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Competitive adjustment processes in labor markets with perfect information but heterogeneous firms and workers are studied. Generalizing results of Shapley and Shubik [**7**], and of Crawford and Knoer [**1**], we show that equilibrium in such markets exists and is stable, in spite of workers' discrete choices among jobs, provided that all workers are gross substitutes from each firm's standpoint. We also generalize Gale and Shapley's [**3**] result that the equilibrium to which the adjustment process converges is biased in favor of agents on the side of the market that makes offers, beyond the class of economies to which it was extended by Crawford and Knoer [**1**]. Finally, we use our techniques to establish the existence of equilibrium in a wider class of markets, and some sensible comparative statics results about the effects of adding agents to the market are obtained.

1. INTRODUCTION

THE ARROW–DEBREU THEORY of general economic equilibrium has long been recognized as a powerful and elegant tool for the analysis of resource allocation in market economies. Not all markets fit equally well into the Arrow–Debreu framework, however. Consider, for example, the labor market—or the housing market, which provides an equally good example for most of our purposes. Essential features of the labor market are pervasive uncertainty about market opportunities on the part of participants, extensive heterogeneity, in the sense that job satisfaction and productivity generally differ (and are expected to differ) interactively and significantly across workers and jobs, and large set-up costs and returns to specialization that typically limit workers to one job.

All of these features can be fitted formally into the Arrow–Debreu framework. State-contingent general equilibrium theory, for example, provides a starting point for studying the effects of uncertainty. But this analysis has been made richer and its explanatory power broadened by the examination of equilibrium with incomplete markets, search theory, and market signaling theory.

The purpose of this paper is to attempt some improvements in another dimension: we study the outcome of competitive sorting processes in markets where complete heterogeneity prevails (or may prevail). To do this, we take as given the implications of set-up costs and returns to specialization by assuming that, while firms can hire any number of workers, workers can take at most one job. We also return to the simplification of perfect information.

In the customary view of competitive markets, agents take market prices as given and respond noncooperatively to them. In this framework equilibrium cannot exist in general unless the goods traded in each market are truly homogeneous; heterogeneity therefore generally requires a very large number of markets. And since these markets are necessarily extremely thin—in many cases containing only a single agent on each side—the traditional stories supporting the plausibility of price-taking behavior are quite strained.

The model we employ resolves the problem of thinness of markets in an intuitively appealing way, by replacing the plethora of noncooperative, price-taking markets by a single market *game*, in which agents negotiate salaries (and, in some interpretations of the model, other endogenous job characteristics) cooperatively. In such a game, the natural notion of equilibrium requires that no firm and group of workers can negotiate an agreement that improves upon the equilibrium agreement for all parties concerned. This equilibrium concept is equivalent to the core in our model since the coalitions that combine a single firm with a group of workers are the only essential ones in the economy.

Furthermore, the usual multiplicity of core allocations is not as serious a problem here because in models of this type, the core and the set of competitive equilibria coincide: noncooperative price-taking behavior and cooperative price-making behavior have identical implications as far as the set of possible equilibrium outcomes is concerned.

Our analysis takes as a starting point the analyses of matching processes in market-like settings in papers by David Gale and Lloyd Shapley [**3**], Shapley and Herbert Scarf [**6**], Shapley and Martin Shubik [**7**], and Vincent Crawford and Elsie Knoer [**1**]. Gale and Shapley [**3**] (and, later, Shapley and Scarf [**6**]) considered the pure matching problem that arises in markets like the marriage market, where agents cannot (in some societies) bribe or compensate each other for matching, so that preferences over the possible matches are exogenous. Gale and Shapley used a "deferred-acceptance" procedure to show the existence of the core for the marriage game (one-to-one matching) and the college-admissions game (groups on one side assigned to single members of the other side). They also established a striking, systematic relationship between market institutions, such as who makes offers, and which of the range of equilibrium outcomes made possible by heterogeneity actually arises. Their process converges to an equilibrium outcome that all members of the side of the market that makes offers (weakly) prefer to any other equilibrium outcome. Shapley and Scarf [**6**] used the method of "top trading cycles," attributed to Gale, to find an equilibrium assignment of indivisible goods to people; they also provided a non-algorithmic proof of the existence of the core using Scarf's [**4**] theorem establishing existence for "balanced" games.

Shapley and Shubik [**7**] continued this line of research by considering the matching problem in a market where there is a divisible good, "money," which agents can use to compensate each other for matching. Money enters agents' utility functions linearly, so that the economy has transferable utility. In this circumstance, an efficient assignment must solve the linear program of maximizing the money (utility) value of the assignment. The duality theory that is available for the linear programming problem was then used to show the existence of core allocations. Shapley and Shubik also showed "that the core must be elongated, with its long axis oriented in the direction of market-wide price trends" [**7**, p. 120], a result reminiscent of the relationship identified by Gale and Shapley [**3**].

