Supplement to[1] "Optimal Bandwidth Selection in Heteroskedasticity-Autocorrelation Robust Testing"

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THIS APPENDIX PROVIDES TECHNICAL RESULTS AND PROOFS FOR THE ABOVE PAPER.

A.1 Technical Lemmas and Supplements

Lemma 1 *Define*

$$c_1^* = 4\int_{-\infty}^{\infty} |k(v)|\, dv,$$

Let Assumption 2 hold, then the cumulants κ_m *of* $\Xi_b - \mu_b$ *satisfy*

$$|\kappa_m| \le 2^m (m-1)!\,(c_1^* b)^{m-1} \ \textit{ for } m \ge 1 \tag{A.1}$$

and its moments $\alpha_m = E\left(\Xi_b - \mu_b\right)^m$ *satisfy*

$$|\alpha_m| \le 2^{2m} m!\,(c_1^* b)^{m-1} \ \textit{ for } m \ge 1. \tag{A.2}$$

Proof of Lemma 1. To find the cumulants of $\Xi_b - \mu_b$, we write

$$\Xi_b = \int_0^1 \int_0^1 k_b^*(r,s)dW(r)dW(s), \tag{A.3}$$

where $k_b^*(r,s)$ is defined by

$$k_b^*(r,s) = k_b(r-s) - \int_0^1 k_b(r-t)dt - \int_0^1 k_b(\tau - s)d\tau + \int_0^1\int_0^1 k_b(t-\tau)dtd\tau.$$

It can be shown that the cumulant generating function of $\Xi_b - \mu_b$ is

$$\ln \phi\,(t) = \sum_{m=1}^{\infty} \kappa_m \frac{(it)^m}{m!}, \tag{A.4}$$

where $\kappa_1 = 0$ and for $m \ge 2$

$$\kappa_m = 2^{m-1}(m-1)! \int_0^1 ... \int_0^1 \left(\prod_{j=1}^m k_b^*(\tau_j, \tau_{j+1})\right) d\tau_1 \cdots d\tau_m, \tag{A.5}$$

with $\tau_1 = \tau_{m+1}$.

[1] We thank Maarten van Kampen for pointing out a calculation error and a typo in the third order expansion on pages 13-15 after the paper was published.

Now

$$\begin{aligned}
&\left| \int_0^1 ... \int_0^1 \left(\prod_{j=1}^{m} k_b^*(\tau_j, \tau_{j+1}) \right) d\tau_1 \cdots d\tau_m \right| \\
&\leq \int_0^1 ... \int_0^1 |k_b^*(\tau_1,\tau_2)k_b^*(\tau_2,\tau_3)\cdots k_b^*(\tau_{m-1},\tau_m)| \, |k_b^*(\tau_m,\tau_1)| \, d\tau_1 \cdots d\tau_m && \text{(A.6)} \\
&\leq 2 \int_0^1 ... \int_0^1 |k_b^*(\tau_1,\tau_2)k_b^*(\tau_2,\tau_3)\cdots k_b^*(\tau_{m-1},\tau_m)| \, d\tau_1 \cdots d\tau_m && \text{(A.7)} \\
&\leq 2 \sup_{\tau_2} \int_0^1 |k_b^*(\tau_1,\tau_2)| \, d\tau_1 \int_0^1 |k_b^*(\tau_2,\tau_3)k_b^*(\tau_3,\tau_4)\cdots k_b^*(\tau_{m-1},\tau_m)| \, d\tau_2 \cdots d\tau_m \\
&\leq 2 \sup_{\tau_2} \int_0^1 |k_b^*(\tau_1,\tau_2)| \, d\tau_1 \sup_{\tau_3} \int_0^1 |k_b^*(\tau_2,\tau_3)| \, d\tau_2 ... \sup_{\tau_m} \int_0^1 [k_b^*(\tau_{m-1},\tau_m)] \, d\tau_{m-1} \\
&= 2 \left(\sup_s \int_0^1 |k_b^*(r,s)| \, dr \right)^{m-1}. && \text{(A.8)}
\end{aligned}$$

But

$$\begin{aligned}
&\sup_s \int_0^1 |k_b^*(r,s)| \, dr \leq 4 \sup_s \int_0^1 |k_b(r-s)| \, dr \\
&= 4 \sup_{s \in [0,1]} \left(\int_{-s}^{1-s} |k_b(v)| \, dv \right) \leq 4 \int_{-\infty}^{\infty} |k_b(v)| \, dv \\
&= bc_1^*. && \text{(A.9)}
\end{aligned}$$

As a result

$$\left| \int_0^1 ... \int_0^1 \left(\prod_{j=1}^{m} k_b^*(\tau_j, \tau_{j+1}) \right) d\tau_1 \cdots d\tau_m \right| \leq 2 \left(c_1^* b \right)^{m-1}, \tag{A.10}$$

so

$$|\kappa_m| \leq 2^m (m-1)! \left(c_1^* b \right)^{m-1}. \tag{A.11}$$

Note that the moments $\{\alpha_j\}$ and cumulants $\{\kappa_j\}$ satisfy the following relationship:

$$\alpha_m = \sum_{\pi} \frac{m!}{(j_1!)^{m_1} (j_2!)^{m_2} \cdots (j_\ell!)^{m_\ell}} \frac{1}{m_1! m_2! \cdots m_\ell!} \prod_{j \in \pi} \kappa_j, \tag{A.12}$$

where the sum is taken over the elements

$$\pi = [\underbrace{j_1, \cdots j_1}_{m_1 \text{ times}}, \underbrace{j_2, \cdots j_2}_{m_2 \text{ times}}, \cdots \underbrace{j_\ell, \cdots j_\ell}_{m_\ell \text{ times}}] \tag{A.13}$$

for some integer ℓ, sequence $\{j_i\}_{i=1}^{\ell}$ such that $j_1 > j_2 > \cdots > j_\ell$ and $m = \sum_{i=1}^{\ell} m_i j_i$.

Combining the preceding formula with (A.11) gives

$$\begin{aligned}
|\alpha_m| &< 2^m m! \left(c_1^* b \right)^{m-1} \sum_{\pi} \frac{(j_1)^{-m_1} (j_2)^{-m_2} \cdots (j_\ell)^{-m_\ell}}{m_1! m_2! \cdots m_\ell!} \\
&\leq 2^{2m} m! \left(c_1^* b \right)^{m-1}, && \text{(A.14)}
\end{aligned}$$

where the last line follows because

$$\sum_{\pi} \frac{(j_1)^{-m_1}(j_2)^{-m_2}\cdots(j_\ell)^{-m_\ell}}{m_1!m_2!\cdots m_\ell!} \leq \sum_{\pi} \frac{1}{m_1!m_2!\cdots m_\ell!} < 2^m. \tag{A.15}$$

■

Lemma 2 *Let Assumptions 2 and 3 hold. When $T \to \infty$ for a fixed b, we have:*

(a)

$$\mu_{bT} = \mu_b + O\left(\frac{1}{T}\right); \tag{A.16}$$

(b)

$$\kappa_{m,T} = \kappa_m + O\left\{\frac{m!2^m}{T^2}(c_1^* b)^{m-2}\right\}, \tag{A.17}$$

uniformly over $m \geq 1$;

(c)

$$\alpha_{m,T} = E(\varsigma_{bT} - \mu_{bT})^m = \alpha_m + O\left\{\frac{m!2^{2m}}{T^2}(c_1^* b)^{m-2}\right\}, \tag{A.18}$$

uniformly over $m \geq 1$.

