Regression with an Evaporating Logarithmic Trend*

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Abstract

Linear regression on an intercept and an evaporating logarithmic trend is shown to be asymptotically collinear to the second order. Consistency results for least squares are given, rates of convergence are obtained and asymptotic normality is established for short memory errors.

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1. Problem

Part A

The time series X_t is generated by the model

$$X_t = \alpha + \frac{\beta}{\ln t} + u_t, \quad t = 2, ..., n \tag{1}$$

where α and β are unknown parameters whose least squares regression estimates are denoted by $\widehat{\alpha}$ and $\widehat{\beta}$, respectively. The error u_t in (1) is assumed to be *iid* $(0, \sigma^2)$ with finite fourth moment.

1. Show that $\widehat{\alpha}$ and $\widehat{\beta}$ are strongly consistent for α and β as $n \to \infty$.
2. Find the asymptotic distribution of $\widehat{\alpha}$ and $\widehat{\beta}$.

Part B

Suppose that u_t in (1) is the linear process

$$u_t = \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}, \quad \text{with } \sum_{j=0}^{\infty} j\,|c_j| < \infty, \tag{2}$$

and where ε_t is *iid* $(0, \sigma^2)$ with finite fourth moment. Explain how you would modify your derivations in Part A to allow for such an error process in the regression model (1).

2. Solution

Part A

Let $z_t = \frac{1}{\ln t} - \frac{1}{n-1}\sum_{t=2}^{n}\frac{1}{\ln t}$. Then

$$\begin{aligned}
\widehat{\beta} - \beta &= \frac{\sum_{t=2}^{n} z_t u_t}{\sum_{t=2}^{n} z_t^2}, \\
\widehat{\alpha} - \alpha &= \frac{\sum_{t=2}^{n} u_t}{n-1} - (\widehat{\beta} - \beta)\frac{\sum_{t=2}^{n} \ln^{-1} t}{n-1}.
\end{aligned}$$

We start with $\widehat{\beta}$ and find an asymptotic representation of the components $\sum_{t=2}^{n} 1/\ln t$ and $\sum_{t=2}^{n} 1/\ln^2 t$ that appear in $\sum_{t=2}^{n} z_t^2$. Using Euler summation (which justifies (3) below) and partial integration (which justifies (4) below) we obtain an asymptotic series representation as follows:

$$\begin{aligned}
\sum_{t=2}^{n}\frac{1}{\ln t} &= \int_2^n \frac{1}{\ln x}dx + O\left(1\right) &(3)\\
&= n\sum_{k=1}^{m}\frac{(k-1)!}{\ln^k n} - 2\sum_{k=1}^{m}\frac{(k-1)!}{\ln^k 2} + m!\int_2^n \frac{1}{\ln^{m+1} x}dx + O(1) &(4)\\
&= n\sum_{k=1}^{m}\frac{(k-1)!}{\ln^k n} + m!\int_2^n \frac{1}{\ln^{m+1} x}dx + O(1). &(5)
\end{aligned}$$

To show that (5) is a valid asymptotic series, we can ignore the $O(1)$ term and by a further application of partial integration to the second term we see that the remainder is proportional to

$$R_{m+1} = \int_2^n \frac{1}{\ln^{m+1} x}dx = \frac{n}{\ln^{m+1} n} - \frac{2}{\ln^{m+1} 2} + (m+1)\int_2^n \frac{1}{\ln^{m+2} x}dx. \quad (6)$$

Let $L = n^{\alpha}$ for some $\alpha \in (0,1)$ and write

$$\int_2^n \frac{1}{\ln^{m+2} x}dx = \int_2^L \frac{1}{\ln^{m+2} x}dx + \int_L^n \frac{1}{\ln^{m+2} x}dx = O\left(n^{\alpha}\right) + \int_L^n \frac{1}{\ln^{m+2} x}dx. \quad (7)$$