In a recent paper, Crawford and Knoer [1] reconsidered the problem addressed by Shapley and Shubik [7]. Using a discrete approximation, they generalized Gale and Shapley's deferred-acceptance procedure to the case where a money good is present. Their algorithm, called the "salary-adjustment process," converges to a core allocation, defined taking into account the discreteness of money. This result was then used to establish existence with perfectly divisible money as well, providing an alternative proof of Shapley and Shubik's [7] result, and allowing significant generalization of it. The new method of proof also shows that the systematic relationship between market institutions and which of the range of equilibrium outcomes arises, originally identified by Gale and Shapley [3], remains valid in markets with money goods.

The Crawford–Knoer salary adjustment process, which was designed to mimic the behavior of real labor markets when firms and workers are well-informed, is quite general in most respects. But the analysis presented there is flawed by two unnecessarily restrictive assumptions about the nature of firms; these are both descriptively implausible and obscure the structure of the model. These assumptions specify that a worker's "productivity" at a given firm is independent of which other workers that firm hires, and that there is a fixed number of workers a given firm wishes to hire, independent of the prevailing salaries and other market data. The main purpose of this paper is to relax these assumptions; this is easily accomplished by a simple modification of the salary-adjustment process. The latter assumption can be dispensed with entirely. The former assumption is replaced by the requirement that the firms' demands for workers satisfy a gross-substitutes condition similar to that frequently employed in general equilibrium theory. Examples are provided to show that the gross-substitutes condition significantly generalizes the separability assumption maintained by Crawford and Knoer [1]. It is further shown, by example, that without our gross-substitutes assumption the core and the competitive equilibrium may fail to exist in this model. While links between gross-substitutes assumptions and the uniqueness and stability of the competitive equilibrium are familiar from general equilibrium theory, to our knowledge it is unusual to find a link between such conditions and the *existence* of competitive equilibrium. We also show that the institutional bias in the modified salary-adjustment process increases or diminishes in the expected way if new firms or workers join the market.

In addition, we argue that the salary-adjustment process can be used to establish sufficient conditions for the existence of the core in some markets where the two-sided, firms-workers division is not natural. Principally, it is a condition analogous to gross substitutes, applied to the characteristic function, that is required. This condition is more readily interpretable economically than the requirement, for instance, that the game be balanced. It is shown to be related to convexity restrictions on firms' technologies.

The paper is organized as follows. Section 2 presents the model and definitions, and Section 3 describes the modified salary-adjustment process. Section 4 establishes the existence of core allocations. Section 5 examines the question of institutional bias and the sensitivity of the equilibrium outcome to the degree of "competition" between firms and workers. Section 6 concludes the paper with a closer examination of the gross-substitutes condition; examples are presented to show that without this condition, the core and the competitive equilibrium may fail to exist even in two-sided markets.

2. THE FORMAL MODEL

There are m workers and n firms, indexed $i = 1, \ldots, m$ and $j = 1, \ldots, n$, respectively. Each firm hires as many workers as it wishes, but each worker is allowed to work only at one firm; further, workers are indifferent about which other workers their firms hire. The analysis seems significantly more difficult without these restrictions, and we are not certain if they could be relaxed without changing the results. Worker i's utility of working for firm j at salary s_{ij} is given by $u^i(j; s_{ij})$, assumed strictly increasing and continuous in its second argument. Firm j's gross product (measured in the same units as the s_{ij}) is given by $y^j(C^j)$, where C^j is the set of indices of the workers firm j hires. Firm j's net profits are therefore given by $\pi^j(C^j; s^j) \equiv y^j(C^j) - \sum_{i \in C^j} s_{ij}$, where $s^j \equiv (s_{1j}, \ldots, s_{mj})$ is the vector of salaries faced by firm j.

Three assumptions about firms' production technologies and workers' preferences are needed. Let $u^i(0; 0)$ stand for worker i's evaluation of unemployment at zero salary. Let σ_{ij} be defined by $u^i(j; \sigma_{ij}) \equiv u^i(0; 0)$; thus, σ_{ij} is the lowest salary at which worker i would ever consider working for firm j. The first assumption requires that for all (i, j) and all C, $i \notin C$,

$$\text{(MP)} \qquad y^j(C \cup \{i\}) - y^j(C) - \sigma_{ij} \geqq 0.$$

This is a natural restriction, since if a worker's marginal product, net of the salary required to compensate him or her for the disutility of work at a given firm, were negative, the firm could agree to let the worker do nothing for a salary of zero.