Proof of Lemma 2. We first calculate $\mu_{bT} = \left(T\omega_T^2\right)^{-1}\text{Trace}(\Omega_T A_T W_b A_T)$. Let $W_b^* = A_T W_b A_T$, then the (i,j)-th element of W_b^* is

$$\begin{aligned}\tilde{k}_b\left(\frac{i}{T},\frac{j}{T}\right) &= k_b\left(\frac{i-j}{T}\right) - \frac{1}{T}\sum_{p=1}^{T} k_b\left(\frac{i-p}{T}\right) \\ &\quad - \frac{1}{T}\sum_{q=1}^{T} k_b\left(\frac{q-j}{T}\right) + \frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T} k_b\left(\frac{p-q}{T}\right).\end{aligned} \tag{A.19}$$

So

$$\begin{aligned}&\text{Trace}(\Omega_T A_T W_b A_T) = \text{Trace}(\Omega_T W_b^*) \\ &= \sum_{1\leq r_1,r_2\leq T}\left\{\gamma(r_1-r_2)\tilde{k}_b\left(\frac{r_1}{T},\frac{r_2}{T}\right)\right\} = \sum_{r_2=1}^{T}\sum_{h_1=1-r_2}^{T-r_2}\gamma(h_1)\tilde{k}_b\left(\frac{r_2+h_1}{T},\frac{r_2}{T}\right) \\ &= \left(\sum_{h_1=1}^{T-1}\sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0}\sum_{r_2=1-h_1}^{T}\right)\gamma(h_1)\tilde{k}_b\left(\frac{r_2+h_1}{T},\frac{r_2}{T}\right).\end{aligned} \tag{A.20}$$

But

$$\sum_{r_2=1}^{T-h_1} \tilde{k}_b\left(\frac{r_2+h_1}{T},\frac{r_2}{T}\right) = \sum_{r_2=1}^{T-h_1} k_b\left(\frac{h_1}{T}\right) - \frac{1}{T}\sum_{r_1=1+h_1}^{T}\sum_{p=1}^{T} k_b\left(\frac{r_1-p}{T}\right)$$
$$-\frac{1}{T}\sum_{r_2=1}^{T-h_1}\sum_{q=1}^{T} k_b\left(\frac{q-r_2}{T}\right) + \sum_{r_2=1}^{T-h_1}\frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T} k_b\left(\frac{p-q}{T}\right)$$
$$= -\frac{1}{T}\sum_{r_1=1}^{T}\sum_{p=1}^{T} k_b\left(\frac{r_1-p}{T}\right) - \frac{1}{T}\sum_{r_2=1}^{T}\sum_{q=1}^{T} k_b\left(\frac{q-r_2}{T}\right)$$
$$+\sum_{r_2=1}^{T}\frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T} k_b\left(\frac{p-q}{T}\right) + Tk_b\left(\frac{h_1}{T}\right) + C(h_1)$$
$$= -\frac{1}{T}\sum_{r=1}^{T}\sum_{s=1}^{T} k_b\left(\frac{r-s}{T}\right) + Tk_b\left(\frac{h_1}{T}\right) + C(h_1)$$
$$= \sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T},\frac{r_2}{T}\right) + T\left\{k_b\left(\frac{h_1}{T}\right) - k_b\left(0\right)\right\} + C(h_1), \qquad \text{(A.21)}$$

where $C(h_1)$ is a function of h_1 satisfying $|C(h_1)| \le h_1$. Similarly,

$$\sum_{r_2=1-h_1}^{T} \tilde{k}_b\left(\frac{r_2+h_1}{T},\frac{r_2}{T}\right) = \sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T},\frac{r_2}{T}\right) + T\left\{k_b\left(\frac{h_1}{T}\right) - k_b\left(0\right)\right\} + C(h_1). \qquad \text{(A.22)}$$

Therefore, $\text{Trace}(\Omega_T A_T W_b A_T)$ is equal to

$$\sum_{h=-T+1}^{T-1}\gamma(h)\sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T},\frac{r_2}{T}\right) + T\sum_{h=-T+1}^{T-1}\gamma(h)\left\{k_b\left(\frac{h}{T}\right) - k_b\left(0\right)\right\} + O\left(1\right)$$
$$= \sum_{h=-T+1}^{T-1}\gamma(h)\sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T},\frac{r_2}{T}\right) + T(bT)^{-q}\sum_{h=-T+1}^{T-1}|h|^q\gamma(h)\left\{\frac{k\left(h/(bT)\right) - k\left(0\right)}{|h/(bT)|^q}\right\} + O\left(1\right)$$
$$= \sum_{h=-T+1}^{T-1}\gamma(h)\sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T},\frac{r_2}{T}\right) + T(bT)^{-q}g_q\sum_{h=-\infty}^{\infty}|h|^q\gamma(h)(1+o(1)) + O(1). \qquad \text{(A.23)}$$

Using

$$\sum_{h=-T+1}^{T-1}\gamma(h) = \omega_T^2(1+O(\frac{1}{T})), \qquad \text{(A.24)}$$

and

$$\frac{1}{T}\sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T},\frac{r_2}{T}\right) = \int_0^1 k_b^*(r,r)dr + O(\frac{1}{T}), \qquad \text{(A.25)}$$

we now have

$$\mu_{bT} = \int_0^1 k_b^*(r,r)dr - (bT)^{-q}g_q\left(\omega_T^{-2}\sum_{h=-\infty}^{\infty}|h|^q\gamma(h)\right)(1+o(1)) + O\left(\frac{1}{T}\right). \qquad \text{(A.26)}$$

By definition, $\mu_b = E\Xi_b = \int_0^1 k_b^*(r,r)dr$ and thus $\mu_{bT} = \mu_b + O\left(T^{-1}\right)$ as desired.

We next approximate $Trace\left[(\Omega_T A_T W_b A_T)^m\right]$ for $m > 1$. The approach is similar to the case $m = 1$ but notationally more complicated. Let $r_{2m+1} = r_1$, $r_{2m+2} = r_2$, and $h_{m+1} = h_1$. Then

$$\begin{aligned}
& Trace\left[(\Omega_T A_T W_b A_T)^m\right] \\
&= \sum_{r_1, r_2, \ldots, r_{2m+1}=1}^{T} \prod_{j=1}^{m} \gamma(r_{2j-1} - r_{2j}) \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+1}}{T}\right) \\
&= \sum_{r_2, r_4, \ldots, r_{2m}=1}^{T} \sum_{h_1=1-r_2}^{T-r_2} \sum_{h_2=1-r_4}^{T-r_4} .. \sum_{h_m=1-r_{2m}}^{T-r_{2m}} \prod_{j=1}^{m} \gamma(h_j) \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}+h_{j+1}}{T}\right) \\
&= \left(\sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T}\right) \cdots \left(\sum_{h_m=1}^{T-1} \sum_{r_{2m}=1}^{T-h_m} + \sum_{h_m=1-T}^{0} \sum_{r_{2m}=1-h_m}^{T}\right) \\
& \prod_{j=1}^{m} \gamma(h_j) \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}+h_{j+1}}{T}\right) \\
&= I + II, \qquad \text{(A.27)}
\end{aligned}$$

where

$$\begin{aligned}
I = & \left(\sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T}\right) \cdots \left(\sum_{h_m=1}^{T-1} \sum_{r_{2m}=1}^{T-h_m} + \sum_{h_m=1-T}^{0} \sum_{r_{2m}=1-h_m}^{T}\right) \\
& \prod_{j=1}^{m} \gamma(h_j) \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right), \qquad \text{(A.28)}
\end{aligned}$$

and

$$\begin{aligned}
II = O\Bigg\{ & \left(\sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T}\right) \cdots \left(\sum_{h_m=1}^{T-1} \sum_{r_{2m}=1}^{T-h_m} + \sum_{h_m=1-T}^{0} \sum_{r_{2m}=1-h_m}^{T}\right) \\
& \prod_{j=1}^{m} |\gamma(h_j)| \left(\frac{|h_{j+1}|}{bT}\right) \Bigg\}. \qquad \text{(A.29)}
\end{aligned}$$

Here we have used

$$\left| \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}+h_{j+1}}{T}\right) - \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right) \right| = O\left(\frac{|h_{j+1}|}{bT}\right). \qquad \text{(A.30)}$$

To show this, note that

$$\begin{aligned}
\frac{1}{T} \sum_{p=1}^{T} k_b\left(\frac{p - r_{2j+2} - h_{j+1}}{T}\right) &= \frac{1}{T} \sum_{p=1-h_{j+1}}^{T-h_{j+1}} k_b\left(\frac{p - r_{2j+2}}{T}\right) \\
&= \frac{1}{T} \sum_{p=1}^{T} k_b\left(\frac{p - r_{2j+2}}{T}\right) + O\left(\frac{|h_{j+1}|}{T}\right), \qquad \text{(A.31)}
\end{aligned}$$

and

$$\left| k_b\left(\frac{r_{2j}-r_{2j+2}-h_{j+1}}{T}\right) - k_b\left(\frac{r_{2j}-r_{2j+2}}{T}\right)\right| = O\left(\frac{|h_{j+1}|}{bT}\right), \tag{A.32}$$

so that

$$\begin{aligned}
&\tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}+h_{j+1}}{T}\right) \\
&= \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right) + k_b\left(\frac{r_{2j}-r_{2j+2}-h_{j+1}}{T}\right) - k_b\left(\frac{r_{2j}-r_{2j+2}}{T}\right) + O(\frac{|h_{j+1}|}{T}) \\
&= \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right) + O\left(\frac{|h_{j+1}|}{bT}\right).
\end{aligned} \tag{A.33}$$