But

$$\int_L^n \frac{1}{\ln^{m+2} x}dx \leq \frac{1}{\ln L}\int_L^n \frac{1}{\ln^{m+1} x}dx \leq \frac{1}{\ln L}\int_2^n \frac{1}{\ln^{m+1} x}dx = \frac{1}{\ln L}R_{m+1}, \quad (8)$$

It follows from (6)-(8) that

$$\begin{aligned}
R_{m+1} &\leq \frac{n}{\ln^{m+1} n} + (m+1)\int_2^n \frac{1}{\ln^{m+2} x}dx \leq \frac{n}{\ln^{m+1} n} + \int_2^L \frac{m+1}{\ln^{m+2} x}dx + \frac{m+1}{\ln L}R_{m+1} \\
&= \frac{n}{\ln^{m+1} n} + \frac{m+1}{\ln L}R_{m+1} + O\left(n^{\alpha}\right)
\end{aligned}$$

so that

$$R_{m+1} = O\left(\frac{n}{\ln^{m+1} n}\right), \tag{9}$$

and thus

$$\sum_{t=2}^{n} \frac{1}{\ln t} = n \sum_{k=1}^{m} \frac{(k-1)!}{\ln^k n} + O(\frac{n}{\ln^{m+1} n}), \tag{10}$$

showing that (5) is a valid asymptotic series.

In a similar way we may establish that

$$\begin{aligned} \sum_{t=2}^{n} \frac{1}{\ln^2 t} &= \int_2^n \frac{1}{\ln^2 x} dx + O(1) && (11) \\ &= n \sum_{k=2}^{m} \frac{(k-1)!}{\ln^k n} + O\left(\frac{n}{\ln^{m+1} n}\right) && (12) \end{aligned}$$

is a valid asymptotic series. Combining (10) and (12), we get

$$\begin{aligned} \sum_{t=2}^{n} z_t^2 &= \sum_{t=2}^{n} \frac{1}{\ln^2 t} - \frac{1}{n-1} \left(\sum_{t=2}^{n} \frac{1}{\ln t}\right)^2 \\ &= \int_2^n \frac{1}{\ln^2 x} dx - \frac{1}{n-1} \left(\int_2^n \frac{1}{\ln x} dx\right)^2 + O(1) && (13) \\ &= \frac{n}{\ln^2 n} + \frac{2n}{\ln^3 n} + \frac{6n}{\ln^4 n} + O\left(\frac{n}{\ln^5 n}\right) && (14) \\ &\quad -\frac{1}{n} \left\{ \frac{n}{\ln n} + \frac{n}{\ln^2 n} + \frac{2n}{\ln^3 n} + O\left(\frac{n}{\ln^4 n}\right) \right\}^2 \\ &= \frac{n}{\ln^4 n} + O\left(\frac{n}{\ln^5 n}\right) \to_{a.s.} \infty, && (15) \end{aligned}$$

observing that we have to go to the third order terms in expansion (14) to avoid degeneracy. We can obtain higher order asymptotic expansions by taking the above process to further terms, leading to the following explicit expression to order $n/\ln^{10} n$

$$\sum_{t=2}^{n} z_t^2 = \frac{n}{\ln^4 n}\left(1 + \frac{8}{\ln n} + \frac{56}{\ln^2 n} + \frac{408}{\ln^3 n} + \frac{3228}{\ln^4 n} + \frac{28032}{\ln^5 n} + \frac{267264}{\ln^6 n}\right) + O\left(\frac{n}{\ln^{11} n}\right). \tag{16}$$

It is immediately apparent from the size of the coefficients in (16) that very large values of n are required before these approximations can be expected to work well. Of course, such approximations are hardly necessary since $\sum_{t=2}^{n} z_t^2$ is amenable to direct calculation or to direct approximation using the logarithmic integrals given in (3) and (11) above. The latter can be evaluated from the following well known series representation of the logarithmic integral (e.g., Gradshteyn and Ryzhik, 1967, 8.213.2, p. 926)

$$\mathrm{li}(y) = \int_0^y \frac{1}{\ln x} dx = \gamma + \ln \ln y + \sum_{k=1}^{\infty} \frac{(\ln y)^k}{k k!}, \quad \text{for } y > 1, \tag{17}$$

