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Online supplement to "Testing for Moderate Explosiveness"

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This supplement presents (i) proofs of the technical lemmas given in Appendix A and (ii) additional Monte Carlo simulation evidence.

SUPPLEMENT A: PROOFS OF THE TECHNICAL LEMMAS IN APPENDIX A

Proof of Lemma A.1: For the first part of the lemma, we use $\lim_{T\to\infty}\ln\left(1+1/T\right)^T=e$ to obtain

$$\rho_T^T=\left(1+\frac{c}{k_T}\right)^T=\left[\left(1+\frac{c}{k_T}\right)^{\frac{k_T}{c}}\right]^{\frac{cT}{k_T}}\to e^{\frac{cT}{k_T}},\text{ as }T\to\infty.$$

Therefore,

$$\frac{\rho_T^{-aT}}{\left(k_T/T\right)^b}=O\left(\frac{\left(T/k_T\right)^b}{e^{ac\left(T/k_T\right)}}\right)=o\left(1\right),$$

as desired. The second part of the lemma can be proved in the same way.□

Proof of Lemma A.2: Part (a). Using summation by parts, we have

$$\begin{aligned}&\left(k_T^{3/2}\rho_T^T\right)^{-1}\sum_{t=1}^T\sum_{j=t}^T\rho_T^{t-1-j}u_j=\left(k_T^{3/2}\rho_T^T\right)^{-1}\sum_{j=1}^T\left(\sum_{t=1}^j\rho_T^{t-1-j}\right)u_j\\&=\left(k_T^{3/2}\rho_T^T\right)^{-1}\sum_{j=1}^T\frac{\rho_T^j-1}{\rho_T^j\left(\rho_T-1\right)}u_j=\frac{1}{c}k_T^{-1/2}\rho_T^{-T}\sum_{j=1}^T\frac{\rho_T^j-1}{\rho_T^j}u_j.\qquad\text{(S.1)}\end{aligned}$$

Now

$$\begin{aligned}&E\left[\left(k_T^{-1/2}\rho_T^{-T}\sum_{j=1}^T\frac{\rho_T^j-1}{\rho_T^j}u_j\right)^2\right]=\sigma^2k_T^{-1}\rho_T^{-2T}\sum_{j=1}^T\left(1-\rho_T^{-j}\right)^2\\&\leq\sigma^2k_T^{-1}\rho_T^{-2T}\sum_{j=1}^T2\left(1+\rho_T^{-2j}\right)=O\left(Tk_T^{-1}\rho_T^{-2T}\right)=o\left(1\right),\end{aligned}$$

by Lemma A.1. Therefore, $\left(k_T^{3/2}\rho_T^T\right)^{-1}\sum_{t=1}^T\sum_{j=t}^T\rho_T^{t-1-j}u_j$ converges in mean-square to 0, and we obtain $\left(k_T^{3/2}\rho_T^T\right)^{-1}\sum_{t=1}^T\sum_{j=t}^T\rho_T^{t-1-j}u_j=o_p\left(1\right).$

Part (b). We write

$$\left(k_T^{3/2}\rho_T^{2T}\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{2(t-1)-j}u_j=\left(k_T^{3/2}\rho_T^{2T}\right)^{-1}\sum_{j=1}^{T}\left(\sum_{t=1}^{j}\rho_T^{2(t-1)-j}\right)u_j$$

$$=\left(k_T^{3/2}\rho_T^{2T}\right)^{-1}\sum_{j=1}^{T}\frac{\rho_T^{2j}-1}{\rho_T^{j}\left(\rho_T^2-1\right)}u_j=O\left(1\right)k_T^{-1/2}\rho_T^{-2T}\sum_{j=1}^{T}\frac{\rho_T^{2j}-1}{\rho_T^{j}}u_j. \tag{S.2}$$

Now, using Lemma A.1, we have

$$E\left[\left(k_T^{-1/2}\rho_T^{-2T}\sum_{j=1}^{T}\frac{\rho_T^{2j}-1}{\rho_T^{j}}u_j\right)^2\right]=\sigma^2k_T^{-1}\rho_T^{-4T}\sum_{j=1}^{T}\left(\rho_T^{j}-\rho_T^{-j}\right)^2$$

$$\leq 2\sigma^2k_T^{-1}\rho_T^{-4T}\sum_{j=1}^{T}\left(\rho_T^{2j}+\rho_T^{-2j}\right)=O\left(k_T^{-1}\rho_T^{-4T}\left(k_T\rho_T^{2T}+k_T\right)\right)=o\left(1\right).$$

Therefore, $\left(k_T^{3/2}\rho_T^{2T}\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{2(t-1)-j}u_j\to 0$ in mean-square, which implies that $\left(k_T^{3/2}\rho_T^{2T}\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{2(t-1)-j}u_j=o_p\left(1\right)$.□