The first term (I) can be written as

$$\begin{aligned}
I = &\left(\sum_{h_1=1-T}^{T-1}\sum_{r_2=1}^{T} - \sum_{h_1=1}^{T-1}\sum_{r_2=T-h_1+1}^{T} - \sum_{h_1=1-T}^{0}\sum_{r_2=1}^{-h_1}\right)\cdots \\
&\left(\sum_{h_m=1-T}^{T-1}\sum_{r_{2m}=1}^{T} - \sum_{h_m=1}^{T-1}\sum_{r_{2m}=T-h_m+1}^{T} - \sum_{h_m=1-T}^{0}\sum_{r_{2m}=1}^{-h_m}\right)\prod_{j=1}^{m}\gamma(h_j)\left\{\tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right)\right\} \\
&= \sum_{\pi}\sum_{h_1,r_2}\cdots\sum_{h_m,r_{2m}}\prod_{j=1}^{m}\gamma(h_j)\left\{\tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right)\right\},
\end{aligned} \tag{A.34}$$

where $\sum_{h_j,r_{2j}}$ is one of the three choices $\sum_{h_j=1-T}^{T-1}\sum_{r_{2j}=1}^{T}$, $-\sum_{h_j=1}^{T-1}\sum_{r_{2j}=T-h_j+1}^{T}$, $-\sum_{h_j=1-T}^{0}\sum_{r_{2j}=1}^{-h_j}$ and $\sum_{\pi}$ is the summation over all possible combinations of $\left(\sum_{h_1,r_2}\cdots\sum_{h_m,r_{2m}}\right)$. The 3^m summands in (A.34) can be divided into two groups with the first group consisting of the summands all of whose r indices run from 1 to T and the second group consisting of the rest. It is obvious that the first group can be written as

$$\left(\sum_{h}\prod_{j=1}^{m}\gamma(h_j)\right)\sum_{r}\left\{\tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right)\right\}.$$

The dominating terms (in terms of the order of magnitude) in the second group are of the forms

$$\sum_{h_1=1-T}^{T-1}\sum_{r_2=1}^{T}\cdots\sum_{h_p=1-T}^{T-1}\sum_{r_{2p}=T-h_p+1}^{T}\cdots\sum_{h_m=1-T}^{T-1}\sum_{r_{2m}=1}^{T}\prod_{j=1}^{m}\gamma(h_j)\left\{\tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right)\right\},$$

or

$$\sum_{h_1=1-T}^{T-1}\sum_{r_2=1}^{T}\cdots\sum_{h_p=1-T}^{T-1}\sum_{r_{2p}=1}^{-h_p}\cdots\sum_{h_m=1-T}^{T-1}\sum_{r_{2m}=1}^{T}\prod_{j=1}^{m}\gamma(h_j)\left\{\tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right)\right\}.$$

These are the summands with only one r index not running from 1 to T. Both of the above terms are bounded by

$$\sum_{h_1=1-T}^{T-1}\sum_{r_2=1}^{T}\cdots\sum_{h_p=1-T}^{T-1}\cdots\sum_{h_m=1-T}^{T-1}\sum_{r_{2m}=1}^{T}\prod_{j=1}^{m}|\gamma(h_j)|\,|h_p|\prod_{j\neq p}\left|\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right|$$
$$\leq\left[\sup_{r_4}\sum_{r_2=1}^{T}\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\right]^{m-2}\left(\sum_{h_j}|\gamma(h_j)|\right)^{m-1}\left(\sum_{h_p}|\gamma(h_p)|\,|h_p|\right),$$

using the same approach as in (A.8). Approximating the sum by an integral and noting that the second group contains $(m-1)$ terms, all of which are of the same order of magnitude as the above typical dominating terms, we conclude that the second group is of order $O\left[2mT^{m-2}\left(c_1^*b\right)^{m-2}\right]$ uniformly over m. As a consequence,

$$I=\left(\sum_h\prod_{j=1}^{m}\gamma(h_j)\right)\sum_r\left\{\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right\}+O\left\{2mT^{m-2}\left(c_1^*b\right)^{m-2}\right\}\tag{A.35}$$

uniformly over m.

The second term (II) is easily shown to be of order $o\left(2mT^{m-2}\left(c_1^*b\right)^{m-2}\right)$ uniformly over m. Therefore

$$\begin{aligned}&Trace\left[(\Omega_T A_T W_b A_T)^m\right]\\&=\left(\sum_h\gamma(h)\right)^m\sum_r\left\{\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right\}+O\left\{2mT^{m-2}\left(c_1^*b\right)^{m-2}\right\}\end{aligned}\tag{A.36}$$

and

$$\begin{aligned}\kappa_{m,T}&=2^{m-1}(m-1)!T^{-m}\left(\omega_T^2\right)^{-m}\text{Trace}\left[(\Omega_T A_T W_b A_T)^m\right]\\&=2^{m-1}(m-1)!\left\{T^{-m}\sum_r\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)+O\left[\frac{2m}{T^2}\left(c_1^*b\right)^{m-2}\right]\right\}\\&=2^{m-1}(m-1)!\left\{\int\prod_{j=1}^{m}\int_0^1k_b^*(\tau_j,\tau_{j+1})d\tau_jd\tau_{j+1}+O\left[\frac{2m}{T^2}\left(c_1^*b\right)^{m-2}\right]\right\}\\&=\kappa_m+O\left\{\frac{m!2^m}{T^2}\left(c_1^*b\right)^{m-2}\right\},\end{aligned}\tag{A.37}$$

uniformly over m.

Finally, we consider $\alpha_{m,T}$. Note that $\alpha_{1,T}=E(\varsigma_{bT}-\mu_{bT})=0$ and

$$\alpha_{m,T}=\sum_\pi\frac{m!}{(j_1!)^{m_1}(j_2!)^{m_2}\cdots(j_k!)^{m_k}}\frac{1}{m_1!m_2!\cdots m_k!}\prod_{j\in\pi}\kappa_{j,T}\tag{A.38}$$

where the summation $\sum_{\pi}$ is defined in (A.12). Combining the preceding formula with part (b) gives

$$\alpha_{m,T} = \alpha_m + O\left\{\frac{2^m}{T^2}\left(c_1^* b\right)^{m-2}\sum_{\pi}\frac{m!}{m_1!m_2!\cdots m_k!}\right\}$$
$$= \alpha_m + O\left\{\frac{m!2^{2m}}{T^2}\left(c_1^* b\right)^{m-2}\right\}, \tag{A.39}$$

uniformly over m, where the last line follows because $\sum_{\pi}\frac{1}{m_1!m_2!\cdots m_k!} < 2^m$. ■

Lemma 3 *Let Assumptions 2 and 3 hold. If* $b \to 0$ *and* $T \to \infty$ *such that* $bT \to \infty$*, then:*
(a)

$$\mu_{bT} = \int_0^1 k_b^*(r,r)dr - (bT)^{-q} g_q\left(\omega_T^{-2}\sum_{h=-\infty}^{\infty}|h|^q\gamma(h)\right)(1+o(1)) + O\left(\frac{1}{T}\right); \tag{A.40}$$

(b)

$$\kappa_{2,T} = 2\int_0^1\int_0^1\left(k_b^*(r,s)\right)^2 drds(1+o(1)) + O\left(\frac{1}{T}\right); \tag{A.41}$$

