Understanding the Fisher Equation*

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ABSTRACT

It is argued that univariate long memory estimates based on ex post data tend to underestimate the persistence of ex ante variables (and, hence, that of the ex post variables themselves) because of the presence of unanticipated shocks whose short run volatility masks the degree of long range dependence in the data. Empirical estimates of long range dependence in the Fisher equation are shown to manifest this problem and lead to an apparent imbalance in the memory characteristics of the variables in the Fisher equation. Evidence in support of this typical underestimation is provided by results obtained with inflation forecast survey data and by direct calculation of the finite sample biases.

To address the problem of bias, the paper introduces a bivariate exact Whittle (BEW) estimator that explicitly allows for the presence of short memory noise in the data. The new procedure enhances the empirical capacity to separate low-frequency behavior from high-frequency fluctuations, and it produces estimates of long range dependence that are much less biased when there is noise contaminated data. Empirical estimates from the BEW method suggest that the three Fisher variables are integrated of the same order, with memory parameter in the range (0.75,1). Since the integration orders are balanced, the ex ante real rate has the same degree of persistence as expected inflation, thereby furnishing evidence against the existence of a (fractional) cointegrating relation among the Fisher variables and, correspondingly, showing little support for a long run form of Fisher hypothesis.

JEL Classification: C13; C14; E40

Key words: Bivariate exact Whittle estimator, exact Whittle estimator, Fisher hypothesis, fractional cointegration, imbalance, log-periodogram regression, perturbed fractional process.

1 Introduction

This study investigates the long run properties of three ex ante Fisher variables including the ex ante real rate, expected inflation and the nominal interest rate. The properties are of intrinsic interest because these variables play a crucial role in determining investment, savings, and indeed virtually all intertemporal decisions. Since both the ex ante real interest rate and expected inflation are not directly observable, it is not a straightforward matter to study their long run behavior. To circumvent the difficulty, most empirical studies use ex post variables as proxies for the ex ante variables. In particular, actual inflation observed ex post is used as a proxy for expected inflation, and the implied ex post real rate, defined as the difference between the nominal interest rate and actual inflation according to the ex post Fisher equation, as a proxy for the ex ante real rate. This practice often leads to controversial results. For example, Rose (1988) concluded that the ex ante real rate is unit root nonstationary by showing that the nominal rate is a unit root process while inflation and inflation forecasting errors are $I(0)$ stationary. In contrast, Mishkin (1992, 1995) found support for an $I(0)$ ex ante real rate by rejecting the null of a unit root in the ex post real rate. Recently, Phillips (1998) showed that the three ex post Fisher components are fractionally integrated, and that the nominal interest rate is more persistent than both the real interest rate and inflation, an outcome that is strikingly at odds with the ex post Fisher equation. According to the ex post Fisher equation $i_t = \pi_{t+1} + r_{t+1}$, where π_{t+1} and r_{t+1} are the realized inflation rate and the ex post real interest rate, respectively, the degree of persistence of i_t is necessarily the same as that of the dominant component of π_{t+1} and r_{t+1}.

This paper attempts to reconcile the findings of Rose (1988) and Mishkin (1992, 1995), and resolve the empirical incompatibility found in Phillips (1998). All three studies estimated or inferred the integration orders of the Fisher variables based on the ex post Fisher equation. We argue here that empirical results obtained in this way can be misleading because the ex post Fisher equation appears unbalanced for the reasons explained below.

First, the timing of the three components are different. The nominal interest rate can be regarded as being set in advance. For example, the widely used three-month Treasury Bill rates are set every Monday and are 'expected' to be relevant over the next three months. Put this way, the nominal interest rate can be regarded as an observable ex ante variable. Therefore, when the Fisher equation is written in the form $i_t = r_{t+1} + \pi_{t+1}$, it expresses an ex ante variable as the sum of two ex post variables. More formally, if $\mathcal{F}_t$ is a filtration representing information at time t, i_t is adapted to the filtration $\mathcal{F}_t$ while π_{t+1} and, in consequence, r_{t+1} are adapted to the filtration $\mathcal{F}_{t+1}$. Interpreted in this way, the Fisher equation implies that the sum of two $\mathcal{F}_{t+1}$-measurable random variables is $\mathcal{F}_t$-measurable, which at first seems puzzling. But the Fisher equation is actually an accounting identity that defines the ex post real rate r_{t+1}. The forces that really determine the nominal interest rate i_t are the expected real rate and expected inflation formed at time t, i.e. $i_t = E_t r_{t+1} + E_t \pi_{t+1}$, where E_t is the expectation operator conditional on the information $\mathcal{F}_t$. By adding and subtracting the $\mathcal{F}_{t+1}$-measurable forecasting errors e_{t+1}, we get $i_t = (E_t r_{t+1} - e_{t+1}) + (E_t \pi_{t+1} + e_{t+1}) = r_{t+1} + \pi_{t+1}$.

Second, the short run dynamics of the three components are different. The nominal interest rate is often less volatile than inflation and the ex post real rate in the short run. The nominal rate is a rate that is *expected* to prevail during some period and is not affected, by definition, by the unexpected shocks that arrive during that period. On the other hand, the inflation rate and ex post real rate are rates that are realized during that period and thus carry the effects of the unexpected shocks over that period.

Third, it can be misleading to infer the integrating order of the real rate in small samples from the ex post Fisher equation as is done in Rose (1988). Due to the presence of possibly large forecasting errors, unit root tests may falsely reject the null that expected inflation contains a unit root, if ex post inflation is used as a proxy for expected inflation. The false rejection, coupled with evidence that the nominal rate contains a unit root, can lead to the false conclusion that the ex ante real rate is an $I(1)$ process. Again, because of forecasting errors, unit root tests are likely to reject the null of a unit root in the ex ante real rate, if the ex post real rate is used as a proxy for the ex ante real rate. This leads to the possibly false conclusion reached by some earlier researchers (e.g., Mishkin, 1992, 1995) that the ex ante real rate is an $I(0)$ process.

The empirical incompatibility found in Phillips (1998) is direct evidence of the apparent imbalance of the ex post Fisher equation. Suppose the forecasting errors are stationary and weakly dependent, and expected inflation $E_t \pi_{t+1}$ follows a fractional process. Then actual inflation follows a perturbed fractional process in the sense that it is the sum of a fractional process and weakly dependent noise. From a statistical perspective, a perturbed fractional process is a long memory process with the same degree of persistence as the original fractional process. However, it can be difficult to estimate the fractional integration parameter even in large samples, especially when the perturbation is volatile because the long memory component gets buried in a lot of short memory noise. In these circumstances, the widely used log-periodogram (LP) estimator (Geweke and Porter-Hudak 1983) and local Whittle estimator (Robinson 1995) suffer substantial downward bias. This bias is large enough to account for the empirical incompatibility that Phillips discovered in the ex post Fisher relation.

Using a new approach, we find evidence that the three Fisher variables are indeed integrated of the same order and are fractionally nonstationary. The evidence presented here takes three forms. First, one cause of the bias is that ex post rather than ex ante variables are observed. If good proxies for the ex ante variables were available, we could perform estimation with these proxies and presumably avoid or at least reduce bias. Of course, the ex post variables can themselves be regarded as proxies for the ex ante variables. However, unexpected subsequent shocks make the ex post variables more volatile than their ex ante counterparts, so these variables may not be such good proxies because of their contamination with short memory effects. This point is especially important because it is the long run properties of the variables that are the focus of interest in the Fisher relation. In search of better proxies than ex post realizations, we employ the inflation forecast from the *Survey of Professional Forecasters* (for details of the survey, see Croushore 1993) as a proxy for expected inflation, and we use the implied real rate forecast as a proxy for the ex ante real rate. Using these variables, we find that the estimated orders of integration are larger than those based on the realized ex post series. This finding corroborates the bias argument and indicates that the true ex ante variables are more persistent than they appear to be from ex post realizations.