Proof of Lemma A.3: We have

$$\left(k_T^{3/2}\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\xi_{t-1}=\left(k_T^{3/2}\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\left(\rho_T^{t-1}\xi_0+\sum_{j=1}^{t-1}\rho_T^{t-1-j}u_j\right)$$

$$=\xi_0\left(k_T^{3/2}\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\rho_T^{t-1}+\left(k_T^{3/2}\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\sum_{j=1}^{t-1}\rho_T^{t-1-j}u_j$$

$$=\left(k_T^{3/2}\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\rho_T^{t-1}\sum_{j=1}^{T}\rho_T^{-j}u_j-\left(k_T^{3/2}\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{t-1-j}u_j+o_p\left(1\right)$$

$$=\left(k_T^{3/2}\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\rho_T^{t-1}\sum_{j=1}^{T}\rho_T^{-j}u_j+o_p\left(1\right), \tag{S.3}$$

by $\xi_0=o_p\left(\sqrt{k_T}\right)$ and Lemma A.2(a). Now

$$\left(k_T^{3/2}\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\rho_T^{t-1}\sum_{j=1}^{T}\rho_T^{-j}u_j=k_T^{-1}\rho_T^{-T}\left(\sum_{t=1}^{T}\rho_T^{t-1}\right)Y_T+o_p\left(1\right)$$

$$=k_T^{-1}\rho_T^{-T}\left(\frac{\rho_T^{T}-1}{\rho_T-1}\right)Y_T+o_p\left(1\right)=\frac{1}{c}Y_T+o_p\left(1\right).$$

Combining the above two results completes the proof of the lemma.□

Proof of Lemma A.4: Part (a). Note that

$$E\left|\left(k_T\rho_T^{T}\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{t-1-j}u_ju_t\right|$$

$$\leq \left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{t-1-j}E\left|u_ju_t\right| \leq \left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{t-1-j}\left(Eu_j^2\right)^{1/2}\left(Eu_t^2\right)^{1/2}$$

$$= var\left(u\right)\left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\frac{1-\rho_T^{t-T-1}}{\rho_T-1} = \frac{var\left(u\right)}{c}\rho_T^{-T}\sum_{t=1}^{T}\left(1-\rho_T^{t-T-1}\right)$$

$$= \frac{var\left(u\right)}{c}\left(T\rho_T^{-T}-\rho_T^{-T}\sum_{t=1}^{T}\rho_T^{t-T-1}\right) = \frac{var\left(u\right)}{c}\left(T\rho_T^{-T}-\rho_T^{-2T}\frac{\rho_T^T-1}{\rho_T-1}\right) = o\left(1\right),$$

by Lemma A.1. Part (a) follows, as convergence in L^1 implies convergence in probability.

Part (b). Under Assumption 3.1, (S.1) still holds, and so

$$\left(k_T^{3/2}\rho_T^T\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{t-1-j}u_j = \frac{1}{c}k_T^{-1/2}\rho_T^{-T}\sum_{j=1}^{T}\frac{\rho_T^j-1}{\rho_T^j}u_j$$

$$= \frac{1}{c}k_T^{-1/2}\rho_T^{-T}\sum_{j=1}^{T}u_j - \frac{1}{c}k_T^{-1/2}\rho_T^{-T}\sum_{j=1}^{T}\rho_T^{-j}u_j$$

$$= O_p\left(\sqrt{T/k_T}\rho_T^{-T}\right) + O_p\left(\rho_T^{-T}\right) = o_p\left(1\right),$$

by Lemmas 3.1(b&c) and A.1.

Part (c). Under Assumption 3.1, (S.2) still holds, and we have

$$\left(k_T^{3/2}\rho_T^{2T}\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{2(t-1)-j}u_j = O(1)k_T^{-1/2}\rho_T^{-2T}\sum_{j=1}^{T}\left(\rho_T^j-\rho_T^{-j}\right)u_j$$

$$= O\left(1\right)k_T^{-1/2}\rho_T^{-T+1}\sum_{j=1}^{T}\rho_T^{-(T-j)-1}u_j + O\left(1\right)k_T^{-1/2}\rho_T^{-2T}\sum_{j=1}^{T}\rho_T^{-j}u_j$$

$$= O_p\left(\rho_T^{-T+1}\right) + O_p\left(\rho_T^{-2T}\right) = o_p\left(1\right),$$

by Lemmas 3.1(a–c).□

Proof of Lemma A.5: We prove Part (c) first.