(c) for $m = 3$ *and* 4*,*

$$\kappa_{m,T} = O\left(b^{m-1}\right) + O\left(\frac{1}{T}\right). \tag{A.42}$$

Proof of Lemma 3. We have proved (A.40) in the proof of Lemma 2 as equation (A.26) holds for both fixed b and decreasing b. It remains to consider $\kappa_{m,T}$ for $m = 2$, 3, and 4. We first consider $\kappa_{2,T} = 2T^{-2}\left(\omega_T^{-4}\right)\text{Trace}\left[\left(\Omega_T A_T W_b A_T\right)^2\right]$. As a first step, we have

$$\begin{aligned}
&\text{Trace}\left[\left(\Omega_T A_T W_b A_T\right)^2\right]\\
&= \sum_{r_1,r_2,r_3,r_4}\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_3}{T}\right)\tilde{k}_b\left(\frac{r_4}{T},\frac{r_1}{T}\right)\right\}\gamma(r_1-r_2)\gamma(r_3-r_4)\\
&= \sum_{r_2=1}^{T}\sum_{h_1=1-r_2}^{T-r_2}\sum_{r_4=1}^{T}\sum_{h_2=1-r_4}^{T-r_4}\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4+h_2}{T}\right)\tilde{k}_b\left(\frac{r_4}{T},\frac{r_2+h_1}{T}\right)\right\}\gamma(h_1)\gamma(h_2)\\
&= \left(\sum_{h_1=1}^{T-1}\sum_{r_2=1}^{T-h_1}+\sum_{h_1=1-T}^{0}\sum_{r_2=1-h_1}^{T}\right)\left(\sum_{h_2=1}^{T-1}\sum_{r_4=1}^{T-h_2}+\sum_{h_2=1-T}^{0}\sum_{r_4=1-h_2}^{T}\right)\\
&\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4+h_2}{T}\right)\tilde{k}_b\left(\frac{r_4}{T},\frac{r_2+h_1}{T}\right)\right\}\gamma(h_1)\gamma(h_2)\\
&:= I_1+I_2+I_3+I_4,
\end{aligned} \tag{A.43}$$

where

$$I_1 = \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-h_1} \sum_{r_4=1}^{T-h_2} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4+h_2}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2+h_1}{T} \right) \right\} \gamma(h_1)\gamma(h_2),$$

$$I_2 = \sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T-h_1} \sum_{h_2=1-T}^{0} \sum_{r_4=1-h_2}^{T} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4+h_2}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2+h_1}{T} \right) \right\} \gamma(h_1)\gamma(h_2),$$

$$I_3 = \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T} \sum_{r_2=1}^{T-h_1} \sum_{r_4=1}^{T-h_2} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4+h_2}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2+h_1}{T} \right) \right\} \gamma(h_1)\gamma(h_2),$$

and

$$I_4 = \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T} \sum_{h_2=1-T}^{0} \sum_{r_4=1-h_2}^{T} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4+h_2}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2+h_1}{T} \right) \right\} \gamma(h_1)\gamma(h_2).$$

We now consider each term in turn. Using equation (A.33), we have

$$\tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4+h_2}{T} \right) = \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4}{T} \right) + O\left(\frac{b\,|h_2|}{T} \right), \tag{A.44}$$

and

$$\tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2+h_1}{T} \right) = \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2}{T} \right) + O\left(\frac{b\,|h_1|}{T} \right). \tag{A.45}$$

It follows from (A.44) and (A.45) that

$$\begin{aligned}
I_1 &= \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-1} \sum_{r_4=1}^{T-1} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4+h_2}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2+h_1}{T} \right) \right\} \gamma(h_1)\gamma(h_2) \\
&\quad + O\left(\sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \left[T\left(|h_1| + |h_2|\right) + |h_1 h_2| \right] |\gamma(h_1)\gamma(h_2)| \right) \\
&= \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-1} \sum_{r_4=1}^{T-1} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4+h_2}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2+h_1}{T} \right) \right\} \gamma(h_1)\gamma(h_2) + O(T) \\
&= \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-1} \sum_{r_4=1}^{T-1} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2}{T} \right) \right\} \gamma(h_1)\gamma(h_2) + O(T) \\
&\quad + O\left\{ \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-1} \sum_{r_4=1}^{T-1} \left| \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4}{T} \right) \right| \left(\frac{b\left(|h_1| + |h_2|\right)}{T} \right) |\gamma(h_1)\gamma(h_2)| \right\} \\
&= \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-1} \sum_{r_4=1}^{T-1} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2}{T} \right) \right\} \gamma(h_1)\gamma(h_2) + O\left(T\right).
\end{aligned} \tag{A.46}$$

Following the same procedure, we can show that

$$I_2 = \sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T} \sum_{h_2=1-T}^{0} \sum_{r_4=1}^{T} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2}{T} \right) \right\} \gamma(h_1)\gamma(h_2) + O(T), \tag{A.47}$$

$$I_3 = \sum_{h_1=1-T}^{0} \sum_{r_2=1}^{T} \sum_{r_2=1}^{T} \sum_{r_4=1}^{T} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2}{T} \right) \right\} \gamma(h_1)\gamma(h_2) + O(T), \tag{A.48}$$

and

$$I_4 = \sum_{h_1=1-T}^{0} \sum_{r_2=1}^{T} \sum_{h_2=1-T}^{0} \sum_{r_4=1}^{T} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4}{T} \right) \tilde{k}_b \left(\frac{r_4}{T}, \frac{r_2}{T} \right) \right\} \gamma(h_1)\gamma(h_2) + O(T). \tag{A.49}$$

As a consequence,

$$\begin{aligned} &\text{Trace} \left[(\Omega_T A_T W_b A_T)^2 \right] \\ &= \sum_{r_2, r_4} \left\{ \tilde{k}_b \left(\frac{r_2}{T}, \frac{r_4}{T} \right) \right\}^2 \left(\sum_{h=1-T}^{T-1} \gamma(h_1) \right)^2 + O(T), \end{aligned} \tag{A.50}$$

and

$$\begin{aligned} \kappa_{2,T} &= 2T^{-2} \left(\omega_T^{-4} \right) Trace \left[(\Omega_T A_T W_b A_T)^2 \right] \\ &= 2 \int_0^1 \int_0^1 \left(k_b^*(r,s) \right)^2 dr ds (1 + o(1)) + O\left(\frac{1}{T} \right). \end{aligned} \tag{A.51}$$

The proof for $\kappa_{m,T}$ for $m = 3$ and 4 is essentially the same except that we use Lemma 1 to obtain the first term $O(b^{m-1})$. The details are omitted. ■

A.2 Proofs of the Main Results

Proof of Theorem 1. Using the independence between $W(1)$ and Ξ_b, we have

$$F_\delta(z) = P \left\{ \left| (W(1) + \delta) \Xi_b^{-1/2} \right| < z \right\} = E \left\{ G_\delta(z^2 \Xi_b) \right\}. \tag{A.52}$$

Taking a fourth-order Taylor expansion of $G_\delta(z^2 \Xi_b)$ around $\mu_b z^2$ yields

$$\begin{aligned} G_\delta(z^2 \Xi_b) &= G_\delta(\mu_b z^2) + \frac{1}{2} \left(G_\delta''(\mu_b z^2) z^4 \right) (\Xi_b - \mu_b)^2 \\ &+ \frac{1}{6} \left(G_\delta'''(\mu_b z^2) z^6 \right) (\Xi_b - \mu_b)^3 + \frac{1}{24} \left(G_\delta^{(4)}(\tilde{\mu}_b z^2) z^8 \right) (\Xi_b - \mu_b)^4, \end{aligned} \tag{A.53}$$

where $\tilde{\mu}_b$ lies on the line segment between μ_b and Ξ_b. Taking expectation on both sides of the equation and using the fact that $\left| G_\delta^{(4)}(\tilde{\mu}_b z^2) z^8 \right| \leq C$ for some constant C, we have

$$\begin{aligned} EG_\delta(z^2 \Xi_b) &= G_\delta(\mu_b z^2) + \frac{1}{2} G_\delta''(\mu_b z^2) E (\Xi_b - \mu_b)^2 z^4 \\ &+ \frac{1}{6} G_\delta'''(\mu_b z^2) \alpha_3 z^6 + O(|\alpha_4|), \end{aligned} \tag{A.54}$$

as $b \to 0$, where the $O(\cdot)$ term holds uniformly over $z \in \mathbb{R}^+$.

In view of Lemma 1, we have

$$|\alpha_4| = O(b^3), \;\; |\alpha_3| \leq |\alpha_4|^{3/4} = O(b^{9/4}) = o(b^2). \tag{A.55}$$

As a consequence,

$$\begin{aligned} F_\delta(z) &= P\left\{\left|(W(1)+\delta)\Xi_b^{-1/2}\right| < z\right\} \\ &= G_\delta(\mu_b z^2) + \frac{1}{2}\left(G''_\delta(\mu_b z^2) z^4\right)\alpha_2 + o(b^2), \end{aligned} \tag{A.56}$$

uniformly over $z \in z \in \mathbb{R}^+$ where

$$\mu_b = E\Xi_b = \int_0^1 k_b^*(r,r)dr = 1 - \int_0^1\int_0^1 k_b(r-s)drds, \tag{A.57}$$

and

$$\begin{aligned} \alpha_2 &= 2\left(\int_0^1\int_0^1 k_b(r-s)drds\right)^2 + 2\int_0^1\int_0^1 k_b^2(r-s)drds \\ &\quad - 4\int_0^1\int_0^1\int_0^1 k_b(r-p)k_b(r-q)drdpdq. \end{aligned} \tag{A.58}$$