where γ is Euler's constant. In the present case we have

$$\int_2^n \frac{1}{\ln x} dx = \mathrm{li}\,(n) - \mathrm{li}\,(2)\,,$$

and

$$\int_2^n \frac{1}{\ln^2 x} dx = \frac{2}{\ln 2} - \frac{n}{\ln n} + \int_2^n \frac{1}{\ln x} dx = \frac{2}{\ln 2} - \frac{n}{\ln n} + [\mathrm{li}\,(n) - \mathrm{li}\,(2)]\,.$$

Observe that $\widehat{\beta} - \beta$ is the same as the error in the OLS estimator of β in the regression

$$X_t = z_t\beta + u_t,$$

where $\{u_t\}$ is a martingale difference sequence with respect to the natural filtration $\mathcal{F}_t$. The persistent excitation condition (15) holds in this regression and so $\widehat{\beta} \rightarrow_{a.s.} \beta$ and $\widehat{\beta}$ is strongly consistent.

Now observe that

$$\begin{aligned} \widehat{\alpha} - \alpha &= \frac{\sum_{t=2}^n u_t}{n-1} - (\widehat{\beta} - \beta)\frac{\sum_{t=2}^n \ln^{-1} t}{n-1} \\ &= o_{a.s.}(1) + o_{a.s.}(1) O_{a.s.}(\ln^{-1} n) \\ &= o_{a.s.}(1), \end{aligned}$$

and so $\widehat{\alpha} \rightarrow_{a.s.} \alpha$.

Next, turn to the asymptotic distribution of $\widehat{\alpha}$ and $\widehat{\beta}$. We have

$$\begin{aligned} \widehat{\beta} - \beta &= \frac{\sum_{t=2}^n z_t u_t}{\sum_{t=2}^n z_t^2} \\ &= \frac{\sum_{t=2}^n z_t u_t}{s_n} \left(\frac{\sqrt{n}}{\ln^2 n}\right)^{-1} (1 + o(1)),\ \ s_n^2 = \sum_{t=2}^n z_t^2. \end{aligned}$$

Let $y_{nt} = z_t u_t / s_n$, then $y_{n2,}\, y_{n_3}, ..., y_{nn}$ are independent random variables with zero means and variances that sum to σ^2. To apply the Liapounoff central limit theorem for $\sum_{t=2}^n y_{nt}$, we need to show that

$$\sum_{t=2}^n E|y_{nt}|^3 \rightarrow 0,\ \text{ as } n \rightarrow \infty.$$

But

$$\sum_{t=2}^n E|y_{nt}|^3 = \frac{\sum_{t=2}^n |z_t|^3 E|u_t|^3}{|s_n|^3} \leq \text{Const.} n \left(\frac{\ln^4 n}{n}\right)^{3/2} = o(1),$$

and so

$$\sum_{t=2}^n y_{nt} = \frac{\sum_{t=2}^n z_t u_t}{s_n} \Rightarrow N(0, \sigma^2).$$

Hence

$$\frac{\sqrt{n}}{\ln^2 n}(\widehat{\beta}-\beta)=\frac{\sum_{t=2}^{n} z_t u_t}{s_n}(1+o_p(1))\Rightarrow N(0,\sigma^2). \tag{18}$$

Note that

$$\begin{aligned}\widehat{\alpha}-\alpha &= \frac{\sum_{t=2}^{n} u_t}{n-1}-(\widehat{\beta}-\beta)\frac{\sum_{t=2}^{n}\ln^{-1} t}{n-1}\\ &= O_p(n^{-1/2})-O_p(n^{-1/2}\ln^2 n\ln^{-1} n)\\ &= O_p(n^{-1/2}\ln n).\end{aligned}$$