$$\left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\xi_{t-1}u_t = \left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\left(\rho_T^{t-1}\xi_0+\sum_{j=1}^{t-1}\rho_T^{t-1-j}u_j\right)u_t$$

$$= \xi_0k_T^{-1}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}u_t + \left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\sum_{j=1}^{t-1}\rho_T^{t-1-j}u_ju_t$$

$$= \left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\sum_{j=1}^{T}\rho_T^{t-1-j}u_ju_t - \left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\sum_{j=t}^{T}\rho_T^{t-1-j}u_ju_t + o_p\left(1\right)$$

$$= \left(k_T\rho_T^T\right)^{-1}\sum_{t=1}^{T}\sum_{j=1}^{T}\rho_T^{t-1-j}u_ju_t + o_p\left(1\right)$$

$$= \left[k_T^{-1/2}\sum_{t=1}^{T}\rho_T^{-(T-t)-1}u_t\right]\left[k_T^{-1/2}\sum_{j=1}^{T}\rho_T^{-j}u_j\right] + o_p\left(1\right) = \tilde{X}_T\tilde{Y}_T + o_p\left(1\right),$$

where we have used $\xi_0 = o_p\left(\sqrt{k_T}\right)$ and Lemmas 3.1 and A.4(a).

Part (a). By squaring $\xi_t = \rho_T \xi_{t-1} + u_t$, we have

$$\xi_t^2 - \xi_{t-1}^2 = \left(\rho_T^2 - 1\right)\xi_{t-1}^2 + 2\rho_T \xi_{t-1} u_t + u_t^2,$$

that is,

$$\left(\rho_T^2 - 1\right)\xi_{t-1}^2 = \xi_t^2 - \xi_{t-1}^2 - 2\rho_T \xi_{t-1} u_t - u_t^2.$$

So

$$\left(\rho_T^2 - 1\right)\sum_{t=1}^T \xi_{t-1}^2 = \xi_T^2 - \xi_0^2 - 2\rho_T \sum_{t=1}^T \xi_{t-1} u_t - \sum_{t=1}^T u_t^2.$$

Using Part (c), we now have

$$\begin{aligned}
&\left(k_T \rho_T^T\right)^{-2} \sum_{t=1}^T \xi_{t-1}^2 = \frac{1}{k_T^2 \rho_T^{2T}\left(\rho_T^2 - 1\right)}\xi_T^2 + o_p(1) \\
&= \frac{1}{k_T^2 \rho_T^{2T}\left(\rho_T^2 - 1\right)}\left(\rho_T^T \xi_0 + \sum_{j=1}^T \rho_T^{T-j} u_j\right)^2 + o_p(1) \\
&= \frac{1}{k_T\left(\rho_T^2 - 1\right)}\left(k_T^{-1/2}\sum_{j=1}^T \rho_T^{-j} u_j\right)^2 + o_p(1) = \frac{1}{2c}\tilde{Y}_T^2 + o_p(1).
\end{aligned}$$

Part (b). The proof is similar to that of Lemma A.3. According to Lemma A.4(b), equation (S.3) still holds under Assumption 3.1. So

$$\begin{aligned}
&\left(k_T^{3/2}\rho_T^T\right)^{-1}\sum_{t=1}^T \xi_{t-1} = \left(k_T^{3/2}\rho_T^T\right)^{-1}\sum_{t=1}^T \rho_T^{t-1}\sum_{j=1}^T \rho_T^{-j} u_j + o_p(1) \\
&= k_T^{-1}\rho_T^{-T}\left(\sum_{t=1}^T \rho_T^{t-1}\right)\tilde{Y}_T + o_p(1) = \frac{1}{c}\tilde{Y}_T + o_p(1).
\end{aligned}$$

□

Proof of Lemma A.6: Part (a). Since $\phi_\ell(\cdot)$ is bounded, we have

$$\left|\frac{1}{k_T\rho_T^T}\sum_{t=1}^T \phi_\ell\left(\frac{t}{T}\right)\rho_T^t\right| = O\left(\frac{1}{k_T\rho_T^T}\sum_{t=1}^T \rho_T^t\right) = O\left(\frac{1}{k_T\rho_T^T}k_T\rho_T^T\right) = O(1),$$

where we have used $\sum_{t=1}^T \rho_T^t = O\left(k_T\rho_T^T\right)$, which holds by elementary calculations.

Part (b). Since

$$\begin{aligned}
&\sum_{t=1}^T \phi_\ell\left(\frac{t}{T}\right)\sum_{j=t}^T \rho_T^{t-1-j}u_j = \sum_{j=1}^T\left(\sum_{t=1}^j \phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)u_j \\
&= \sum_{j=1}^T\left(\sum_{t=1}^j \phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)C(1)\varepsilon_j + \sum_{j=1}^T\left(\sum_{t=1}^j \phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\left(\tilde{\varepsilon}_{j-1} - \tilde{\varepsilon}_j\right) \\
&= \sum_{j=1}^T\left(\sum_{t=1}^j \phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)C(1)\varepsilon_j + \phi_\ell\left(\frac{1}{T}\right)\rho_T^{-1}\tilde{\varepsilon}_0 - \sum_{t=1}^T \phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-T}\tilde{\varepsilon}_T
\end{aligned}$$