We proceed to develop an asymptotic expansion of μ_b and α_2 as $b \to 0$. Let

$$\mathcal{K}_1(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)\exp(-i\lambda x)dx, \;\; \mathcal{K}_2(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k^2(x)\exp(-i\lambda x)dx, \tag{A.59}$$

then

$$k(x) = \int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\exp(i\lambda x)d\lambda, \;\; k^2(x) = \int_{-\infty}^{\infty}\mathcal{K}_2(\lambda)\exp(i\lambda x)d\lambda. \tag{A.60}$$

For the integral that appears in both μ_b and α_2, we have

$$\begin{aligned} &\int_0^1\int_0^1 k_b(r-s)drds \\ &= \int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\int_0^1\int_0^1 \exp(\frac{i\lambda(r-s)}{b})drdsd\lambda \\ &= \int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\left(\int_0^1 \exp(\frac{i\lambda r}{b})dr\right)\left(\int_0^1 \exp(-\frac{i\lambda s}{b})ds\right) \\ &= \int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\left(\frac{b^2}{\lambda^2}\left(\left(1-\cos\left(\frac{\lambda}{b}\right)\right)^2 + \left(\sin\left(\frac{\lambda}{b}\right)\right)^2\right)\right)d\lambda \\ &= b\int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\, b\left(\frac{\sin\frac{\lambda}{2b}}{\frac{\lambda}{2}}\right)^2 d\lambda \\ &= 2\pi b\mathcal{K}_1(0) + 4b^2\int_{-\infty}^{\infty}\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}\left(\sin\frac{\lambda}{2b}\right)^2 d\lambda, \end{aligned} \tag{A.61}$$

where the last equality holds because

$$\int_{-\infty}^{\infty}\left(\frac{\lambda}{2b}\right)^{-2}\left(\sin\frac{\lambda}{2b}\right)^{2}d\lambda = 2\int_{-\infty}^{\infty}x^{-2}\sin^{2}x dx = 2\pi. \tag{A.62}$$

Now,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}\left(\sin\frac{\lambda}{2b}\right)^2 d\lambda \\
&= \int_{-\infty}^{\infty}\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}\left(\left(\sin\frac{\lambda}{2b}\right)^2-\frac{1}{2}\right)d\lambda + \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}d\lambda \\
&= -\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}\right)\left(\cos\frac{1}{b}\lambda\right)d\lambda + \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}d\lambda \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}\right)d\lambda + o(1)
\end{aligned} \tag{A.63}$$

as $b \to 0$, where we have used the Riemann-Lebesgue lemma. In view of the symmetry of $k(x)$, $\mathcal{K}_1(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)\cos(\lambda x)dx$, and, therefore, (A.61) and (A.63) lead to

$$\begin{aligned}
&\int_0^1\int_0^1 k_b(r-s)drds \\
&= 2\pi b\mathcal{K}_1(0) + 2b^2\int_{-\infty}^{\infty}\left(\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}\right)d\lambda \\
&= 2\pi b\mathcal{K}_1(0) + b^2\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(x)\frac{\cos\lambda x - 1}{\lambda^2}dxd\lambda \\
&= 2\pi b\mathcal{K}_1(0) - b^2\int_{-\infty}^{\infty}k(x)\left|x\right|dx. \\
&= bc_1 + b^2c_3 + o(b^2).
\end{aligned} \tag{A.64}$$

Similarly,

$$\int_0^1\int_0^1 k_b^2(r-s)drds = bc_2 + b^2c_4 + o(b^2). \tag{A.65}$$

Next,

$$\begin{aligned}
&\int_0^1 k_b(r-s)ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\mathcal{K}(\lambda)\int_0^1\left\{\exp(\frac{i\lambda(r-s)}{b}) + \exp(-\frac{i\lambda(r-s)}{b})\right\}dsd\lambda \\
&= \int_{-\infty}^{\infty}\mathcal{K}(\lambda)\int_0^1\cos\left(\frac{\lambda(r-s)}{b}\right)dsd\lambda \\
&= b\int_{-\infty}^{\infty}\mathcal{K}(\lambda)\frac{1}{\lambda}\left(\left\{\sin\left(\frac{\lambda(r-1)}{b}\right)-\sin\left(\frac{\lambda r}{b}\right)\right\}\right)d\lambda \\
&= b\int_{-\infty}^{\infty}\mathcal{K}(xb)\frac{1}{x}\left(\left\{\sin(x(r-1))-\sin(xr)\right\}\right)dx,
\end{aligned} \tag{A.66}$$

so

$$\int_0^1\int_0^1\int_0^1 k_b(r-p)k_b(r-q)drdpdq$$
$$= b^2\int_0^1\left[\int_{-\infty}^{\infty}\mathcal{K}(xb)\frac{1}{x}\left(\{\sin(x(r-1))-\sin(xr)\}\right)dx\right]^2 dr$$
$$= b^2\mathcal{K}^2(0)\int_0^1\left(\int_{-\infty}^{\infty}\frac{1}{x}\sin(x(r-1))dx-\int_{-\infty}^{\infty}\frac{1}{x}\sin(xr)dx\right)^2 dr$$
$$= b^2\mathcal{K}^2(0)\int_0^1\left(\int_{-\infty}^{\infty}\frac{\sin(x(r-1))}{x(r-1)}d(x(r-1))-\int_{-\infty}^{\infty}\frac{1}{xr}\sin(xr)dxr\right)^2 dr$$
$$= b^2\mathcal{K}^2(0)\int_0^1\left(2\int_{-\infty}^{\infty}\frac{1}{y}\sin(y)dy\right)^2 dr = c_1^2b^2. \tag{A.67}$$

Combining (A.64), (A.65), and (A.67) yields

$$\mu_b = 1-bc_1-b^2c_3+o(b^2), \tag{A.68}$$

and

$$\alpha_2 = 2bc_2+b^2(2c_4-2c_1^2)+o(b^2). \tag{A.69}$$

Now

$$\begin{aligned}F_\delta(z) &= G_\delta(\mu_bz^2)+\frac{1}{2}\left(G_\delta''(\mu_bz^2)z^4\right)\alpha_2+o(b^2)\\
&= G_\delta(z^2)-G_\delta'(z^2)z^2bc_1-G_\delta'(z^2)z^2b^2c_3+\frac{1}{2}G_\delta''(z^2)z^4c_1^2b^2\\
&+G_\delta''(z^2)z^4bc_2+\frac{1}{2}G_\delta''(z^2)z^4b^2(2c_4-2c_1^2)\\
&+\frac{1}{2}G_\delta'''(z^2)z^6\left(-bc_1\right)(2bc_2)+o(b^2)\\
&= G_\delta(z^2)+\left[c_2G_\delta''(z^2)z^4-c_1G_\delta'(z^2)z^2\right]b\\
&-\left(G_\delta'(z^2)z^2c_3-\frac{1}{2}G_\delta''(z^2)z^4(2c_4-c_1^2)+G_\delta'''(z^2)z^6c_1c_2\right)b^2+o(b^2)\end{aligned} \tag{A.70}$$

uniformly over $z\in\mathbb{R}^+$, where the uniformity holds because $\left|G_\delta'(z^2)z^2\right|\le C$, $\left|G_\delta''(z^2)z^4\right|<C$ and $\left|G_\delta''(z^2)z^6\right|<C$ for some constant C. This completes the proof of the theorem. ■

Proof of Corollary 2. Using a power series expansion, we have

$$\begin{aligned}F_0(z_{\alpha,b}) &= D(z_{\alpha,b}^2)+\left[c_2D''(z_{\alpha,b}^2)z_{\alpha,b}^4-c_1D'(z_{\alpha,b}^2)z_{\alpha,b}^2\right]b\\
&-\left(D'(z_{\alpha,b}^2)z_{\alpha,b}^2c_3-\frac{1}{2}D''(z_{\alpha,b}^2)z_{\alpha,b}^4(2c_4-c_1^2)+D'''(z_{\alpha,b}^2)z_{\alpha,b}^6c_1c_2\right)b^2+o\left(b^2\right)\\
&= D(z_\alpha^2)+D'(z_\alpha^2)\left(z_{\alpha,b}^2-z_\alpha^2\right)+\frac{1}{2}D''(z_\alpha^2)\left(z_{\alpha,b}^2-z_\alpha^2\right)^2\\
&+\left[c_2D''(z_\alpha^2)z_\alpha^4-c_1D'(z_\alpha^2)z_\alpha^2\right]b\\
&+\left[c_2D'''(z_\alpha^2)z_\alpha^4+2c_2D''(z_\alpha^2)z_\alpha^2-c_1D''(z_\alpha^2)z_\alpha^2-c_1D'(z_\alpha^2)\right]\left(z_{\alpha,b}^2-z_\alpha^2\right)b\\
&-\left(D'(z_\alpha^2)z_\alpha^2c_3-\frac{1}{2}D''(z_\alpha^2)z_\alpha^4(2c_4-c_1^2)+D'''(z_\alpha^2)z_\alpha^6c_1c_2\right)b^2+o(b^2),\end{aligned} \tag{A.71}$$

i.e.