Second, we calculate the bias effects explicitly using asymptotic expressions. Asymptotic theory shows that the asymptotic bias of the LP estimator depends on the ratio of the long run variance of the forecasting errors and that of the innovations that drive the inflation forecasts. Using inflation and the inflation forecasts, we evaluate the ratio and calculate the asymptotic bias of the LP estimator. The evidence points to a substantial asymptotic bias and these findings are supported by simulation evidence from finite samples. Furthermore, we investigate the bias of the exact Whittle (EW) estimator of Shimotsu and Phillips (2002a), which is more efficient and more widely applicable (to stationary and nonstationary fractional series) than the LP estimator. The EW estimator produces the same empirical incompatibility as the LP estimator. Simulations show that the EW estimator also has a substantial downward bias.

Third, we introduce a bivariate exact Whittle (BEW) estimator that accounts for the possible presence of additive perturbations in the data. The estimator resembles the bivariate Whittle estimator of Lobato (1999) but involves an additional term in the approximation of the spectral density matrix and uses an exact version of the local Whittle likelihood (see Phillips, 1999, and Shimotsu and Phillips, 2002a). Simulations show that the BEW estimator has a significantly smaller bias than the LP and EW estimators. Applying the BEW estimator to the ex post data, we find the BEW estimates are significantly higher than LP and EW estimates, which again lends strong support for the small sample bias argument. Moreover, the empirical estimates suggest that the three Fisher variables are integrated of the same order, with memory parameter in the range (0.75,1).

The BEW estimator also provides a framework for testing the equality of the integration orders of the three Fisher components. Applying this approach, we find that we can not reject the null that inflation and the real rate are integrated of the same order. Since the integration orders are balanced, the ex ante real rate has the same degree of persistence as expected inflation, thereby furnishing evidence against the existence of a (fractional) cointegrating relation among the Fisher variables and, correspondingly, showing little support for a long run form of Fisher hypothesis.

The rest of the paper is organized as follows. Section 2 gives LP and EW estimates of the fractional integration parameters using ex post data and confirms the empirical incompatibility described above. Section 3 presents evidence from the *Survey of Professional Forecasters* to show that the ex post variables are more volatile than the ex ante variables, and that the LP and EW estimates based on the ex post data are substantially downward biased. Section 4 introduces the BEW estimator and provides further evidence that the LP and EW estimates are biased downward. This section also develops and implements a test of the equality of long memory in the Fisher components. Section 5 concludes.

2 Memory Estimation and the Apparent Empirical Incompatibility of the Fisher Components

We calculate three-month inflation rates using the US monthly CPI (all commodities, with no adjustment) and take the US three-month Treasury Bill rate as the nominal interest rate[1]. Instead of using monthly overlapping data as in Phillips (1998), we compute and employ quarterly non-overlapping data in order to make them conform to the data from the *Survey of Professional Forecasters*. In addition, the use of quarterly data avoids the possibly spurious serial correlation resulting from the horizon of the variables being longer than the observation interval. The timing of the data is as follows: data are collected in January, April, July and October each year. A January interest rate observation uses the end of January three-month TB rate. A January observation of the three-month inflation rate is calculated from the January to April CPI data.

Using quarterly data in the US over the period 1934:1-1999:4, we employ both the exact Whittle and the log-periodogram approaches to estimate the fractional differencing parameters. The advantages of the exact local Whittle estimator are its robustness to non-stationarity, its consistency and asymptotic normality for all values of d. However, its properties are unknown in the presence of additive short memory disturbances. The log-periodogram estimator is easy to implement and has been to shown to be consistent and asymptotic normal (Sun and Phillips 2002) for both fractional processes and perturbed fractional processes. However, the LP estimator is inconsistent when $d > 1$ (Kim and Phillips, 1999). In this case, two popular approaches are to difference or taper the data (Velasco 1999a&b, Lobato and Velasco 2000). It is well known that tapering distorts the trajectory of the data and inflates the asymptotic variance. We therefore adopt the former approach and difference the data using the filter $(1-L)^{1/2}$. This half-difference filter has been used in earlier research, e.g. Gil-Alana and Robinson (1997). To apply a fractional filter such as $(1-L)^{1/2}$ to a given time series of fixed length, we assume that the prehistorical values of the time series are zero (see the footnote below). Since the choice of the fractional filter $(1-L)^{1/2}$ is somewhat arbitrary, it is worthwhile using

[1] **CPI**: Bureau of Labor Statistics, *Monthly Labor Review.* Code: CUUROOOOSA0. http://www.bls.gov/data/home.htm

Three-month Treasury Bill Rate: Board of Governors of the Federal Reserve System, Federal Reserve Bulletin. Code: TB3MS. http://research.stlouisfed.org/fred/

alternate filters and checking robustness. In the empirical work reported below, similar estimates were obtained after applying the filters $(1-L)^{0.75}$ and $(1-L)$.

2.1 EW and LP Estimation

The exact Whittle estimator was proposed in Phillips (1999) and its asymptotic theory was developed in Shimotsu and Phillips (2002a). For a fractional process defined as[2]

$$x_t = (1-L)^{-d} w_t = \sum_{k=0}^{t-1} \frac{\Gamma(d+k)}{\Gamma(d)\Gamma(k+1)} w_{t-k}, \tag{1}$$

exact Whittle estimation of the memory parameter d involves maximizing the following Whittle log-likelihood function

$$Q_m(G,d) = \frac{1}{m} \sum_{j=1}^{m} \left(\log G\lambda_j^{-2d} + \frac{1}{G} I_{\Delta^d(x)}(\lambda_j) \right), \tag{2}$$

where $I_{\Delta^d(x)}(\lambda_j)$ is the periodogram of the filtered time series $(1-L)^d x_t$ defined on the band of m fundamental frequencies $\{\lambda_j = 2\pi j/n,\ j = 1, ..., m\}$ with $\frac{m}{n} + \frac{1}{m} \to 0$, so that the band concentrates on the zero frequency as the sample size $n \to \infty$.

Shimotsu and Phillips (2002a) show that the exact Whittle estimator $(\widehat{G}_{EW}, \widehat{d}_{EW})$ is consistent and that $\widehat{d}_{EW}$ has the following limiting distribution as $n \to \infty$

$$\sqrt{m}(\widehat{d}_{EW} - d) \xrightarrow{d} N(0, 1/4), \tag{3}$$

for all values of d. The robustness of the asymptotic properties of $\widehat{d}_{EW}$ is especially useful when the domain of the true order of fractional integration is controversial. The EW estimate also provides guidance on the order of the fractional difference that can render the data stationary.

[2] Two main approaches have been used in the literatue to define a fractional process x_t. The first, which is adopted in Hosking (1981), among others, defines a stationary fractional process as an infinite order moving average of innovations: $x_t = \sum_{k=0}^{\infty} \frac{\Gamma(d+k)}{\Gamma(d)\Gamma(k+1)} w_{t-k}$; and defines a nonstationary $I(d)$ process as the partial sum of an $I(d-1)$ process (Hurvich and Ray 1995, Velasco 1999). The second, which is used in Robinson (1994) and Phillips (1999) truncates the fractional difference filter and defines $x_t = \sum_{k=0}^{t-1} \frac{\Gamma(d+k)}{\Gamma(d)\Gamma(k+1)} w_{t-k}$ for all values of d. For a more detailed discussion of the definitions and their implications, see Shimotsu and Phillips (2002b).