$$+\sum_{j=1}^{T-1}\left[(\rho_T^{-1}-1)\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}+\phi_\ell\left(\frac{j+1}{T}\right)\rho_T^{-1}\right]\tilde{\varepsilon}_j$$

$$=\sum_{j=1}^{T}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)C(1)\varepsilon_j+O\left(k_T^{-1}\sum_{j=1}^{T-1}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\tilde{\varepsilon}_j\right)+o_p\left(\sqrt{T}k_T\rho_T^T\right),$$

it suffices to show that

$$\sum_{j=1}^{T}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\varepsilon_j=o_p\left(\sqrt{T}k_T\rho_T^T\right),$$

and

$$k_T^{-1}\sum_{j=1}^{T-1}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\tilde{\varepsilon}_j=o_p\left(\sqrt{T}k_T\rho_T^T\right).$$

But

$$\begin{aligned}
&var\left[\sum_{j=1}^{T}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\varepsilon_j\right]\\
&=O(1)\sum_{j=1}^{T}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)^2=O(1)\sum_{j=1}^{T}\left(\sum_{t=1}^{j}\rho_T^{t-1-j}\right)^2\\
&=O\left(k_T^2\right)\sum_{j=1}^{T}\left(1-\rho_T^{-j}\right)^2=O\left(Tk_T^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&var\left[\sum_{j=1}^{T-1}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\tilde{\varepsilon}_j\right]\\
&=\sum_{j=1}^{T-1}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)^2 var\left(\tilde{\varepsilon}_j\right)\\
&\quad+2\sum_{j<s}^{T-1}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\left(\sum_{t=1}^{s}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-s}\right)cov\left(\tilde{\varepsilon}_j,\tilde{\varepsilon}_s\right)\\
&=O\left(k_T^2\right)\sum_{j=1}^{T-1}\left(1-\rho_T^{-j}\right)^2\\
&\quad+\left[2\sum_{j<s}^{T-1}\sum_{k=0}^{\infty}\tilde{c}_k\tilde{c}_{(s-j)+k}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\left(\sum_{t=1}^{s}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-s}\right)\right]var\left(\varepsilon_j\right)\\
&=O\left(Tk_T^2\right)+O\left(\sum_{j<s}^{T-1}\left(\sum_{k=0}^{\infty}\tilde{c}_k^2\right)\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\left(\sum_{t=1}^{s}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-s}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= O\left(Tk_T^2\right) + O\left(\left[\sum_{j=1}^{T-1}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\right]^2\right) \\
&= O\left(Tk_T^2\right) + O\left(\left[\sum_{j=1}^{T-1}\left(\sum_{t=1}^{j}\rho_T^{t-1-j}\right)\right]^2\right) \\
&= O\left(Tk_T^2\right) + O\left(T^2k_T^2\right) = O\left(T^2k_T^2\right).
\end{aligned}
$$

So

$$
\sum_{j=1}^{T}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\varepsilon_j = O_p\left(\sqrt{T}k_T\right) = o_p\left(\sqrt{T}k_T\rho_T^T\right),
$$

and

$$
k_T^{-1}\sum_{j=1}^{T-1}\left(\sum_{t=1}^{j}\phi_\ell\left(\frac{t}{T}\right)\rho_T^{t-1-j}\right)\tilde{\varepsilon}_j = O_p\left(T\right) = o_p\left(\sqrt{T}k_T\rho_T^T\right),
$$

as desired.□

SUPPLEMENT B: MONTE CARLO SIMULATION EVIDENCE

As mentioned in the main text, we conduct two sets of Monte Carlo simulations. The first set is based on i.i.d. errors while the second set is based on weakly dependent errors. The main text has reported and discussed the simulation results for the case with $\alpha = 0.5$ and $T = 100$. In this section, we consider additional combinations of α and T, and provide more evidence for the conclusions given in the main text.

Tables S.1–S.3 report the empirical size and power results of the t_{PM}, t_{WY}, and t_{MED} tests under i.i.d. errors for $\alpha = \{0.3, 0.5, 0.8\}$ and $T \in \{100, 150\}$. First, as is clear from the tables, the size performance of the three tests based on i.i.d. Gaussian errors is qualitatively similar to that based on i.i.d. uniform errors. For example, the empirical size of the t_{PM} test is 5.0% in the case wherein $\alpha = 0.3$, $T = 100$, and $\mu_T = 0$ under i.i.d. Gaussian errors, while the corresponding size under i.i.d. uniform errors is 4.7%. Similarly, the t_{MED} test has a size of 5.1% under i.i.d. Gaussian errors and 5.3% under i.i.d. uniform errors when $\alpha = 0.5$, $T = 150$, and $\mu_T = T^{-\alpha/4}$. This is in line with our theoretical analysis that normality of the errors is not necessary for these tests.

