	0.0645
0.50	0.1341	0.0711	0.1053	0.0598
0.75	0.2208	0.1015	0.1671	0.0824
0.90	0.4040	0.1105	0.2962	0.0932

Table 2: Empirical Rejection Probabilities of the Tests based on IM-OLS, TA-OLS, and FM-OLS for Endogenous Cointegrated Homogeneous Regression

$T = 100$				
	Case (i)		Case (ii)	
ρ	IM-OLS	TA-OLS	FM-OLS	TA-OLS
0.05	0.0542	0.0532	0.0685	0.0520
0.25	0.0836	0.0629	0.0936	0.0648
0.50	0.1359	0.0591	0.1601	0.0567
0.75	0.2595	0.0617	0.3743	0.0591
0.90	0.3346	0.0781	0.7632	0.0841
$T = 200$				
	Case (i)		Case (ii)	
ρ	IM-OLS	TA-OLS	FM-OLS	TA-OLS
0.05	0.0537	0.0517	0.0532	0.0514
0.25	0.0763	0.0643	0.0745	0.0629
0.50	0.0958	0.0621	0.1007	0.0623
0.75	0.1702	0.0607	0.1912	0.0623
0.90	0.3236	0.0598	0.4966	0.0627

Table 3: Country Codes and the Corresponding Country Names

Country Codes	**Country Names**
AUS, AUT, BEL, CAN, DEN, FIN, FRA, HOL, ITA, JAP, NOR, UK, USA	Australia, Austria, Belgium, Canada, Denmark, Finland, France, Netherlands, Italy, Japan, Norway, United Kingdom, United States of America

Table 4: Estimation Results for the Turning Points ($= \exp((-(\hat{\beta}_{0,1}/2\hat{\beta}_{0,2})))$) of Carbon Dioxide Emissions

	Turning Points ($)		
Country	TA-OLS	FM-OLS	LS
AUS	25,925	26,680	28,257
AUT	23,885	26,553	30,855
BEL	11,863	11,784	12,181
CAN	19,223	20,508	21,497
DEN	16,216	16,276	15,980
FIN	17,556	17,935	18,163
FRA	11,893	11,781	12,105
HOL	17,698	18,424	17,746
ITA	22,150	21,624	22,221
JAP	19,198	19,442	20,872
NOR	32,345	34,567	38,378
UK	20,042	21,316	23,431
USA	21,333	21,552	21,934

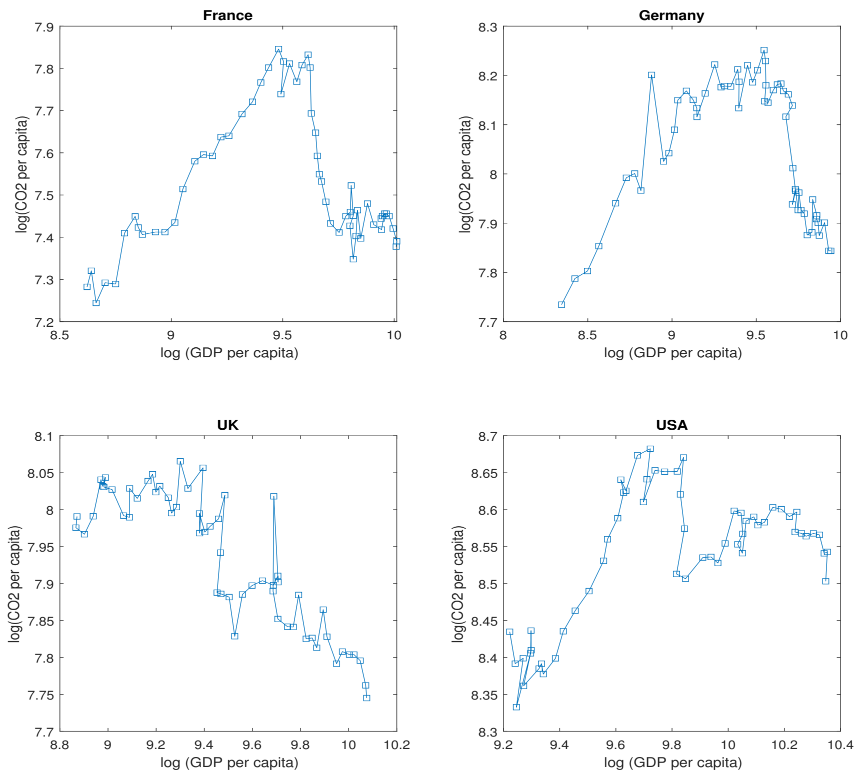


Figure 1: Plots of the logarithm of per capita CO_2 against the logarithm of per capita GDP for selected countries.