$$
\begin{aligned}
&D'(z_\alpha^2)\left(z_{\alpha,b}^2 - z_\alpha^2\right) + \frac{1}{2}D''(z_\alpha^2)\left(z_{\alpha,b}^2 - z_\alpha^2\right)^2 \\
&+ \left[c_2 D''(z_\alpha^2) z_\alpha^4 - c_1 D'(z_\alpha^2) z_\alpha^2\right] b \\
&+ \left[c_2 D'''(z_\alpha^2) z_\alpha^4 + 2c_2 D''(z_\alpha^2) z_\alpha^2 - c_1 D''(z_\alpha^2) z_\alpha^2 - c_1 D'(z_\alpha^2)\right]\left[\left(z_{\alpha,b}^2 - z_\alpha^2\right)\right] b \\
&- \left(D'(z_\alpha^2) z_\alpha^2 c_3 - \frac{1}{2}D''(z_\alpha^2) z_\alpha^4 (2c_4 - c_1^2) + D'''(z_\alpha^2) z_\alpha^6 c_1 c_2\right) b^2 + o(b^2) = 0.
\end{aligned} \tag{A.72}
$$

Let

$$
z_{\alpha,b}^2 = z_\alpha^2 + k_1 b + k_2 b^2 + o(b^2), \tag{A.73}
$$

then

$$
\begin{aligned}
&\left[c_2 D''(z_\alpha^2) z_\alpha^4 - c_1 D'(z_\alpha^2) z_\alpha^2\right] b + D'(z_\alpha^2) k_1 b \\
&- \left(D'(z_\alpha^2) z_\alpha^2 c_3 - \frac{1}{2}D''(z_\alpha^2) z_\alpha^4 (2c_4 - c_1^2) + D'''(z_\alpha^2) z_\alpha^6 c_1 c_2\right) b^2 \\
&+ \left[c_2 D'''(z_\alpha^2) z_\alpha^4 + 2c_2 D''(z_\alpha^2) z_\alpha^2 - c_1 D''(z_\alpha^2) z_\alpha^2 - c_1 D'(z_\alpha^2)\right] k_1 b^2 \\
&+ \frac{1}{2}D''(z_\alpha^2) k_1^2 b^2 + D'(z_\alpha^2) k_2 b^2 + o(b^2) = 0.
\end{aligned} \tag{A.74}
$$

This implies that

$$
k_1 = -\frac{1}{D'(z_\alpha^2)}\left[c_2 D''(z_\alpha^2) z_\alpha^4 - c_1 D'(z_\alpha^2) z_\alpha^2\right], \tag{A.75}
$$

and

$$
\begin{aligned}
k_2 = -\tfrac{1}{D'(z_\alpha^2)}\Big[&-\left(D'(z_\alpha^2) z_\alpha^2 c_3 - \tfrac{1}{2}D''(z_\alpha^2) z_\alpha^4 (2c_4 - c_1^2) + D'''(z_\alpha^2) z_\alpha^6 c_1 c_2\right) \\
&+ \left(c_2 D'''(z_\alpha^2) z_\alpha^4 + 2c_2 D''(z_\alpha^2) z_\alpha^2 - c_1 D''(z_\alpha^2) z_\alpha^2 - c_1 D'(z_\alpha^2)\right) k_1 \\
&+ \tfrac{1}{2}D''(z_\alpha^2) k_1^2\Big].
\end{aligned} \tag{A.76}
$$

Now

$$
D'(z) = \frac{z^{-1/2} e^{-z/2}}{\Gamma(1/2)\sqrt{2}},\ \ D''(z) = \frac{1}{4\sqrt{\pi} z^2}\left(-\sqrt{2z} e^{-\frac{1}{2}z} - z^{\frac{3}{2}}\sqrt{2} e^{-\frac{1}{2}z}\right), \tag{A.77}
$$

$$
D'''(z) = \frac{1}{8\sqrt{\pi} z^{\frac{7}{2}}}\left(3z\sqrt{2} e^{-\frac{1}{2}z} + 2z^2\sqrt{2} e^{-\frac{1}{2}z} + z^3\sqrt{2} e^{-\frac{1}{2}z}\right) \tag{A.78}
$$

and thus

$$
\frac{D''(z^2)}{D'(z^2)} = \frac{1}{4z^3}\left(-2z - 2z^3\right),\ \ \frac{D'''(z^2)}{D'(z^2)} = \frac{1}{4z^4}\left(2z^2 + z^4 + 3\right). \tag{A.79}
$$

Hence

$$
k_1 = \left(c_1 + \frac{1}{2}c_2\right) z_\alpha^2 + \frac{1}{2}c_2 z_\alpha^4, \tag{A.80}
$$

and

$$
\begin{aligned}
k_2 = &\left(\frac{1}{2}c_1^2 + \frac{3}{2}c_1 c_2 + \frac{3}{16}c_2^2 + c_3 + \frac{1}{2}c_4\right) z_\alpha^2 \\
&+ \left(-\frac{1}{2}c_1^2 + \frac{3}{2}c_1 c_2 + \frac{9}{16}c_2^2 + \frac{1}{2}c_4\right) z_\alpha^4 + \left(\frac{5}{16}c_2^2\right) z_\alpha^6 - \left(\frac{1}{16}c_2^2\right) z_\alpha^8
\end{aligned} \tag{A.81}
$$

as desired.

It follows from $z^2_{\alpha,b} = z^2_\alpha + k_1 b + k_2 b^2 + o(b^2)$ that

$$\begin{aligned} z_{\alpha,b} &= z_\alpha \left(1 + \frac{1}{2}\frac{k_1 b + k_2 b^2}{z^2_\alpha} - \frac{1}{8}\frac{k_1^2 b^2}{z^4_\alpha}\right) + o(b^2) \\ &= z_\alpha + \frac{1}{2}\frac{k_1}{z_\alpha} b + \left(\frac{1}{2}\frac{k_2}{z_\alpha} - \frac{1}{8}\frac{k_1^2}{z^3_\alpha}\right) b^2 + o(b^2) \\ &= z_\alpha + k_3 b + k_4 b^2 + o(b^2) \end{aligned} \tag{A.82}$$

where

$$k_3 = \frac{1}{2}\left[\left(c_1 + \frac{1}{2}c_2\right) z_\alpha + \frac{1}{2}c_2 z^3_\alpha\right] \tag{A.83}$$

and

$$\begin{aligned} k_4 = &\left(\frac{1}{2}c_3 + \frac{1}{4}c_4 + \frac{5}{8}c_1 c_2 + \frac{1}{8}c_1^2 + \frac{1}{16}c_2^2\right) z_\alpha \\ &+ \left(-\frac{1}{4}c_1^2 + \frac{1}{4}c_4 + \frac{5}{8}c_1 c_2 + \frac{7}{32}c_2^2\right) z^3_\alpha + \frac{1}{8}c_2^2 z^5_\alpha - \frac{1}{32}c_2^2 z^7_\alpha \end{aligned} \tag{A.84}$$