The second feature is that when $\mu_T \neq 0$, both the t_{WY} test and t_{MED} test have quite accurate size. Take the case with $\alpha = 0.5$, $T = 100$, $\mu_T = T^{-\alpha/4}$, and $u_t \sim i.i.d.N(0,1)$ as an example. The empirical size of the t_{MED} test is 5.5%, whereas the corresponding size of the t_{WY} test is 5.6%. Note that the asymptotic distribution of the t_{WY} statistic is simulated by employing the true parameter values. The standard normal approximation to the distribution of the t_{MED} statistic appears to be very accurate.

When $\mu_T = 0$, as Tables S.1–S.3 show, the t_{PM} test has satisfactory size performance for at least the cases with $\alpha = 0.3$ and $\alpha = 0.5$. This is expected, as the t_{PM} statistic is based on a regression without an intercept. In such cases, we observe that the size performance of the t_{MED} test is not worse than that of the t_{PM} test. For example, when $\alpha = 0.5$, $T = 150$, and $\mu_T = 0$, the null rejection probabilities of the t_{PM} and t_{MED} tests are around 5% for both Gaussian and uniform errors. We also notice that these two tests have some size distortion when α is large and close to 1. This is not surprising because when $\alpha \to 1$, the ME root $\rho_T = 1 + 1/T^{\alpha}$ will approach a near unit root, a scenario that is not accommodated by our asymptotic theory. But the size distortion decreases as T increases or as μ_T departs farther away from zero. In fact, as μ_T becomes larger, the $t_{\rm PM}$ test suffers from increasing size distortion while the $t_{\rm MED}$ test enjoys a good size control. For example, the empirical size of the $t_{\rm MED}$ test is 5.4% when $\alpha = 0.8$, $T = 150$, $\mu_T = 1$, and $u_t \sim i.i.d.U(-\sqrt{3}, \sqrt{3})$, which is much closer to the nominal level than that of the $t_{\rm PM}$ test. According to the size accuracy, the $t_{\rm MED}$ test dominates the $t_{\rm PM}$ test.

Finally, the $t_{\rm MED}$ test is more powerful than the $t_{\rm PM}$ test in our simulation experiments. For example, when $\alpha = 0.8$, $T = 150$, $\mu_T = T^{-\alpha/4}$, and $u_t \sim i.i.d.U(-\sqrt{3}, \sqrt{3})$, the size-adjusted power of the $t_{\rm PM}$ test is 71.0% while that of the $t_{\rm MED}$ test is 75.5%. As α decreases, the local-to-unity alternative departs more from the null of moderate explosiveness, and the power of the tests approaches 100%. This explains why the $t_{\rm PM}$ and $t_{\rm MED}$ tests always reject when $\alpha = 0.3$ and 0.5. Our simulation evidence clearly shows that the $t_{\rm MED}$ test outperforms the $t_{\rm PM}$ test in terms of both size accuracy and and power performance.

Tables S.4–S.6 report the empirical size and power results of the $t_{\rm MED}$ and $\tilde{t}_{\rm MED}$ tests under both the AR design and MA design. The results for the sample size $T = 100$ are similar to those for $T = 150$. In view of the size accuracy, the $t_{\rm MED}$ test performs well when $\theta = 0.00$, as there is no autocorrelation. However, this test has large size distortion when θ is different from 0. The size distortion increases significantly as θ becomes larger. In contrast, the size distortion of the $\tilde{t}_{\rm MED}$ test is substantially smaller than that of the standard $t_{\rm MED}$ test. For example, in the case wherein $\alpha = 0.3$, $T = 150$, and $\theta = 0.75$, the size results of $\tilde{t}_{\rm MED}$ are 4.0% under the AR design and 5.9% under the MA design, respectively, both of which are quite smaller than 36.3% and 15.4%, the corresponding size levels of $t_{\rm MED}$. Other parameter configurations also lead to the observation that the $\tilde{t}_{\rm MED}$ test is more accurate and is therefore preferred when the errors are serially correlated. This result is consistent with our asymptotic theory. Ignoring the autocorrelation leads to an inaccurate test.

Tables S.4–S.6 show that the size-adjusted power of the $\tilde{t}_{\rm MED}$ test is close to that of the $t_{\rm MED}$ test in both the AR and MA cases. Take the case with $\alpha = 0.5$, $T = 100$, and $\theta = 0.75$ as an example. The $\tilde{t}_{\rm MED}$ test has a power of 97.1% under the AR design, whereas the corresponding power of the $t_{\rm MED}$ test is 99.3%. Under the MA design, the power level of both the $\tilde{t}_{\rm MED}$ and $t_{\rm MED}$ tests reaches 100%. Given these observations, we can conclude that the $\tilde{t}_{\rm MED}$ test achieves size accuracy with only very small power loss.