LP regression involves linear least squares over the same frequency band $(\lambda_j : j = 1, 2, ..., m)$ leading to the regression equation

$$\log I_{\Delta^{\tilde{d}}(x)}(\lambda_j) = \widehat{\alpha} - \widehat{\beta} \ln |1 - \exp(i\lambda_j)|^2 + error \tag{4}$$

for some $\widetilde{d}$ corresponding to preliminary fractional differencing of the data. The LP estimate $\widehat{d}_{LP}$ of d is then obtained by adding $\widetilde{d}$ back in to the estimate $\widehat{\beta}$, giving $\widehat{\beta} + \widetilde{d}$. In our empirical work reported below, we took $\widetilde{d}$ to be $0.5, 0.75$ and 1, and found that the estimates for different $\widetilde{d}$ matched fairly closely. We therefore only report the case for $\widetilde{d} = 0.5$. Sun and Phillips (2002) show that the LP estimator is consistent and has the following limiting distribution even in the presence of perturbations

$$\sqrt{m}(\widehat{d}_{LP} - d) \xrightarrow{d} N(0, \pi^2/24). \tag{5}$$

2.2 Empirical Memory Estimates for the Fisher Components

Since the EW and LP estimates depend on the choice of bandwidth m, several different bandwidths were used. Figs. 1(a) and 1(b) present the empirical estimates of (d_i, d_π, d_r), the long memory parameters for the three ex post Fisher components, using the LP estimator and the EW estimator, respectively. Both the EW estimates and the LP estimates appear fairly robust to the choice of m. A salient feature of both figures is that the 95% confidence bands for d_i and d_π do not overlap each other while the confidence bands for d_r and d_π are almost indistinguishable. The EW and LP estimates suggest that $d_i > d_\pi$ and $d_r = d_\pi$, a configuration that is incompatible with the ex post Fisher equation. For, in a model where the three fractionally integrated variables $y_t \equiv I(d_y)$, $x_t \equiv I(d_x)$ and $z_t \equiv I(d_z)$ satisfy the linear relationship $y_t = x_t - z_t$, the long run behavior of y_t is characterized by the dominant component of x_t and z_t, e.g., if $d_x > d_z$, then $d_y = d_x$. In the present case, $r_{t+1} = i_t - \pi_{t+1}$. So, if $d_i > d_\pi$, then $d_r > d_\pi$. This conclusion is clearly at odds with the empirical estimates.

The estimates obtained here are similar to those reported in Phillips (1998) where the empirical incompatibility of the long run behavior of the Fisher components was discovered. Phillips used the local Whittle estimator (Robinson 1995) that was originally proposed for stationary fractional processes, extending it to the nonstationary case $d \in (1/2, 1]$. So, local Whittle, exact local Whittle and LP estimators all reveal the same empirical incompatibility. To further check the robustness of the result, we re-estimated the fractional parameters using only the post war data ranging from 1948:1 to 1999:4, finding that the subsample EW and LP estimates were close to those based on the full sample and that the empirical incompatibility remains.

3 Understanding the Empirical Imbalance in the Fisher Equation

The estimates reported in the previous section are all based on the ex post time series[3], which are either directly observable or indirectly available from the ex post Fisher equation. We argue that the empirical estimates based on the ex post data underestimate the true degree of persistence of the underlying ex ante variables and hence that of the ex post variables themselves. Since the long run properties of the underlying variables are the focus of interest, the ex post variables can be regarded as proper proxies for the ex ante variables. On theoretical grounds, this will be true as long as the forecasting errors are stationary and weakly dependent (i.e., have short memory). However, when the unexpected shocks are so large that the forecasting errors have greater variation than the innovations that drive the ex ante variables, the actual variables observed ex post may appear to be less persistent than they really are because the slowly moving nature of the persistent component is buried in the volatile short run fluctuations. This interpretation gains support from the evidence presented below from inflation forecasts.

3.1 Results from Inflation Forecasts

Under the assumption of rational expectations, the realized inflation rate differs from the expected inflation rate by an unexpected shock, i.e.

$$\pi_{t+1} = \pi_t^e + e_{t+1}, \tag{6}$$

where the unexpected shock (forecasting error) e_{t+1} is a martingale difference process. If we further assume that $\{\pi_t^e\}$ is a fractional process that is uncorrelated with $\{e_t\}$, then the realized inflation rate is a fractional process with uncorrelated additive disturbances. Such a process is called a perturbed fractional process and is studied in Sun and Phillips

[3]To aviod confusion, we should note that we sometimes refer to the nominal rate as an ex post variable because it has ex ante features and is observable ex post.

(2002). The uncorrelatedness between the forecasting errors and the innovations that drive the inflation forecast seems plausible. This is supported by a simple calculation of cross correlation coefficients using the data on inflation and inflation forecasts.

The strong dependence in the inflation expectations data is consistent with Fisher's original study (Fisher, 1930). Fisher found that the duration of the expectation formation process was long and that realized inflation was quite volatile. He constructed inflation expectations series by taking moving averages of realized inflation over as many as 15 to 40 years. Here we employ modern techniques to model the same phenomenon, using a persistent (long memory) process to model expected inflation and additive disturbances (representing unexpected shocks) to allow for the greater volatility of realized inflation. When there is large variation in the unexpected shock component, realized inflation appears less persistent because the slow moving component is less evident in the time series. Therefore, estimates of strong dependence tend to be downward biased with the bias depending on the relative variation in the forecasts and the unexpected shocks, as shown by Sun and Phillips (2002).

To compare variation, we need to obtain the expected inflation rates. Prior studies of this issue can be grouped into two categories. One models expectation formation explicitly and then estimates expected inflation from the observed time series of realized (ex post) values (e.g. Hamilton 1985). The other uses survey data on inflation forecasts or inflation expectations. Several surveys are available and among these *the Survey of Professional Forecasters* is the oldest quarterly survey of macroeconomic forecasts in the United States.[4] The survey respondents include a diverse group of forecasters who share one thing in common: they forecast as part of their current jobs. Hence it is reasonable to believe that their forecasts represent an overview of expectations about macroeconomic activity in general and expected inflation in particular. This position is supported by the study of Keane and Runkle (1990). In analyzing the characteristics of these forecasts, Keane and Runkle found that they were unable to reject the hypothesis that the price level forecasts are unbiased and rational.

Using data from the *Survey of Professional Forecasters*, we extract a quarterly series

[4] The survey began in 1968 and was conducted by the American Statistical Association and the National Bureau of Economic Research. The Federal Reserve Bank of Philadelphia took over the survey in 1990. The survey is publicly available at no cost and is often reported in major newspapers and financial newswires. For more information see http://www.phil.frb.org/econ/spf/

of expected inflation rates from 1981:4 to 1999:3. Fig. 2 graphs this expected inflation series against that of realized inflation. Expected inflation appears much smoother than realized inflation, revealing the volatility induced by the presence of unexpected shocks in realized inflation rates. Over the time period shown, there are spikes in realized inflation corresponding to both positive and negative shocks. The volatility of these shocks in realized inflation obscures the slow moving component of expected inflation and makes the realized inflation series appear to be less persistent.