Thus, the term $(\widehat{\beta}-\beta)(n-1)^{-1}\sum_{t=2}^{n}\ln^{-1} t$ dominates the asymptotics of $\widehat{\alpha}$ and we deduce that

$$\sqrt{n}\ln^{-1} n(\widehat{\alpha}-\alpha)=-\frac{\sqrt{n}}{\ln^2 n}(\widehat{\beta}-\beta)+o_p(1)=_d N(0,\sigma^2). \tag{19}$$

In view of (18) and (19), we have the following joint asymptotics:

$$\begin{pmatrix} n^{1/2}\ln^{-1} n(\widehat{\alpha}-\alpha)\\ n^{1/2}\ln^{-2} n(\widehat{\beta}-\beta)\end{pmatrix}\Rightarrow N(0,\Sigma)\text{ with }\Sigma=\sigma^2\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}. \tag{20}$$

Remarks

1. The limit distribution (20) is singular and the components $n^{1/2}\ln^{-1} n(\widehat{\alpha}-\alpha)$ and $n^{1/2}\ln^{-2} n(\widehat{\beta}-\beta)$ are perfectly negatively correlated as $n\to\infty$. Observe that the rate of convergence of $\widehat{\alpha}$ exceeds that of $\widehat{\beta}$, by virtue of the fact that the signal from the intercept is stronger than the signal from the evaporating logarithmic regressor $\ln^{-1} t$.

2. Result (20) gives first order asymptotics. As is apparent from (16), asymptotic series expansions in the present model involve factors of $\frac{1}{\ln n}$ in contrast to the more usual $\frac{1}{\sqrt{n}}$ and, correspondingly, they deliver only very slow improvements in the first order asymptotics. The asymptotic variance $\frac{\sigma^2\ln^4 n}{n}$ and higher order approximations based on (16) are therefore poor approximations to the variance of $\widehat{\beta}$ even for quite large n. To illustrate, for values of the sample size $n\in[10^2,10^4]$ and for $\sigma^2=1$, Fig. 1 provides graphs of the exact variance of $\widehat{\beta}$, the asymptotic variance, the two-term and three-term series approximations to the variance based on the first few terms of (16), and direct calculation of the logarithmic integral representations of the variance obtained from (13) and (17). Apparently, only the latter are adequate for sample sizes n in this range.

3. The theory developed here is part of a general theory of regression on slowly varying regressors, a subject that has recently been studied in Phillips (2000).