Table 5: TA-OLS Coefficients with 95% Confidence Intervals

Country	$\hat{\beta}_{0,1}$	95% CI for $\beta_{0,1}$	$\hat{\beta}_{0,2}$	95% CI for $\beta_{0,2}$
AUS	12.1629	[8.6629, 15.6629]	-0.5985	[-0.7824, -0.4146]
AUT	5.8547	[1.9911, 9.7182]	-0.2880	[-0.4967, -0.0793]
BEL	9.4405	[2.9312, 15.9497]	-0.5026	[-0.8504, -0.1548]
CAN	12.9825	[5.7404, 20.2246]	-0.6583	[-1.0381, -0.2785]
DEN	23.9747	[16.2082, 31.7412]	-1.2367	[-1.6452, -0.8283]
FIN	20.2463	[13.1517, 27.3409]	-1.0350	[-1.4181, -0.6518]
FRA	19.4696	[10.6470, 28.2922]	-1.0377	[-1.5094, -0.5660]
HOL	16.6963	[9.3788, 24.0139]	-0.8536	[-1.2409, -0.4663]
ITA	15.8374	[11.5176, 20.1573]	-0.8007	[-1.0361, -0.5654]
JAP	5.9537	[3.3736, 8.5337]	-0.2855	[-0.4289, -0.1421]
NOR	16.8327	[10.0928, 23.5725]	-0.8491	[-1.2047, -0.4935]
UK	4.3306	[-0.2112, 8.8723]	-0.2403	[-0.4800, -0.0006]
USA	11.3811	[3.8142, 18.9481]	-0.5713	[-0.9580, -0.1847]

Table 6: FM-OLS Coefficients with 95% Confidence Intervals

Country	$\hat{\beta}_{0,1}$	95% CI for $\beta_{0,1}$	$\hat{\beta}_{0,2}$	95% CI for $\beta_{0,2}$
AUS	11.8073	[8.9543, 14.6603]	-0.6025	[-0.8043, -0.4006]
AUT	5.6466	[2.5744, 8.7188]	-0.3192	[-0.5422, -0.0963]
BEL	9.5546	[4.4767, 14.6325]	-0.5411	[-0.9078, -0.1744]
CAN	10.7404	[4.8068, 16.6740]	-0.6826	[-1.0686, -0.2966]
DEN	23.2411	[17.7675, 28.7147]	-1.2630	[-1.6820, -0.8441]
FIN	19.6522	[14.5893, 24.7151]	-1.0525	[-1.4560, -0.6489]
FRA	18.6251	[10.7618, 26.4883]	-1.0929	[-1.5849, -0.6009]
HOL	14.6872	[8.8261, 20.5482]	-0.8866	[-1.2764, -0.4969]
ITA	6.9044	[4.9938, 8.8149]	-0.3399	[-0.4643, -0.2156]
JAP	16.0276	[13.9593, 18.0958]	-0.8370	[-1.0816, -0.5924]
NOR	5.7771	[3.6115, 7.9426]	-0.2985	[-0.4438, -0.1533]
UK	15.1188	[9.6151, 20.6224]	-0.8733	[-1.2395, -0.5071]
USA	10.2724	[6.2604, 14.2844]	-0.5821	[-0.9734, -0.1909]