Fig. 3(a) graphs the LP estimates of the long memory parameter in expected and realized inflation using the quarterly data over 1981:4–1999:3. Although the sample size is small, the results are indicative. We observe a significant difference between the two estimates, especially when m is large. This is consistent with the fact that when m is larger, the estimator is less able to avoid contamination from short memory (higher frequency) effects arising from sources such as a stationary disturbance. Fig. 3(a) also graphs the EW estimates and it is clear that the same qualitative comparison between the expected and realized series applies in the case of these EW estimates.

In much the same way as for inflation, the ex post real rate is more volatile than the ex ante real rate because of the presence of the additional short memory component. Fig. 3(b) shows the differences in the two estimates obtained from the ex ante and ex post real rate series. Again, the long memory parameter estimates for the ex ante real rate are generally larger than those for the ex post real rate. The differences are particularly marked in the case of the EW estimates.

In sum, the empirical estimates obtained in Fig. 3 suggest that expected inflation and the ex ante real rate may be just as persistent as the nominal rate. Under rational expectations, ex post and ex ante variables are characterized by the same degree of persistence because they differ by unanticipated shocks. Thus, under this assumption and according to these estimates, actual inflation and the real rate observed ex post may be as persistent as the nominal rate.

3.2 Evaluating the Small Sample Bias

The last subsection used inflation forecasts to estimate the long memory parameter directly. This subsection uses the forecasting data to evaluate the small sample biases of the LP and EW estimates when additive perturbations are present.

We start by assuming that expected inflation follows a fractional process:

$$\pi_t^e = \mu + (1-L)^{-d} w_t = \mu + \sum_{k=0}^{\infty} \frac{\Gamma(d+k)}{\Gamma(d)\Gamma(k+1)} w_{t-k}, \tag{7}$$

and that the forecasting errors e_τ are uncorrelated with π_s^e for all τ and s. Sun and Phillips (2002) show that, under certain regularity conditions, the LP estimator based on actual inflation, $\pi_{t+1} = \pi_t^e + e_{t+1}$, is consistent and asymptotically normal. The limiting distribution is

$$\sqrt{m}(\hat{d}_{LP} - d) \Rightarrow N(b_{LP}, \frac{\pi^2}{24}), \tag{8}$$

where the asymptotic bias effect

$$b_{LP} = -(2\pi)^{2d} \left(\frac{f_w(0)}{f_e(0)} \right)^{-1} \frac{d}{(2d+1)^2} \frac{m^{2d}}{n^{2d}} \sqrt{m}. \tag{9}$$

The asymptotic bias b_{LP} is always negative, just as one would expect when there is short memory contamination. The magnitude of the bias obviously depends on the Signal-Noise (SN) ratio $f_w(0)/f_e(0)$, which is the ratio of the long run variance of the innovations that drive expected inflation to that of the forecasting errors. Again, this is unsurprising, since the ratio measures the underlying force of expected inflation shocks relative to that of the forecasting errors. Because of the presence of the bias in (8), the larger is the force of the forecasting errors, the more difficult it is to recover good estimates of the long memory parameter from ex post observations.

Asymptotic results analogous to (8) for the EW estimator are not available in the literature and to derive such results is beyond the scope of the present paper. However, in related work without the effect of perturbations, Andrews and Sun (2002) show that the local Whittle estimator has the same asymptotic bias, but smaller asymptotic variance than the LP estimator for stationary long memory processes. We conjecture these results continue to hold for nonstationary perturbed long memory processes. This conjecture is supported by the simulation study reported in Table 2 below.

To evaluate the asymptotic bias $b_{LP}/\sqrt{m}$, we estimate the forecasting errors e_{t+1} by $\pi_{t+1} - \pi_t^e$ and the innovations w_t by $(1-L)^{\widehat{d}_\pi} \widetilde{\pi}_t^e$, where $\widetilde{\pi}_t^e$ is the demeaned inflation forecast. To estimate the long run variances $f_e(0)$ and $f_w(0)$, we employ the following formula:

$$lrvar = \gamma(0) + 2\sum_{j=1}^{p} (1 - \frac{j}{p+1}) \gamma(j), \tag{10}$$

where $\gamma(k)$ is the k-th autocovariance function. The SN ratio $f_w(0)/f_e(0)$ can then be calculated as the ratio of the estimates of the long-run variances. The estimated ratio evidently depends on the choices of p and $\hat{d}_\pi$. Table 1 presents estimates of the inverted SN ratio obtained in this way for various selections of p and d_π. It shows that the variation in unexpected shocks is indeed relatively very large. Large variation in unexpected shocks leads to large small sample bias. For example, when $n = 264, m = n^{1/2}$, $f_e(0)/f_w(0) = 12$ and $d = 0.8$, the asymptotic bias $b_{LP}/\sqrt{m}$ is -0.3105, or 39%.

Table 1: Estimates of the Inverted Signal-Noise Ratio

	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$	$d = 1$
$p = 1$	6.69	7.35	7.84	8.18	8.37
$p = 3$	6.24	7.61	8.99	10.34	11.60
$p = 5$	5.74	7.36	9.12	10.91	12.67
$p = 7$	5.93	8.05	10.54	13.33	16.30
$p = 9$	6.14	8.55	11.41	14.62	17.99
$p = 11$	6.22	8.94	12.33	16.31	20.68
$p = 13$	6.60	9.63	13.44	17.92	22.80

To examine the effectiveness of the asymptotic results for finite samples, we conduct a Monte Carlo simulation. Let $x_t = (1-L)^{-d} u_t$ and $y_t = x_t + e_t$ where $u_t \sim iid\ N(0, 1)$ and $e_t \sim iid\ N(0, 12)$. These variances are chosen to calibrate to the variation observed in the data. For each replication with sample size $n = 264$ and $m = n^{1/2}$, we estimate the long memory parameter using the original process $\{x_t\}$ and using the perturbed process $\{y_t\}$. Table 2 gives the averages and standard errors of the LP and EW estimates obtained from 1000 replications. As expected, the EW estimator has more or less the same finite sample bias but smaller variance than the LP estimator. Both the EW estimator and the LP estimator have a large finite sample bias. Thus, on bias grounds alone, estimates around 0.55 obtained from ex post data as shown in Fig. 1 could come from a model where the true memory parameter is as large as 0.8. When the bias is so large, memory parameter estimates obtained from ex post inflation and the real interest rate series can therefore be seriously misleading.

Table 2: Average Estimates Using Original Series and Perturbed Series.

		$d = 0.5$	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$	$d = 1$
LP	$\widehat{d}_x$	0.5028	0.6133	0.6971	0.8053	0.9003	1.0095
	Std($\widehat{d}_x$)	(0.2013)	(0.2018)	(0.2059)	(0.2011)	(0.2057)	(0.2132)
	$\widehat{d}_y$	0.2249	0.3326	0.4432	0.5778	0.6944	0.8227
	Std($\widehat{d}_y$)	(0.2170)	(0.2033)	(0.2168)	(0.2069)	(0.2059)	(0.2146)
EW	$\widehat{d}_x$	0.4805	0.5823	0.6752	0.7826	0.8752	0.9768
	Std($\widehat{d}_x$)	(0.1821)	(0.1780)	(0.1699)	(0.1750)	(0.1712)	(0.1789)
	$\widehat{d}_y$	0.2084	0.3072	0.4212	0.5512	0.6611	0.7826
	Std($\widehat{d}_y$)	(0.1676)	(0.1691)	(0.1737)	(0.1664)	(0.1632)	(0.1688)

4 Further Evidence using a new Bivariate Exact Local Whittle Estimator

The small sample biases discussed above arise because expected inflation and the ex ante real rate are not directly observable. Of course, we can use survey data on expectations such as that from *the Survey of Professional Forecasters* as proxies. However, time series of expectations data like the inflation forecasts series we have used earlier are not long series, so empirical estimates based on them may not be very accurate, particularly for a parameter that characterizes long range dependence in the data. In this section, therefore, we explore the structure of the Fisher equation further and propose a new estimator that is based on the ex post data to achieve bias reduction.