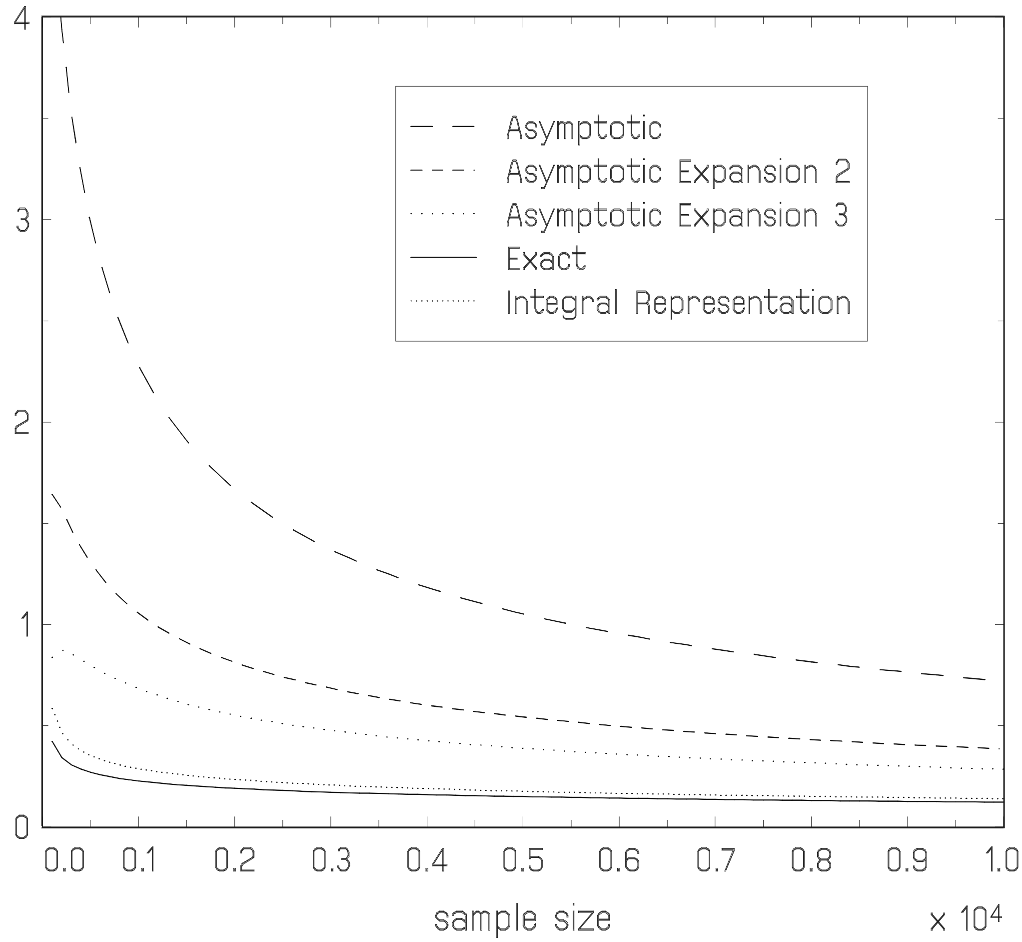

Figure 1: $\sigma^2 / \sum_{t=1}^{n} z_t^2$ and Asymptotic and Integral Approximations

Part B

Using the Phillips-Solo (1992) device, we have

$$u_t = C(L)\varepsilon_t = C(1)\varepsilon_t + \widetilde{\varepsilon}_{t-1} - \widetilde{\varepsilon}_t,$$

where $\widetilde{\varepsilon}_t = \widetilde{C}(L)\varepsilon_t$, $\widetilde{C}(L) = \sum_{u=0}^n \widetilde{C}_u L^u$, $\widetilde{C}_u = \sum_{s=u+1}^n c_s$. Then, we can write

$$\begin{aligned} \sum_{t=2}^n z_t u_t &= \sum_{t=2}^n z_t C(1)\varepsilon_t + \sum_{t=2}^n z_t \widetilde{\varepsilon}_{t-1} - \sum_{t=2}^n z_t \widetilde{\varepsilon}_t \\ &= C(1)\sum_{t=2}^n z_t\varepsilon_t + \sum_{t=2}^{n-1}(z_{t+1} - z_t)\widetilde{\varepsilon}_t + z_2\widetilde{\varepsilon}_1 - z_n\widetilde{\varepsilon}_n. \end{aligned}$$

It follows that

$$\widehat{\beta} - \beta = \frac{C(1)\sum_{t=2}^n z_t\varepsilon_t}{\sum_{t=2}^n z_t^2} + \frac{\sum_{t=2}^{n-1}(z_{t+1} - z_t)\widetilde{\varepsilon}_t}{\sum_{t=2}^n z_t^2} + \frac{z_2\widetilde{\varepsilon}_1}{\sum_{t=2}^n z_t^2} - \frac{z_n\widetilde{\varepsilon}_n}{\sum_{t=2}^n z_t^2}.$$

Note that the first term satisfies the limit theory given in the earlier part. If we prove that the last three terms converge strongly to zero, then we are done with the strong consistency of $\widehat{\beta}$. To obtain the asymptotic distribution, we need to consider $(\sum_{t=2}^n z_t^2)^{1/2}(\widehat{\beta} - \beta)$. We prove that the last three terms, so normalized, are $o_p(1)$, and then the asymptotic distribution is determined by the first term. The following three results are given first.