■

Proof of Corollary 3. For notational convenience, let

$$p^{(1)}(z^2_\alpha) = \left(c_1 + \frac{c_2}{2}\right) z^2_\alpha + \left(\frac{c_2}{2}\right) z^4_\alpha, \tag{A.85}$$

and then $z^2_{\alpha,b} = z^2_\alpha + p^{(1)}(z^2_\alpha) b + o(b)$. We have

$$\begin{aligned} &1 - EG_\delta(z^2_{\alpha,b}\Xi_b) \\ &= 1 - G_\delta\left(z^2_\alpha + p^{(1)}(z^2_\alpha) b\right) - \left\{c_2 G''_\delta\left(z^2_\alpha + p^{(1)}(z^2_\alpha) b\right)\left[z^2_\alpha + b p^{(1)}(z^2_\alpha)\right]^2\right\} b \\ &+ c_1\left\{G'_\delta\left(z^2_\alpha + p^{(1)}(z^2_\alpha) b\right)\left(z^2_\alpha + p^{(1)}(z^2_\alpha) b\right)\right\} b + o(b) \\ &= 1 - G_\delta(z^2_\alpha) - G'_\delta(z^2_\alpha) p^{(1)}(z^2_\alpha) b - \left[c_2 G''_\delta(z^2_\alpha) z^4_\alpha - c_1 G'_\delta(z^2_\alpha) z^2_\alpha\right] b + o(b) \\ &= 1 - G_\delta(z^2_\alpha) - G'_\delta(z^2_\alpha)\left[\left(c_1 + \frac{c_2}{2}\right) z^2_\alpha + \left(\frac{c_2}{2}\right) z^4_\alpha\right] b - \left[c_2 G''_\delta(z^2_\alpha) z^4_\alpha - c_1 G'_\delta(z^2_\alpha) z^2_\alpha\right] b + o(b) \\ &= 1 - G_\delta(z^2_\alpha) - G'_\delta(z^2_\alpha)\left[\left(c_1 + \frac{c_2}{2}\right) z^2_\alpha + \left(\frac{c_2}{2}\right) z^4_\alpha\right] b - \left[c_2 G''_\delta(z^2_\alpha) z^4_\alpha - c_1 G'_\delta(z^2_\alpha) z^2_\alpha\right] b + o(b) \\ &= 1 - G_\delta(z^2_\alpha) - \left(\frac{c_2}{2} G'_\delta(z^2_\alpha) z^4_\alpha + c_2 G''_\delta(z^2_\alpha) z^4_\alpha + \frac{c_2}{2} G'_\delta(z^2_\alpha) z^2_\alpha\right) b + o(b) \\ &= 1 - G_\delta(z^2_\alpha) - c_2\left(\frac{1}{2} G'_\delta(z^2_\alpha) z^4_\alpha + G''_\delta(z^2_\alpha) z^4_\alpha + \frac{1}{2} G'_\delta(z^2_\alpha) z^2_\alpha\right) b + o(b). \end{aligned} \tag{A.86}$$

Note that

$$G'_\delta(z) = \sum_{j=0}^{\infty} \frac{\left(\delta^2/2\right)^j}{j!} e^{-\delta^2/2} \frac{z^{j-\frac{1}{2}} e^{-z/2}}{\Gamma(j + 1/2) 2^{j+1/2}}, \tag{A.87}$$

and

$$G_{\delta}''(z) = \sum_{j=0}^{\infty} \frac{\left(\delta^2/2\right)^j}{j!} e^{-\delta^2/2} \left((j - \frac{1}{2}) \frac{1}{z} \frac{z^{j-\frac{1}{2}} e^{-z/2}}{\Gamma(j+1/2)2^{j+1/2}} - \frac{1}{2} \frac{z^{j-\frac{1}{2}} e^{-z/2}}{\Gamma(j+1/2)2^{j+1/2}} \right)$$

$$= \left(-\frac{1}{2z} - \frac{1}{2} \right) G_{\delta}'(z) + \sum_{j=0}^{\infty} \frac{\left(\delta^2/2\right)^j}{j!} e^{-\delta^2/2} \frac{z^{j-1/2} e^{-z/2}}{\Gamma(j+1/2)2^{j+1/2}} \frac{j}{z}$$

$$= -\frac{1}{2} G_{\delta}'(z) \left(\frac{1}{z} + 1 \right) + K_{\delta}(z), \tag{A.88}$$

so that

$$\frac{1}{2} G_{\delta}'(z_{\alpha}^2) z_{\alpha}^4 + G_{\delta}''(z_{\alpha}^2) z_{\alpha}^4 + \frac{1}{2} G_{\delta}'(z_{\alpha}^2) z_{\alpha}^2 = z_{\alpha}^4 K_{\delta}\left(z_{\alpha}^2\right), \tag{A.89}$$

and

$$1 - EG_{\delta}(z_{\alpha,b}^2 \Xi_b) = 1 - G_{\delta}(z_{\alpha}^2) - c_2 z_{\alpha}^4 K_{\delta}\left(z_{\alpha}^2\right) b + o(b), \tag{A.90}$$

completing the proof of the corollary. ■

Proof of Theorem 4. It follows from Lemma 3 that when $b \to 0$,

$$\begin{aligned} \alpha_{2,T} &= \kappa_{2,T} = 2bc_2(1+o(1)) + O\left(T^{-1}\right), \\ \alpha_{3,T} &= \kappa_{3,T} = O(b^2) + O\left(T^{-1}\right), \\ \alpha_{4,T} &= \kappa_{4,T} + 3\kappa_{2,T}^2 = O(b^2) + O\left(T^{-1}\right), \end{aligned} \tag{A.91}$$

and

$$\mu_{bT} = \mu_b - (bT)^{-q} g_q \left(\omega_T^{-2} \sum_{h=-\infty}^{\infty} |h|^q \gamma(h) \right) (1+o(1)) + O\left(T^{-1}\right), \tag{A.92}$$

Thus, as $b \to 0$

$$\begin{aligned} F_{T,\delta}(z) &= P\left\{ \left| \sqrt{T}\left(\hat{\beta} - \beta_0\right)/\hat{\omega}_b \right| \le z \right\} = E\left\{ G_{\delta}(z^2 \varsigma_{bT}) \right\} + O\left(T^{-1}\right) \\ &= G_{\delta}(\mu_{bT} z^2) + \frac{1}{2} G_{\delta}''(\mu_{bT} z^2) z^4 \alpha_{2,T} + o(b) \\ &= G_{\delta}(\mu_{bT} z^2) + \frac{1}{2} G_{\delta}''(\mu_{bT} z^2) z^4 \left(2bc_2\right) + o(b) + O\left(T^{-1}\right) \\ &= G_{\delta}(\mu_b z^2) + G_{\delta}'(\mu_b z^2) z^2 \left(\mu_{bT} - \mu_b\right) + bc_2 G_{\delta}''(\mu_b z^2) z^4 \\ &+ o(b) + O\left(T^{-1}\right), \end{aligned} \tag{A.93}$$

uniformly over $z \in \mathbb{R}^+$, using (A.91) and (A.92). But

$$\begin{aligned} G_{\delta}(\mu_b z^2) &= G_{\delta}(z^2) + G_{\delta}'(z^2) z^2 \left(\mu_b - 1\right) + o(b) \\ &= G_{\delta}(z^2) - bc_1 G_{\delta}'(z^2) z^2 + o(b), \end{aligned} \tag{A.94}$$

uniformly over $z \in \mathbb{R}^+$, and

$$\begin{aligned}
&G'_\delta(\mu_b z^2) z^2 \left(\mu_{bT} - \mu_b\right) \\
&= \left(G'_\delta(z^2) + O\left(b\right)\right) z^2 \left\{ -g_q \left(\omega_T^{-2} \sum_{h=-\infty}^{\infty} |h|^q \gamma(h) \right) (bT)^{-q} (1 + o(1)) + O\left(T^{-1}\right) \right\} \\
&= -g_q \left(\omega_T^{-2} \sum_{h=-\infty}^{\infty} |h|^q \gamma(h) \right) G'_\delta(z^2) z^2 (bT)^{-q} (1 + o(1)) + o(b) + O\left(T^{-1}\right), \qquad \text{(A.95)}
\end{aligned}$$

uniformly over $z \in \mathbb{R}^+$. So

$$\begin{aligned}
F_{T,\delta}(z) &= G_\delta(z^2) + \left(c_2 G''_\delta(\mu_b z^2) z^4 - c_1 G'_\delta(z^2) z^2\right) b - g_q d_{qT} G'_\delta(z^2) z^2 (bT)^{-q} \\
&\quad + o(b + (bT)^{-q}) + O\left(T^{-1}\right), \qquad \text{(A.96)}
\end{aligned}$$

uniformly over $z \in \mathbb{R}^+$, as desired. ■

Proof of Corollary 5. Part (a) Using Theorem 4, we have, as $b + 1/T + 1/(bT) \to 0$

$$\begin{aligned}
F_{T,0}(z_{\alpha,b}) &= D(z_{\alpha,b}^2) + \left[c_2 D''(z_{\alpha,b}^2) z_{\alpha,b}^4 - c_1 D'(z_{\alpha,b}^2) z_{\alpha,b}^2\right] b \\
&\quad - g_q d_{qT} D'(z_{\alpha,b}^2) z_{\alpha,b}^2 (bT)^{-q} + O\left(T^{-1}\right) + o\left(b + (bT)^{-q}\right) \\
&= F_0(z_{\alpha,b}) - g_q d_{qT} D'(z_{\alpha,b}^2) z_{\alpha,b}^2 (bT)^{-q} + O\left(T^{-1}\right) + o\left(b + (bT)^{-q}\right) \qquad \text{(A.97)} \\
&= 1 - \alpha - g_q d_{qT} D'(z_\alpha^2) z_\alpha^2 (bT)^{-q} + O\left(T^{-1}\right) + o\left(b + (bT)^{-q}\right).
\end{aligned}$$