4.1 The Bivariate Exact Whittle Estimator

Observe that the ex post real rate and the realized inflation rate can be represented in system format as

$$\begin{aligned} r_{t+1} &= r_t^e - e_{t+1}, \\ \pi_{t+1} &= \pi_t^e + e_{t+1}. \end{aligned} \tag{11}$$

Under the assumption that r_t^e and π_t^e are fractional processes, both r_{t+1} and π_{t+1} are perturbed fractional processes. Furthermore, the perturbations are from the same source, i.e. the unexpected inflation shocks. Therefore, we can expect that it is more efficient to estimate the fractional parameters jointly.

Assuming that the ex ante variables and forecasting errors are uncorrelated, the spectral density matrix $f(\lambda)$ of $x_t := (r_t, \pi_t)'$ satisfies

$$f(\lambda) \sim \Lambda G \Lambda^* + \kappa H \text{ as } \lambda \to 0+, \tag{12}$$

where $\Lambda = diag(e^{\frac{\pi}{2}d_1 i}\lambda^{-d_1}, e^{\frac{\pi}{2}d_2 i}\lambda^{-d_2})$, $d = (d_1, d_2)'$, $\kappa = \sigma_e^2/(2\pi)$, G is a symmetric positive definite real matrix,

$$H = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{13}$$

and the affix $*$ denotes complex conjugate transpose. Define $\Delta^d(x_t) = (\Delta^{d_1} r_t, \Delta^{d_2}\pi_t)'$ and

$$I_{\Delta^d(x)}(\lambda_j) = \frac{1}{2\pi n}|\sum_{t=1}^{n} \Delta^d(x_t)\exp(it\lambda_j)|^2,\ \Lambda_j = diag(e^{\frac{\pi}{2}d_1 i}\lambda_j^{-d_1}, e^{\frac{\pi}{2}d_2 i}\lambda_j^{-d_2}). \tag{14}$$

Then the (negative) exact local Whittle likelihood is

$$Q_m(G,\kappa,d) = \frac{1}{m}\sum_{j=1}^{m}\left(\log|\Lambda_j G \Lambda_j^* + \kappa H| + tr\left[\left(G + \kappa\Lambda_j^{-1} H \Lambda_j^{*-1}\right)^{-1} I_{\Delta^d(x)}(\lambda_j)\right]\right).$$

Minimizing $Q_m(G,\kappa,d)$ yields the Bivariate Exact Whittle (BEW) estimator:

$$\left(\widehat{G}_{BEW}, \widehat{\kappa}_{BEW}, \widehat{d}_{BEW}\right) = \arg\min Q_m(G,\kappa,d). \tag{15}$$

When x_t is a scalar time series, both G and $I_{\Delta^d(x)}$ reduce to positive scalars. In this case, we get the univariate exact Whittle (UEW) estimator. Observe that the UEW estimator is different from the univariate EW estimator of Shimotsu and Phillips (2002a) because, unlike the EW estimator, the UEW estimator takes account of the additive perturbations in the observed series.

The BEW estimator is motivated by the bivariate Whittle (BW) estimator of Lobato (1999), the exact Whittle (EW) estimator of Shimotsu and Phillips (2002a), and the nonlinear log-periodogram (NLP) estimator of Sun and Phillips (2002). In view of the established properties of the latter three estimators, we expect, under certain regularity conditions, the BEW estimator to be more efficient than the corresponding univariate exact Whittle estimator, to be consistent and asymptotically normal for all values of d, and to be less biased than Lobato's BW estimator in the presence of stationary perturbations.

A theoretical development of the asymptotic properties of the BEW is beyond the scope of the present paper and are left for future research. Instead, we provide some simulation evidence here to justify the new estimator and reveal its finite sample performance in relation to existing procedures. To save space, we only consider the following data generating process

$$z_{1t} = (1-L)^{-d_1} v_{1t} - \varepsilon_t, \quad (16)$$

$$z_{2t} = (1-L)^{-d_2} v_{2t} + \varepsilon_t, \quad (17)$$

where $d_1 = d_2$ are long memory parameters, $\{v_{1t}\}, \{v_{2t}\}$ and $\{\varepsilon_t\}$ are independent and each is iid $N(0, 12)$. For each simulated sample of size $n = 264$, we estimate d_1 and d_2 using the EW, LP and BEW estimators with bandwidth $m = \sqrt{n}$. The EW and LP estimators are based on the individual time series $\{z_{1t}\}$ and $\{z_{2t}\}$, whereas the BEW estimator is based on the bivariate series $\left\{(z_{1t}, z_{2t})'\right\}$.

Table 3 presents the average estimates and the standard deviations (in parentheses) using 500 simulation repetitions. We note the following two main features of the simulation results. First, the BEW estimator achieves substantial bias reduction, producing results that are only slightly downward biased. By contrast the EW and LP estimates both show very significant downward bias, amounting to as much as 50% in some cases. Second, the variance of the BEW estimator appears to lie between that of the EW and LP estimators. It is not so surprising that the BEW estimates have greater variance than the EW estimates. Because it utilizes the system structure, the BEW estimate is expected to be more efficient than the corresponding UEW estimator, which does not make use of the system formulation. On the other hand, the UEW estimator can be expected to have larger variance than the EW estimator because UEW estimation involves the extra parameter arising from the perturbation effects. Apparently, the latter factor outweighs the former, which leads to the BEW estimator having larger variance than the EW estimator.

Table 3: Finite Sample Performance of the EW, LP and BEW Estimators, $n = 264$

	EW estimator		LP estimator		BEW estimator	
(d_1, d_2)	$\widehat{d}_1$	$\widehat{d}_2$	$\widehat{d}_1$	$\widehat{d}_2$	$\widehat{d}_1$	$\widehat{d}_2$
$(0.4, 0.4)$	0.1383	0.1440	0.1306	0.1490	0.3800	0.3779
	(0.1330)	(0.1291)	(0.2068)	(0.2060)	(0.1730)	(0.1749)
$(0.6, 0.6)$	0.3146	0.3109	0.3273	0.3194	0.5678	0.5761
	(0.1579)	(0.1615)	(0.2025)	(0.2065)	(0.1864)	(0.1903)
$(0.8, 0.8)$	0.5418	0.5392	0.5709	0.5670	0.7422	0.7449
	(0.1645)	(0.1776)	(0.2056)	(0.2140)	(0.1921)	(0.1813)
$(1.0, 1.0)$	0.7892	0.7843	0.8417	0.8302	0.9369	0.9319
	(0.1570)	(0.1630)	(0.1983)	(0.2093)	(0.2040)	(0.2000)

4.2 Empirical Estimation and Inference using BEW

We now use the BEW estimator to estimate the fractional difference parameters using actual inflation and the ex post real rate. Fig. 4 presents the results. Compared with Fig. 1, we find that the BEW estimates are significantly higher than both the EW and LP estimates. For the bandwidths considered, the estimated integration orders are larger than 0.75 and center around 0.9. So, the BEW estimates appear to fall in the same general range as the estimated integration order of the nominal interest rate (based on either EW or LP estimation). These empirical estimates of the long range dependence of the Fisher components are therefore generally consistent with the Fisher equation.