(a)

$$\frac{z_2\widetilde{\varepsilon}_1}{\sum_{t=2}^n z_t^2} = o_{a.s}(1) \text{ and } \frac{z_2\widetilde{\varepsilon}_1}{(\sum_{t=2}^n z_t^2)^{1/2}} = o_p(1).$$

Proof. These are immediate since $\sum_{t=2}^n z_t^2 \to \infty$ and $(\sum_{t=2}^n z_t^2)^{1/2} \to \infty$. ■

(b)

$$\frac{z_n\widetilde{\varepsilon}_n}{\sum_{t=2}^n z_t^2} = o_{a.s}(1) \text{ and } \frac{z_n\widetilde{\varepsilon}_n}{(\sum_{t=2}^n z_t^2)^{1/2}} = o_p(1).$$

Proof. Note that z_n is bounded and for any $\delta > 0$

$$\begin{aligned} P\left(\left|\frac{z_n\widetilde{\varepsilon}_n}{\sum_{t=2}^n z_t^2}\right| > \delta\right) &\le \frac{z_n^2 E(\widetilde{\varepsilon}_n^2)}{(\sum_{t=2}^n z_t^2)^2\delta^2} \\ &= O\left(\left(\sum_{t=2}^n z_t^2\right)^{-2}\right), \end{aligned}$$

so $\sum_n P(|\frac{z_n\widetilde{\varepsilon}_n}{\sum_{t=2}^n z_t^2}| > \delta) < \infty$ and the first result follows. The second follows because $P(|\frac{z_n\widetilde{\varepsilon}_n}{(\sum_{t=2}^n z_t^2)^{1/2}}| > \delta) = O((\sum_{t=2}^n z_t^2)^{-1}) \to 0$. ■

(c)

$$\frac{\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t}{\sum_{t=2}^{n} z_t^2} = o_{a.s}(1) \text{ and } \frac{\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t}{(\sum_{t=2}^{n} z_t^2)^{1/2}} = o_p(1)$$

Proof. Note that $\{\widetilde{\varepsilon}_t\}$ has finite fourth moment because

$$||\widetilde{\varepsilon}_t||_4 := \left(E(\widetilde{\varepsilon}_t^4)\right)^{1/4} = ||\sum_{j=0}^{t}\widetilde{C}_{t-j}\varepsilon_j||_4 \le \sum_{j=0}^{t}|\widetilde{C}_{t-j}|\cdot||\varepsilon_j||_4 < \infty.$$

Here we have employed the fact that $\sum_{j=0}^{t}|\widetilde{C}_{t-j}| < \infty$, which follows from the assumption that $\sum_{j=0}^{\infty} j\,|c_j| < \infty$.

Next,

$$\begin{aligned}
E\left(\sum_{t=2}^{n}\widetilde{\varepsilon}_t^2\right)^2 &= E\left(\sum_{t=2}^{n}\widetilde{\varepsilon}_t^4 + 2\sum_{\substack{s,\,t=2\\ s<t}}^{n}\widetilde{\varepsilon}_s^2\widetilde{\varepsilon}_t^2\right)\\
&\le O(n) + 2\sum_{\substack{s,\,t=2\\ s<t}}^{n}\left[E(\widetilde{\varepsilon}_s^4)\right]^{1/2}\left[E(\widetilde{\varepsilon}_t^4)\right]^{1/2} = O(n^2).
\end{aligned}$$

Therefore, for any $\delta > 0$,

$$\begin{aligned}
&P\left(|\frac{\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t}{\sum_{t=2}^{n} z_t^2}| > \delta\right)\\
\le\ & \frac{E(\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t)^4}{(\sum_{t=2}^{n} z_t^2)^4\delta^4} \le \frac{E\left[\sum_{t=2}^{n-1}(z_{t+1}-z_t)^2(\sum_{t=2}^{n-1}\widetilde{\varepsilon}_t^2)\right]^2}{(\sum_{t=2}^{n} z_t^2)^4\delta^4}\\
=\ & \frac{E\left(\sum_{t=2}^{n-1}(z_{t+1}-z_t)^2\right)^2\left(\sum_{t=2}^{n-1}\widetilde{\varepsilon}_t^2\right)^2}{(\sum_{t=2}^{n} z_t^2)^4\delta^4} = \frac{\left(\sum_{t=2}^{n-1}(z_{t+1}-z_t)^2\right)^2 O(n^2)}{(\sum_{t=2}^{n} z_t^2)^4\delta^4}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{t=2}^{n-1}(z_{t+1}-z_t)^2 &= \sum_{t=2}^{n-1}(\ln^{-1}(t+1)-\ln^{-1}t)^2\\
&= \sum_{t=2}^{n-1}\ln^2(1+\frac{1}{t})[\ln^{-2}t][\ln^{-2}(t+1)]\\
&< \text{Const.}\sum_{t=2}^{n}t^{-2} < \infty,
\end{aligned}$$