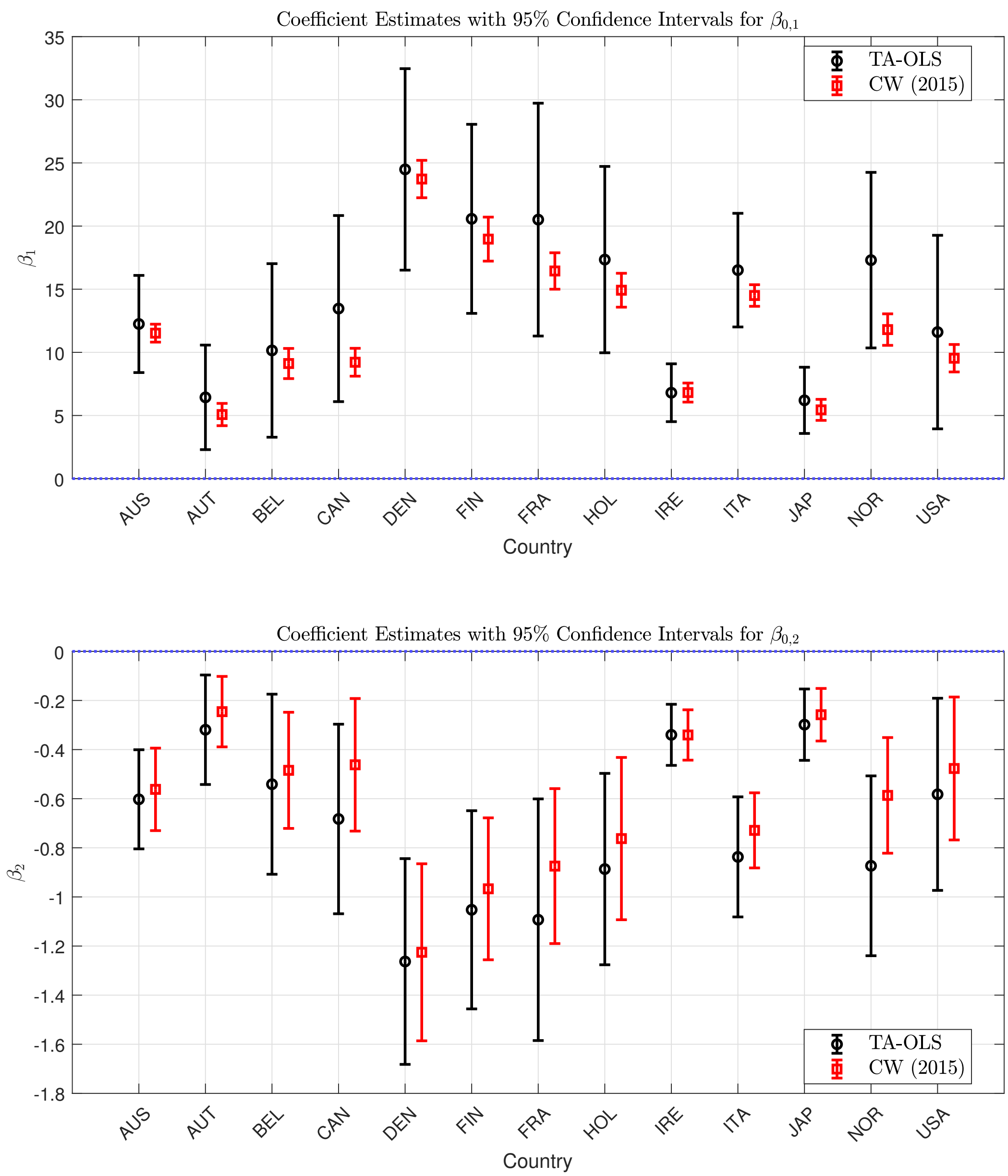
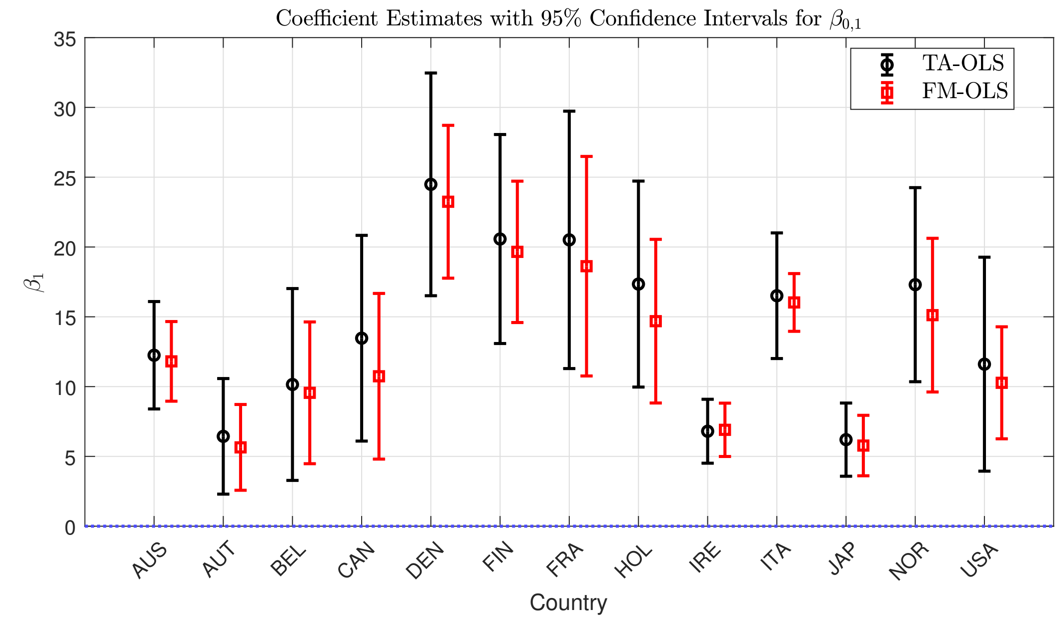