Table S.1. Size and power under i.i.d. errors: the case with $\alpha = 0.3$.

		Size ($\rho = 1 + 1/T^{\alpha}$)			Power ($\rho = 1 + 1/T$)	
		t_{PM}	t_{WY}	t_{MED}	t_{PM}	t_{MED}
$T = 100$			(a) i.i.d. Gaussian errors			
	$\mu_T = 0$	0.050	0.056	0.055	1.000	1.000
	$\mu_T = T^{-\alpha/2}$	0.254	0.058	0.055	1.000	1.000
	$\mu_T = T^{-\alpha/4}$	0.397	0.060	0.055	1.000	1.000
	$\mu_T = 1$	0.601	0.058	0.055	1.000	1.000
			(b) i.i.d. uniform errors			
	$\mu_T = 0$	0.047	0.049	0.046	1.000	1.000
	$\mu_T = T^{-\alpha/2}$	0.251	0.049	0.046	1.000	1.000
	$\mu_T = T^{-\alpha/4}$	0.393	0.048	0.046	1.000	1.000
	$\mu_T = 1$	0.596	0.048	0.046	1.000	1.000
$T = 150$			(a) i.i.d. Gaussian errors			
	$\mu_T = 0$	0.050	0.052	0.051	1.000	1.000
	$\mu_T = T^{-\alpha/2}$	0.259	0.052	0.052	1.000	1.000
	$\mu_T = T^{-\alpha/4}$	0.428	0.053	0.051	1.000	1.000
	$\mu_T = 1$	0.663	0.057	0.052	1.000	1.000
			(b) i.i.d. uniform errors			
	$\mu_T = 0$	0.051	0.061	0.054	1.000	1.000
	$\mu_T = T^{-\alpha/2}$	0.253	0.057	0.054	1.000	1.000
	$\mu_T = T^{-\alpha/4}$	0.420	0.058	0.054	1.000	1.000
	$\mu_T = 1$	0.652	0.055	0.053	1.000	1.000

Note: This table reports the empirical size and size-adjusted power of 5% tests with 5,000 Monte Carlo replications. In the i.i.d. Gaussian group, $u_t \sim i.i.d.N(0,1)$, while in the i.i.d. uniform group, $u_t \sim i.i.d.U(-\sqrt{3}, \sqrt{3})$. Different parameter combinations are configured to conduct simulations for the null of moderate explosiveness $\rho = 1 + 1/T^{\alpha}$ against the alternative of local-to-unity $\rho = 1 + 1/T$.

Table S.2. Size and power under i.i.d. errors: the case with $\alpha = 0.5$.

		Size ($\rho = 1 + 1/T^{\alpha}$)			Power ($\rho = 1 + 1/T$)	
		t_{PM}	t_{WY}	t_{MED}	t_{PM}	t_{MED}
$T = 100$			(a) i.i.d. Gaussian errors			
	$\mu_T = 0$	0.050	0.053	0.056	1.000	1.000
	$\mu_T = T^{-\alpha/2}$	0.277	0.058	0.055	1.000	1.000
	$\mu_T = T^{-\alpha/4}$	0.641	0.056	0.055	1.000	1.000
	$\mu_T = 1$	0.969	0.058	0.055	1.000	1.000
			(b) i.i.d. uniform errors			
	$\mu_T = 0$	0.049	0.047	0.049	1.000	1.000
	$\mu_T = T^{-\alpha/2}$	0.276	0.049	0.049	1.000	1.000
	$\mu_T = T^{-\alpha/4}$	0.637	0.047	0.048	1.000	1.000
	$\mu_T = 1$	0.973	0.047	0.048	1.000	1.000
$T = 150$			(a) i.i.d. Gaussian errors			
	$\mu_T = 0$	0.050	0.056	0.051	1.000	1.000
	$\mu_T = T^{-\alpha/2}$	0.289	0.054	0.051	1.000	1.000
	$\mu_T = T^{-\alpha/4}$	0.689	0.053	0.051	1.000	1.000
	$\mu_T = 1$	0.992	0.053	0.051	1.000	1.000
			(b) i.i.d. uniform errors			
	$\mu_T = 0$	0.051	0.053	0.053	1.000	1.000
	$\mu_T = T^{-\alpha/2}$	0.276	0.053	0.053	1.000	1.000
	$\mu_T = T^{-\alpha/4}$	0.678	0.053	0.053	1.000	1.000
	$\mu_T = 1$	0.991	0.053	0.053	1.000	1.000

Note: This table reports the empirical size and size-adjusted power of 5% tests with 5,000 Monte Carlo replications. In the i.i.d. Gaussian group, $u_t \sim i.i.d.N(0,1)$, while in the i.i.d. uniform group, $u_t \sim i.i.d.U(-\sqrt{3}, \sqrt{3})$. Different parameter combinations are configured to conduct simulations for the null of moderate explosiveness $\rho = 1 + 1/T^{\alpha}$ against the alternative of local-to-unity $\rho = 1 + 1/T$.