The empirical estimates in Figs. 1 and 4 show that inflation and the real rate may well be integrated of the same order. To investigate this further, we formally test the null $H_0 : d_\pi = d_r$ against the alternative $H_1 : d_\pi > d_r$. To do so, we construct the test statistic $T = \sqrt{m}(\hat{d}_\pi - \hat{d}_r)$, where $(\hat{d}_\pi, \hat{d}_r)$ is the BEW estimate of (d_π, d_r), rejecting the null if T is larger than some critical value.

Since an asymptotic theory of the BEW estimator has not yet been established, we use simulations to compute the critical value for a given size. The experiment is designed as follows. The data generating processes for r_t and π_t are

$$r_{t+1} = r_t^e - e_{t+1} = (1 - L)^{-d_r} u_t - e_{t+1}, \tag{18}$$

$$\pi_{t+1} = \pi_t^e + e_{t+1} = (1 - L)^{-d_\pi} v_t + e_{t+1}, \tag{19}$$

where $d_r = d_\pi = d_0$, $\{e_t\}$ is *iid* $N(0, \sigma_e^2)$ and (u_t, v_t) follows a fourth order vector autoregressive model. Since the long memory cannot be precisely estimated, we use a range of long memory parameters. For each value of $d_0 = 0.5, 0.6, 0.7, 0.8, 0.9, 1$, we first apply the fractional difference operators $(1-L)^{d_0}$ to the forecasted inflation rates (extracted from the *Survey of Professional Forecasters*) and the forecasted real rates (calculated by subtracting forecasted inflation from nominal rates) to get $\{u_t, v_t\}$ and then use them to estimate the VAR in the DGP. The choice of bandwidth in the bivariate Gaussian semiparametric estimator necessarily involves judgement. Although too large a value of m causes contamination from high frequencies, too small a value of m leads to imprecision in estimation. Hence, several values of m are employed. For each value of $m = n^l$, $l = 1/2, 2/3$ and $3/4$, we perform 500 replications with sample size $n = 264$. Table 4 contains the critical values so obtained for $l = 1/2, 2/3$ and size of 5% and 10%.

Table 4: Empirical Critical Values for the T Test

		$d_0 = 0.5$	$d_0 = 0.6$	$d_0 = 0.7$	$d_0 = 0.8$	$d_0 = 0.9$	$d_0 = 1.0$
5%	$l = 1/2$	1.4829	1.7076	1.7600	1.8710	1.8994	1.9060
	$l = 2/3$	0.9174	0.9994	1.0037	0.9775	0.9125	0.9156
10%	$l = 1/2$	1.0538	1.2628	1.3379	1.2526	1.2807	1.2625
	$l = 2/3$	0.7015	0.7467	0.7096	0.7345	0.6365	0.6530

The power properties of this test are also examined using simulations. The data generating processes are the same as before except $d_\pi > d_r$. For each combination of d_π and d_r such that $d_\pi = d_r + (k-1)/10$, $k = 1, ..., 6$, we first apply the fractional difference operator $(1-L)^{d_0}$ with $d_0 = d_r$ to the forecasted inflation rates and the forecasted real rates to get $\{u_t, v_t\}$ and then use them to estimate the VAR in the DGP. We report the powers based on 500 replications when the size is 10%, $n = 264$ and $m = n^{1/2}$ in Table 5. Apparently, the power increases with the difference between d_π and d_r and is reasonably high when the difference is greater than 0.4.

Table 5: The Power of the T test

	k					
	0	1	2	3	4	5
$d_1 = 0.5$	10.00	19.80	32.20	59.00	81.40	93.50
$d_1 = 0.6$	10.20	15.20	25.60	47.40	71.20	89.00
$d_1 = 0.7$	10.00	16.80	27.00	45.00	68.00	82.80
$d_1 = 0.8$	10.20	17.40	29.40	47.20	65.80	83.60
$d_1 = 0.9$	9.80	15.60	29.40	46.80	67.20	81.20

We now perform this test in the empirical application using the ex post data. The results for m at equispaced points in the interval $[10, 90]$ are tabulated in Table 6. For all values of $m \geq 10$, including the values not presented here, $\hat{d}_\pi$ is only slightly larger than $\hat{d}_r$. Using the critical values given in Table 4 (and those not reported here for intermediate values), we cannot reject the null at the level of 10%. According to this evidence, therefore, inflation and the real rate are integrated of the same order. From the ex ante Fisher equation, the nominal rate also has the same order of integration. As a consequence, all three ex ante and ex post Fisher components share the same degree of persistence. The last row of Table 6 reports the restricted estimates when we impose the restriction that $d_\pi = d_r$. As might be expected, the restricted estimates fall between the unrestricted ones, which are presented in the first two rows in Table 6. Combining this with the estimates based on the univariate nominal rate series, these empirical findings suggest that the integration order of the Fisher components lies between 0.75 and 1.

Table 6: Bivariate Estimates and Tests: Ex Post Data

m	10	20	30	40	50	60	70	80	90
$\widehat{d}_\pi$	0.8645	0.8266	0.8961	0.9298	0.9993	0.8974	0.9268	0.9516	0.9780
$\widehat{d}_r$	0.6610	0.7700	0.8537	0.8907	0.9624	0.8694	0.9105	0.9327	0.9516
T	0.6504	0.2530	0.2320	0.2468	0.2608	0.2175	0.1358	0.1690	0.2505
$\widehat{d}$	0.7572	0.8181	0.8958	0.9244	0.9876	0.8929	0.9271	0.9501	0.9717

Equality of the integration orders of the three Fisher components has important implications for the long run Fisher hypothesis, which states that the nominal rate moves with the expected inflation rate in the long run. Since both variables appear to be fractionally nonstationarity, validity of the long run Fisher hypothesis requires the existence of a fractional cointegrating relationship between these variables. In addition, the full Fisher effect implies the cointegrating vector must be $(1, -1)$. In effect, therefore, the long run Fisher hypothesis requires that the residual in this relationship, viz., $i_t - \pi_t^e$, be less persistent than i_t and π_t^e. However, this residual is the ex ante real rate, which according to the evidence above is as persistent as π_t^e. It follows from these findings that the relationship is not cointegrating and the long run Fisher hypothesis does not hold.

5 Conclusions

Many empirical studies in the past have investigated the orders of integration of the Fisher equation variables. Given the recent development of robust semiparametric estimation methods for stationary and nonstationary long memory, this practice is now being extended to include analyses of the degree of persistence using fractional models and estimates of the degree of long run dependence in each of the series. For time series that may be perturbed by weakly dependent noise, we show here that estimating the degree of persistence is more difficult because of the presence of a strong downward bias in conventional estimates of long memory.

In the present context, ex post data can be viewed as noisy observations of the ex ante variables. Our findings reveal that conventional semiparametric estimation using ex post data substantially underestimates the true degree of persistence in the ex ante variables and, hence, that of the ex post variables themselves. The bivariate exact Whittle estimator introduced here explicitly allows for the presence of additive perturbations or short memory noise in the data. This new estimator enhances our capacity to separate low-frequency behavior from high-frequency fluctuations and give us estimates of long range dependence that are much less biased when there is noise contaminated data. Evidence based on this new estimator supports the hypothesis that the three Fisher components are integrated of the same order. Accordingly, we find little support for the presence of a cointegrating relation among the Fisher variables and, therefore, little support for the long run Fisher hypothesis.

References

[1] Andrews DWK and Sun Y. 2002. Adaptive Local Polynomial Whittle Estimation of Long-range Dependence. Cowles Foundation Discussion Paper #1384, Yale University.