So

$$1 - F_{T,0}(z_{\alpha,b}) - \alpha = g_q d_{qT} D'(z_\alpha^2) z_\alpha^2 (bT)^{-q} + O\left(T^{-1}\right) + o\left(b + (bT)^{-q}\right). \qquad \text{(A.98)}$$

Part (b) Plugging $z_{\alpha,b}^2$ into

$$\begin{aligned}
F_{T,\delta}(z) &= G_\delta(z^2) + \left[c_2 G''_\delta(\mu_b z^2) z^4 - c_1 G'_\delta(z^2) z^2\right] b \\
&\quad - g_q d_{qT} G'_\delta(z^2) z^2 (bT)^{-q} + o\left\{b + (bT)^{-q}\right\} + O\left(T^{-1}\right) \qquad \text{(A.99)}
\end{aligned}$$

yields

$$\begin{aligned}
&P\left(\left| \frac{\sqrt{T}(\hat{\beta} - \beta_0)}{\hat{\omega}_b} \right|^2 \geq z_{\alpha,b}^2 \right) \\
&= 1 - G_\delta(z_{\alpha,b}^2) - \left[c_2 G''_\delta(z_{\alpha,b}^2) z_{\alpha,b}^4 - c_1 G'_\delta(z_{\alpha,b}^2) z_{\alpha,b}^2\right] b \\
&+ g_q d_{qT} G'_\delta(z_{\alpha,b}^2) z_{\alpha,b}^2 (bT)^{-q} + O\left(T^{-1}\right) + o\left(b + (bT)^{-q}\right) \qquad \text{(A.100)} \\
&= 1 - G_\delta(z_\alpha^2) - c_2 z_\alpha^4 K_\delta\left(z_\alpha^2\right) b \\
&+ g_q d_{qT} G'_\delta(z_\alpha^2) z_\alpha^2 (bT)^{-q} + O\left(T^{-1}\right) + o\left(b + (bT)^{-q}\right),
\end{aligned}$$

where the last equality follows as in the proof of Corollary 3. ■

Proof of Theorem 6. First, since $D(\cdot)$ is a bounded function, we have

$$P\left\{\left|W(1)\Xi_b^{-1/2}\right| \le z\right\} = \lim_{B\to\infty} ED(z^2\Xi_b)1\left\{|\Xi_b-\mu_b|\le B\right\}$$

$$= \lim_{B\to\infty} E\sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)\left(\Xi_b-\mu_b\right)^m z^{2m}1\left\{|\Xi_b-\mu_b|\le B\right\}$$

$$= \lim_{B\to\infty}\sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)\alpha_m z^{2m}1\left\{|\Xi_b-\mu_b|\le B\right\}, \tag{A.101}$$

where the last line follows because the infinite sum $\sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)\alpha_m z^{2m}$ converges uniformly to $D(z^2\Xi_b)$ when $|\Xi_b-\mu_b|\le B$. Uniformity holds because $D(\cdot)$ is infinitely differentiable with bounded derivatives.

Since $D(z^2)$ decays exponentially as $z^2\to\infty$, there exists a constant C such that $\left|D^{(m)}(\mu_b z^2)z^{2m}\right| < C$ for all m. Using this and Lemma 1, we have

$$\left|\sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)\alpha_m z^{2m}\right|$$

$$\le C\sum_{m=1}^{\infty}\frac{1}{m!}|\alpha_m| \le C\sum_{m=1}^{\infty}\frac{1}{m!}2^{2m}m!\left(c_1^*b\right)^{m-1}$$

$$= C\left(c_1^*b\right)^{-1}\sum_{m=1}^{\infty}\left(4c_1^*b\right)^m < \infty, \tag{A.102}$$

provided that $b<1/(4c_1^*)$. As a consequence, the operation $\lim_{B\to\infty}$ can be moved inside the summation sign in (A.101), giving

$$P\left\{\left|W(1)\Xi_b^{-1/2}\right| \le z\right\} = \sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)\alpha_m z^{2m}, \tag{A.103}$$

when $b<1/(4c_1^*)$.

$$F_{T,\delta}(z) := P\left\{\left|\sqrt{T}\left(\hat\beta-\beta_0\right)/\hat\omega_b\right| \le z\right\}$$

$$= E\left\{G_\delta(z^2\hat\omega_b^2/\tilde\omega_T^2)\right\} = E\left\{G_\delta(z^2\varsigma_{bT})\right\} + O\left(T^{-1}\right), \tag{A.104}$$

Second, it follows from

$$F_{T,\delta}(z) = P\left\{\left|\sqrt{T}\left(\hat\beta-\beta_0\right)/\hat\omega_b\right| \le z\right\} = E\left\{G_\delta(z^2\varsigma_{bT})\right\} + O\left(T^{-1}\right), \tag{A.105}$$

that

$$P\left\{\left|\sqrt{T}\left(\hat\beta-\beta_0\right)/\hat\omega_b\right| \le z\right\} = E\left\{D(z^2\varsigma_{bT})\right\} + O\left(T^{-1}\right). \tag{A.106}$$

But

$$E\left\{D(z^2\varsigma_{bT})\right\} = \sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_{bT}z^2)\alpha_{m,T}z^{2m}, \tag{A.107}$$

where the right hand side converges to $E\left\{D(z^2\varsigma_{bT})\right\}$ uniformly over T because

$$\alpha_{m,T} = \alpha_m + O\left\{\frac{2^{2m} m!}{T^2}\left(c_1^* b\right)^{m-2}\right\},$$

uniformly over m by Lemma 2, $\left|D^{(m)}(\mu_{bT} z^2) z^{2m}\right| < C$ for some constant C, and thus

$$\left|\sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_{bT} z^2)\alpha_{m,T} z^{2m}\right| \leq C \sum_{m=1}^{\infty} \frac{1}{m!}\left|\alpha_m\right| + \frac{C}{T^2}\sum_{m=1}^{\infty} 2^{2m}\left(c_1^* b\right)^{m-2} < \infty,$$

when $b < 1/(4c_1^*)$. Therefore

$$P\left\{\left|\sqrt{T}\left(\hat{\beta}-\beta_0\right)/\hat{\omega}_b\right| \leq z\right\} = \sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_{bT} z^2)\alpha_{m,T} z^{2m} + O(\frac{1}{T}), \tag{A.108}$$

uniformly over $z \in \mathbb{R}^+$.

It follows from (A.103) and (A.108) that

$$\begin{aligned}
&\left|F_{T,0}(z) - F_0(z)\right| \\
&= \left|P\left\{\left|\sqrt{T}\left(\hat{\beta}-\beta\right)/\hat{\omega}_b\right| \leq z\right\} - P\left\{\left|W(1)\Xi_b^{-1/2}\right| < z\right\}\right| \\
&= \left|\sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_{bT} z^2)\alpha_{m,T} z^{2m} - \sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2)\alpha_m z^{2m}\right| + O(\frac{1}{T}) \qquad \text{(A.109)} \\
&= \left|\sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2)\alpha_{m,T} z^{2m} - \sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2)\alpha_m z^{2m}\right| + O(\frac{1}{T})
\end{aligned}$$

uniformly over $z \in \mathbb{R}$, where the second equality holds because

$$D^{(m)}(\mu_{bT} z^2) = D^{(m)}(\mu_b z^2) + O\left(D^{(m+1)}(\mu_b z^2) z^2/T\right),$$

uniformly over $z \in \mathbb{R}$ and

$$\left|\sum_{m=1}^{\infty} \frac{1}{m!} D^{(m+1)}(\mu_b z^2)\alpha_{m,T} z^{2m+2}\right| < \infty.$$

Therefore,

$$\begin{aligned}
\left|F_{T,0}(z) - F_0(z)\right| &= \left|\sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2)\left(\alpha_{m,T} - \alpha_m\right) z^{2m}\right| + O(\frac{1}{T}) \\
&= O\left\{\frac{1}{T^2}\sum_{m=1}^{\infty} 2^{2m}\left(c_1^* b)\right)^{m-2}\right\} + O\left(\frac{1}{T}\right) \\
&= O\left(\frac{1}{T}\right),
\end{aligned} \tag{A.110}$$

uniformly over $z \in \mathbb{R}^+$ as desired. ■

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