Table S.3. Size and power under i.i.d. errors: the case with $\alpha = 0.8$.

		Size ($\rho = 1 + 1/T^{\alpha}$)			Power ($\rho = 1 + 1/T$)	
		t_{PM}	t_{WY}	t_{MED}	t_{PM}	t_{MED}
$T = 100$				(a) i.i.d. Gaussian errors		
	$\mu_T = 0$	0.118	0.256	0.261	0.133	0.142
	$\mu_T = T^{-\alpha/2}$	0.180	0.121	0.123	0.175	0.196
	$\mu_T = T^{-\alpha/4}$	0.795	0.048	0.048	0.407	0.553
	$\mu_T = 1$	1.000	0.052	0.051	0.978	0.999
				(b) i.i.d. uniform errors		
	$\mu_T = 0$	0.113	0.241	0.257	0.136	0.142
	$\mu_T = T^{-\alpha/2}$	0.181	0.104	0.111	0.161	0.194
	$\mu_T = T^{-\alpha/4}$	0.807	0.044	0.047	0.399	0.527
	$\mu_T = 1$	1.000	0.048	0.053	0.982	1.000
$T = 150$				(a) i.i.d. Gaussian errors		
	$\mu_T = 0$	0.118	0.240	0.246	0.207	0.192
	$\mu_T = T^{-\alpha/2}$	0.202	0.109	0.114	0.257	0.257
	$\mu_T = T^{-\alpha/4}$	0.883	0.051	0.054	0.695	0.741
	$\mu_T = 1$	1.000	0.051	0.055	1.000	1.000
				(b) i.i.d. uniform errors		
	$\mu_T = 0$	0.118	0.235	0.235	0.200	0.196
	$\mu_T = T^{-\alpha/2}$	0.197	0.114	0.114	0.273	0.282
	$\mu_T = T^{-\alpha/4}$	0.880	0.052	0.054	0.710	0.755
	$\mu_T = 1$	1.000	0.053	0.054	1.000	1.000

Note: This table reports the empirical size and size-adjusted power of 5% tests with 5,000 Monte Carlo replications. In the i.i.d. Gaussian group, $u_t \sim i.i.d.N(0,1)$, while in the i.i.d. uniform group, $u_t \sim i.i.d.U(-\sqrt{3}, \sqrt{3})$. Different parameter combinations are configured to conduct simulations for the null of moderate explosiveness $\rho = 1 + 1/T^{\alpha}$ against the alternative of local-to-unity $\rho = 1 + 1/T$.

Table S.4. Size and power in the presence of autocorrelated errors: the case with $\alpha = 0.3$.

		Size ($\rho = 1 + 1/T^{\alpha}$)		Power ($\rho = 1 + 1/T$)	
		t_{MED}	$\tilde{t}_{\mathrm{MED}}$	t_{MED}	$\tilde{t}_{\mathrm{MED}}$
$T = 100$			(a) AR design		
	$\theta = 0.00$	0.056	0.059	1.000	1.000
	$\theta = 0.25$	0.123	0.061	1.000	1.000
	$\theta = 0.50$	0.221	0.054	1.000	1.000
	$\theta = 0.75$	0.356	0.045	1.000	1.000
			(b) MA design		
	$\theta = 0.00$	0.055	0.059	1.000	1.000
	$\theta = 0.25$	0.108	0.056	1.000	1.000
	$\theta = 0.50$	0.143	0.058	1.000	1.000
	$\theta = 0.75$	0.155	0.056	1.000	1.000
$T = 150$			(a) AR design		
	$\theta = 0.00$	0.052	0.055	1.000	1.000
	$\theta = 0.25$	0.119	0.056	1.000	1.000
	$\theta = 0.50$	0.215	0.053	1.000	1.000
	$\theta = 0.75$	0.363	0.040	1.000	1.000
			(b) MA design		
	$\theta = 0.00$	0.051	0.056	1.000	1.000
	$\theta = 0.25$	0.103	0.055	1.000	1.000
	$\theta = 0.50$	0.144	0.055	1.000	1.000
	$\theta = 0.75$	0.154	0.059	1.000	1.000

Note: This table reports the empirical size and size-adjusted power of 5% tests with 5,000 Monte Carlo replications. In the AR design, $u_t = \theta u_{t-1} + \sqrt{1-\theta^2} e_{1,t}$, while in the MA design, $u_t = \theta e_{2,t-1} + \sqrt{1-\theta^2} e_{2,t}$, where $e_{1,t} \sim i.i.d.N(0,1)$ and $e_{2,t} \sim i.i.d.N(0,1)$. Different parameter combinations are configured to conduct simulations for the null of moderate explosiveness $\rho = 1 + 1/T^{\alpha}$ against the alternative of local-to-unity $\rho = 1 + 1/T$.