[2] Croushore D. 1993. Introducing: the Survey of Professional Forecasters. *Business Review*, the Federal Reserve Bank of Philadelphia, 1993, Nov. 3-15.

[3] Fisher I. 1930. *The Theory of Interest.* New York: A. M. Kelly.

[4] Geweke J and Porter-Hudak S. 1983. The estimation and application of long-memory time series models. *Journal of Time Series Analysis* **4:** 221–237.

[5] Gil-Alaña LA and Robinson PM. 1997. Testing of unit root and other nonstationary hypotheses in macroeconomic time series, *Journal of Econometrics*, **80**: 241-268.

[6] Hamilton JD. 1985. Uncovering financial market expectations of inflation. *Journal of Political Economy* **93**: 1224-1241.

[7] Hurvich CM and Ray BK. 1995. Estimation of the Memory Parameter for Nonstationary or Noninvertible Fractionally Integrated Processes. *Journal of Time Series Analysis* **16**: 17–42.

[8] Keane M and Runkle DE. 1990. Testing the rationality of price forecasts: new evidence from panel data. *American Economic Review* **80**: 714-735.

[9] Kim CS and Phillips PCB. 1999. Log periodogram regression: the nonstationary case. Mimeographed, Cowles Foundation, Yale University.

[10] Lobato IN. 1999. A semiparametric two step estimator in a multivariate long memory model. *Journal of Econometrics* **90**: 129-153.

[11] Lobato IN and Velasco C. 2000. Long Memory in Stock Market Trading Volume. *Journal of Business and Economics Statistics*, **18**: 410-427.

[12] Maynard A and Phillips PCB. 2001. Rethinking an old empirical puzzle: econometric evidence on the forward discount anomaly. *Journal of Applied Econometrics* **16**: 671-708.

[13] Mishkin FS. 1992. Is the Fisher effect for real? *Journal of Monetary Economics* **30**: 195-215.

[14] Mishkin FS. 1995. Nonstationarity of regressors and test on the real rate behavior. *Journal of Business and Economics Statistics* **13**: 47-51.

[15] Phillips PCB. 1998. Econometric analysis of Fisher's equation. Cowles Foundation Discussion Paper #1180, Yale University.

[16] Phillips PCB. 1999. Discrete Fourier transforms of fractional processes. Cowles Foundation Discussion Paper #1243, Yale University.

[17] Robinson PM. 1994. Time series with strong dependence. In *Advances in Econometrics, Sixth World Congress* Sims CA (eds), **1**: 47-96. Cambridge: Cambridge University Press.

[18] Robinson PM. 1995. Gaussian semiparametric estimation of long range dependence. *Annals of Statistics* **23**: 1630-1661.

[19] Rose AK. 1988. Is the real interest rate stable? *Journal of Finance* **43**: 1095-1112.

[20] Shimotsu K and Phillips PCB. 2002a. Exact local Whittle estimation of fractional integration. Cowles Foundation Discussion Paper #1367, Yale University.

[21] Shimotsu K and Phillips PCB. 2002b. Local Whittle estimation of fractional integration and some of its variants. Mimeograph. Yale University.

[22] Sun Y. and Phillips, PCB 2002. Nonlinear log-periodogram regression estimator for perturbed fractional processes. Cowles Foundation Discussion Paper #1366, Yale University.

[23] Velasco C. 1999a. Gaussian semiparametric estimation of non-stationary time series. *Journal of Time Series Analysis* **20:** 87-127.

[24] Velasco C. 1999b. Non-stationary log-periodogram regression. *Journal of econometrics* **91:** 325-371.