we have

$$P(|\frac{\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t}{\sum_{t=2}^{n} z_t^2}| > \delta) = O(n^{-2}\ln^8 n)$$

and so $\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t/\sum_{t=2}^{n} z_t^2 = o_{a.s}(1)$. Similarly,

$$\begin{aligned}
&P\left(\left|\frac{\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t}{(\sum_{t=2}^{n} z_t^2)^{1/2}}\right| > \delta\right) \\
\leq\ & \frac{E\left[\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t)\right]^2}{\sum_{t=2}^{n} z_t^2\delta^2} \\
\leq\ & \frac{E\sum_{t=2}^{n-1}(z_{t+1}-z_t)^2\widetilde{\varepsilon}_t^2 + 2\sum_{s>t}(|z_{t+1}-z_t||z_{s+1}-z_s|E|\widetilde{\varepsilon}_t\widetilde{\varepsilon}_s|)}{\sum_{t=2}^{n} z_t^2\delta^2} \\
\leq\ & \frac{E\sum_{t=2}^{n-1}(z_{t+1}-z_t)^2\widetilde{\varepsilon}_t^2 + 2\sum_{s>t}\left(|z_{t+1}-z_t||z_{s+1}-z_s|(E|\widetilde{\varepsilon}_t|^2)^{1/2}(E|\widetilde{\varepsilon}_s|^2)^{1/2}\right)}{\sum_{t=2}^{n} z_t^2\delta^2} \\
=\ & \text{Const.}\frac{(\sum_{t=2}^{n-1}|z_{t+1}-z_t|)^2}{\sum_{t=2}^{n} z_t^2\delta^2} = \text{Const.}\frac{\ln^2 n}{\sum_{t=2}^{n} z_t^2\delta^2} \to 0.
\end{aligned}$$

The last equality follows as $\sum_{t=2}^{n-1}|z_{t+1}-z_t| = O(\sum_{t=2}^{n} 1/t) = O(\ln n)$. Hence $(\sum_{t=2}^{n} z_t^2)^{-1/2}\left(\sum_{t=2}^{n-1}(z_{t+1}-z_t)\widetilde{\varepsilon}_t\right) = o_p(1)$. ■

Combining results (a), (b), and (c), we have

$$\widehat{\beta} - \beta = o_{a.s.}(1), \tag{21}$$

and then

$$(\sum_{t=2}^{n} z_t^2)^{1/2}(\widehat{\beta}-\beta) = (\sum_{t=2}^{n} z_t^2)^{1/2}\frac{C(1)\sum_{t=2}^{n} z_t\varepsilon_t}{\sum_{t=2}^{n} z_t^2} + o_p(1) =_d N(0, C(1)^2\sigma^2). \tag{22}$$

With (21), (22) and $\sum_{t=2}^{n} u_t/(n-1) \to_{a.s.} 0$ (e.g. Phillips and Solo, 1992), the previous arguments can now be repeated to obtain the strong consistency and the asymptotic distribution of $\widehat{\alpha}$. Specifically

$$\begin{pmatrix} n^{1/2}\ln^{-1} n(\widehat{\alpha}-\alpha) \\ n^{1/2}\ln^{-2} n(\widehat{\beta}-\beta) \end{pmatrix} \Rightarrow N(0,\Sigma) \text{ with } \Sigma = (C(1))^2\sigma^2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

3. References

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