Table S.5. Size and power in the presence of autocorrelated errors: the case with $\alpha = 0.5$.

		Size ($\rho = 1 + 1/T^{\alpha}$)		Power ($\rho = 1 + 1/T$)	
		t_{MED}	$\tilde{t}_{\text{MED}}$	t_{MED}	$\tilde{t}_{\text{MED}}$
$T = 100$		(a) AR design			
	$\theta = 0.00$	0.053	0.057	1.000	1.000
	$\theta = 0.25$	0.134	0.068	1.000	1.000
	$\theta = 0.50$	0.242	0.070	1.000	0.998
	$\theta = 0.75$	0.413	0.081	0.993	0.971
		(b) MA design			
	$\theta = 0.00$	0.055	0.060	1.000	1.000
	$\theta = 0.25$	0.114	0.060	1.000	1.000
	$\theta = 0.50$	0.157	0.066	1.000	1.000
	$\theta = 0.75$	0.169	0.066	1.000	1.000
$T = 150$		(a) AR design			
	$\theta = 0.00$	0.052	0.053	1.000	1.000
	$\theta = 0.25$	0.127	0.064	1.000	1.000
	$\theta = 0.50$	0.243	0.064	1.000	0.999
	$\theta = 0.75$	0.430	0.074	0.998	0.996
		(b) MA design			
	$\theta = 0.00$	0.051	0.055	1.000	1.000
	$\theta = 0.25$	0.111	0.061	1.000	1.000
	$\theta = 0.50$	0.155	0.063	1.000	1.000
	$\theta = 0.75$	0.169	0.062	1.000	1.000

Note: This table reports the empirical size and size-adjusted power of 5% tests with 5,000 Monte Carlo replications. In the AR design, $u_t = \theta u_{t-1} + \sqrt{1-\theta^2} e_{1,t}$, while in the MA design, $u_t = \theta e_{2,t-1} + \sqrt{1-\theta^2} e_{2,t}$, where $e_{1,t} \sim i.i.d.N(0,1)$ and $e_{2,t} \sim i.i.d.N(0,1)$. Different parameter combinations are configured to conduct simulations for the null of moderate explosiveness $\rho = 1 + 1/T^{\alpha}$ against the alternative of local-to-unity $\rho = 1 + 1/T$.

Table S.6. Size and power in the presence of autocorrelated errors: the case with $\alpha = 0.8$.

		Size ($\rho = 1 + 1/T^{\alpha}$)		Power ($\rho = 1 + 1/T$)	
		t_{MED}	$\tilde{t}_{\text{MED}}$	t_{MED}	$\tilde{t}_{\text{MED}}$
$T = 100$		(a) AR design			
	$\theta = 0.00$	0.049	0.051	0.556	0.545
	$\theta = 0.25$	0.121	0.061	0.525	0.468
	$\theta = 0.50$	0.250	0.077	0.398	0.346
	$\theta = 0.75$	0.459	0.111	0.272	0.195
		(b) MA design			
	$\theta = 0.00$	0.048	0.051	0.552	0.550
	$\theta = 0.25$	0.099	0.056	0.531	0.488
	$\theta = 0.50$	0.144	0.065	0.489	0.427
	$\theta = 0.75$	0.157	0.065	0.483	0.404
$T = 150$		(a) AR design			
	$\theta = 0.00$	0.055	0.057	0.763	0.748
	$\theta = 0.25$	0.126	0.069	0.666	0.631
	$\theta = 0.50$	0.252	0.074	0.512	0.461
	$\theta = 0.75$	0.465	0.103	0.330	0.266
		(b) MA design			
	$\theta = 0.00$	0.054	0.055	0.740	0.733
	$\theta = 0.25$	0.106	0.062	0.688	0.656
	$\theta = 0.50$	0.150	0.066	0.630	0.577
	$\theta = 0.75$	0.163	0.066	0.615	0.561

Note: This table reports the empirical size and size-adjusted power of 5% tests with 5,000 Monte Carlo replications. In the AR design, $u_t = \theta u_{t-1} + \sqrt{1-\theta^2} e_{1,t}$, while in the MA design, $u_t = \theta e_{2,t-1} + \sqrt{1-\theta^2} e_{2,t}$, where $e_{1,t} \sim i.i.d.N(0,1)$ and $e_{2,t} \sim i.i.d.N(0,1)$. Different parameter combinations are configured to conduct simulations for the null of moderate explosiveness $\rho = 1 + 1/T^{\alpha}$ against the alternative of local-to-unity $\rho = 1 + 1/T$.