Figure 1: Empirical Estimates of d

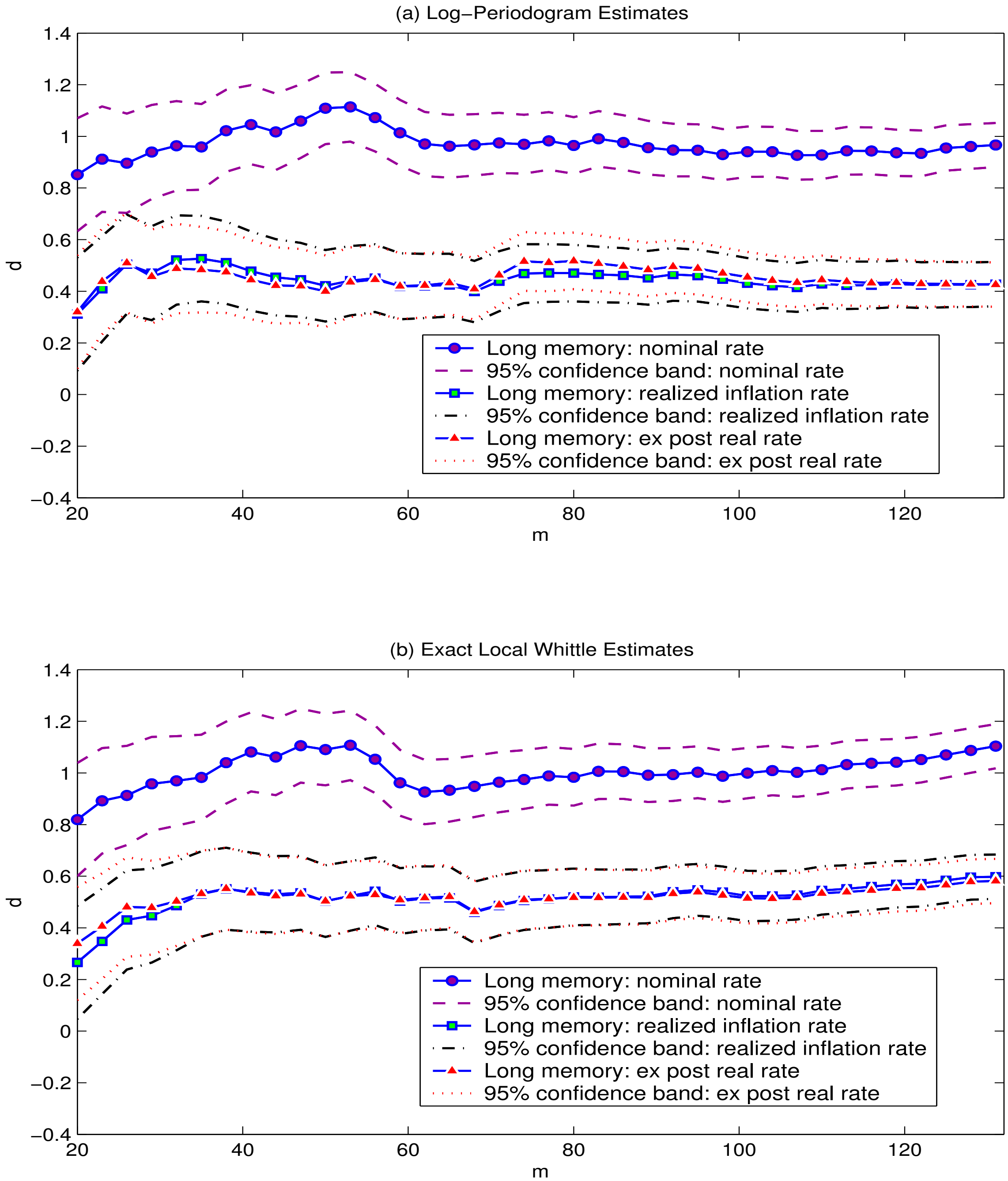


Figure 2: Expected and Realized Inflation

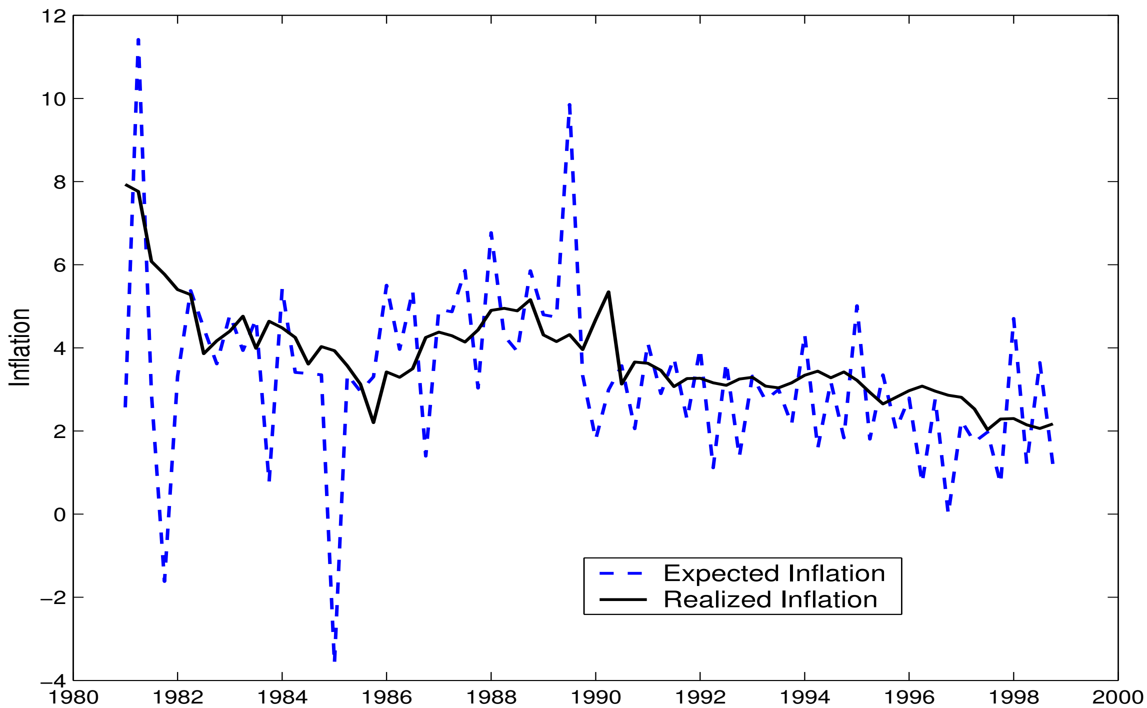
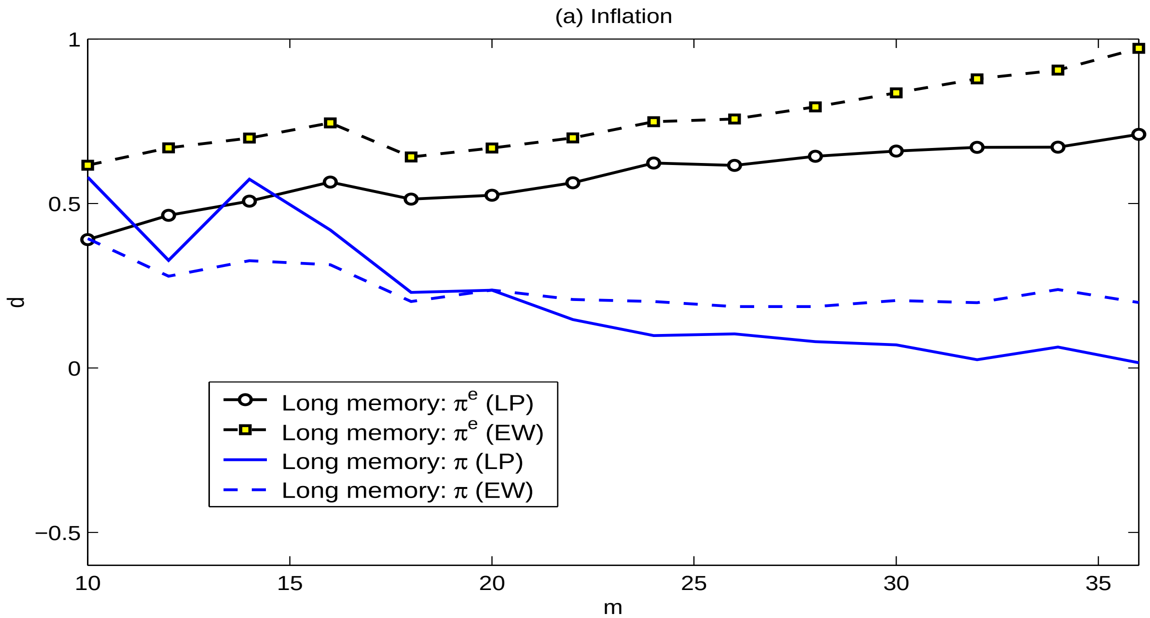


Figure 3: Empirical Estimates of d Based on Inflation Forecasts



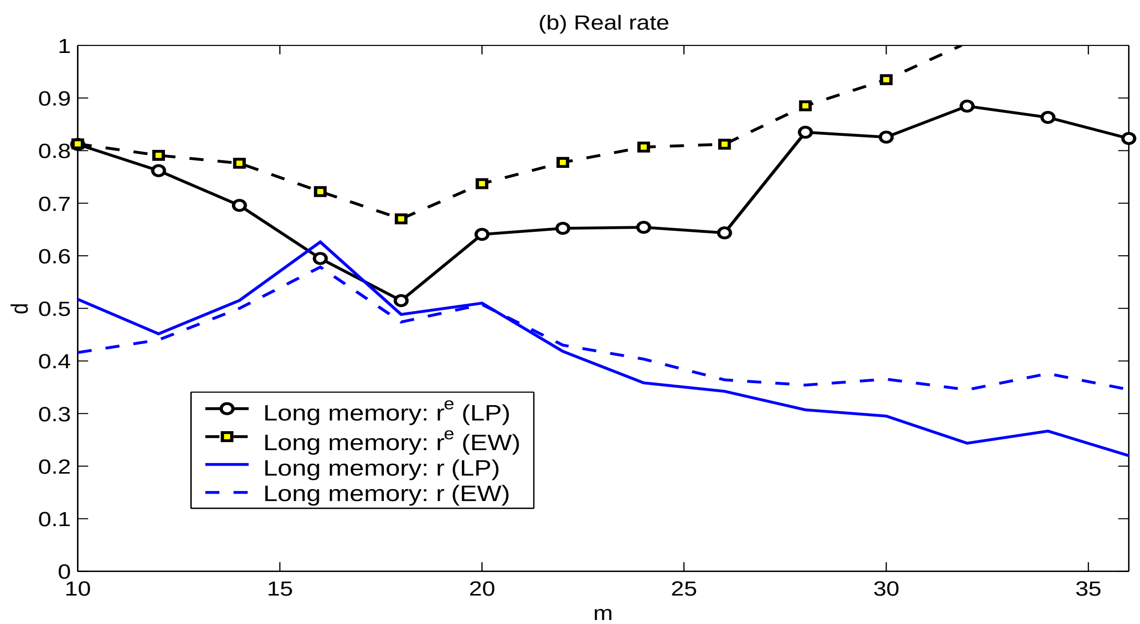


Figure 4: Empirical Estimates of d: BEW Estimator