Figure 2: TA-OLS estimates and LS estimates (based on Chan and Wang (2015)) and the 95% confidence intervals with $K = 12$ for the quadratic cointegrated model in (24)



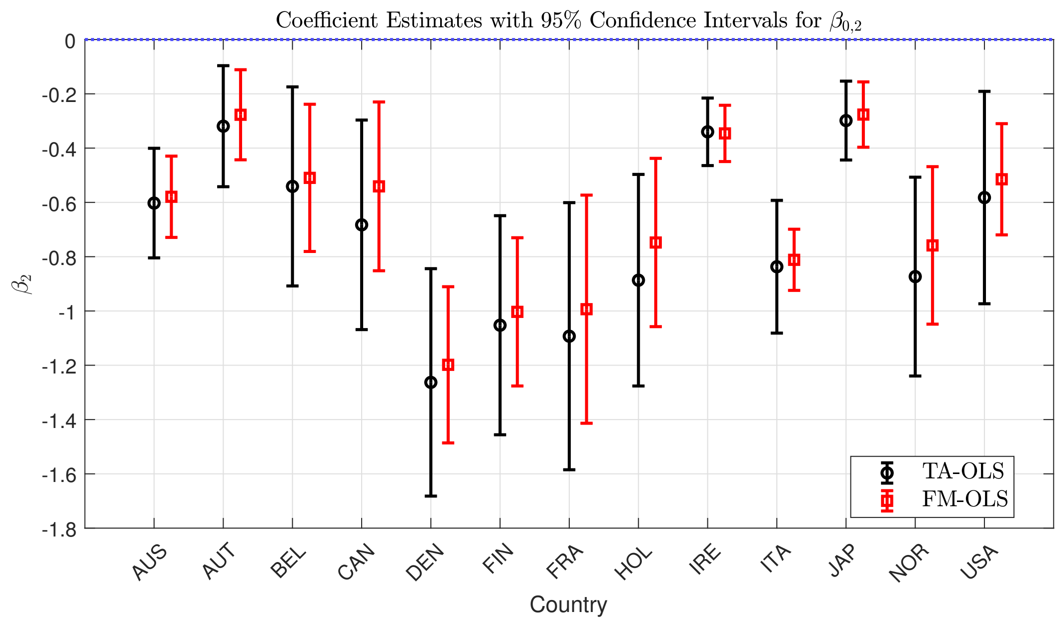


Figure 3: TA-OLS estimates and FM-OLS estimates (based on Wagner and Hong (2016)) and the 95% confidence intervals with $K = 12$ for the quadratic cointegrated model in (24)

Table 7: Chan and Wang (2015)'s LS Coefficients with 95% Confidence Intervals

Country	$\hat{\beta}_{0,1}$	95% CI for $\beta_{0,1}$	$\hat{\beta}_{0,2}$	95% CI for $\beta_{0,2}$
AUS	11.5200	[10.8100, 12.2300]	-0.5620	[-0.7300, -0.3940]
AUT	5.0755	[4.1930, 5.9580]	-0.2455	[-0.3890, -0.1020]
BEL	9.1160	[7.9220, 10.3100]	-0.4845	[-0.7210, -0.2480]
CAN	9.2175	[8.1140, 10.3210]	-0.4620	[-0.7320, -0.1920]
DEN	23.7235	[22.2420, 25.2050]	-1.2255	[-1.5860, -0.8650]
FIN	18.9670	[17.2250, 20.7090]	-0.9670	[-1.2560, -0.6780]
FRA	16.4430	[15.0000, 17.8860]	-0.8745	[-1.1900, -0.5590]
HOL	14.9205	[13.5790, 16.2620]	-0.7625	[-1.0930, -0.4320]
ITA	6.8160	[6.0620, 7.5700]	-0.3405	[-0.4430, -0.2380]
JAP	14.5015	[13.6470, 15.3560]	-0.7290	[-0.8820, -0.5760]
NOR	5.4465	[4.6140, 6.2790]	-0.2580	[-0.3650, -0.1510]
UK	11.8025	[10.5550, 13.0500]	-0.5865	[-0.8220, -0.3510]
USA	9.5360	[8.4490, 10.6230]	-0.4770	[-0.7680, -0.1860]

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