Our second restriction is a "no-free-lunch" assumption:

$$\text{(NFL)} \qquad y^j(\emptyset) = 0 \qquad \text{for all } j.$$

Finally, we require that all workers be *gross substitutes* from the standpoint of each firm. More formally, let $M^j(s^j)$ denote the set of solutions to the problem

$$\text{(A)} \qquad \max_C \pi^j(C; s^j),$$

where the maximum is taken over all possible sets of workers. Consider two vectors of salaries s^j and $\tilde{s}^j$ facing firm j. Let $T^j(C^j) \equiv \{i \mid i \in C^j \text{ and } \tilde{s}_{ij} = s_{ij}\}$. Then we require

(GS) for every firm j, if $C^j \in M^j(s^j)$ and $\tilde{s}^j \geqq s^j$, then there exists $\tilde{C}^j \in M^j(\tilde{s}^j)$ such that $T^j(C^j) \subseteq \tilde{C}^j$.

Assumption (GS) requires that all workers be (weak) gross substitutes to each firm, in the sense that increases in other workers' salaries can never cause a firm to withdraw an offer from a worker whose salary has not risen.[1] Later, we present examples to show that, while (GS) is implied by additive separability of the production technology, it is significantly more general. It is worth noting that, in general, there will be markets with discrete measurement of salaries that satisfy (GS), while the market with the same firms and workers, but with continuous measurement of salaries, does not; this distinction will be relevant below.

The natural notion of equilibrium in markets like those considered here requires, as pointed out above, that no firm and group of workers can negotiate an agreement that is better than the equilibrium agreement for all parties involved in the negotiations. This is equivalent to the usual definition of the core in our model since the coalitions of one firm and a group of workers are the only *essential* coalitions in the economy: coalitions that consist exclusively of firms or workers can accomplish nothing, and any other coalition can be broken down with no loss into subunits, each containing one firm. We shall refer to this equilibrium notion as the "core," in order to preserve the distinction between it and the equilibria of the dynamic adjustment processes studied below.

When salaries are perfectly divisible, this notion is equivalent to the competitive equilibrium as well, because the simplicity of the set of essential coalitions and the discreteness of matching allow a price system to be constructed to support any core allocation. To see this, note that any competitive equilibrium allocation is also a core allocation (given the existence of a divisible good agents can use to compensate each other), for the usual reasons. Conversely, any core allocation is also competitive, because no coalition, and therefore no essential coalition, can improve upon it. Competitive prices—or salaries, as we shall call them below—can therefore be constructed to support the core allocation by taking the salaries implicit in the agreements made by matched firms and workers as given, and setting all other salaries at levels that make any worker indifferent between the firm he or she is matched with and every other firm. (Recall that salaries can be chosen independently, if necessary, for each firm-worker pair.) These salaries are competitive, since workers are all indifferent at them, and any firm that was not employing an optimal set of workers, taking salaries as given, would also be in a position to construct a coalition to improve upon the core allocation. This contradiction establishes the equivalence.

Like Crawford and Knoer, we distinguish between two different notions of the core, and further between the core in markets with discretely variable salaries and in the continuous market to which those discrete markets converge as salaries become more finely variable.

[1] As Franklin Fisher has pointed out to us, our treatment of agents' preferences is more closely related to that of producer theory than that of consumer theory. Since in producer theory there is no distinction between gross and net substitutes, "substitutes," rather than "gross substitutes," might be a more natural terminology. We use "gross substitutes" because it more clearly suggests the nature of our assumption when applied to consumers.

The following definitions are the appropriate modifications of the parallel concepts used by Crawford and Knoer, who motivate them.

D1: An *individually rational allocation* is an assignment of workers to firms together with a salary schedule such that, if $f: \{1, \ldots, m\} \to \{1, \ldots, n\}$ is the function that represents the assignments (so that $f(i)$ is the firm to which worker i is assigned) and $C^j \equiv \{i \mid j = f(i)\}$ (so that C^j is the set of workers hired by firm j),

$$(1) \qquad s_{if(i)} \geqq \sigma_{if(i)}, \quad \text{and}$$

$$(2) \qquad \pi^j(C^j; s^j) \equiv y^j(C^j) - \sum_{i \in C^j} s_{ij} \geqq 0.$$

D2: A (*discrete*) *strict core allocation* is an individually rational allocation $(f; s_{1f(1)}, \ldots, s_{mf(m)})$ such that there are no firm-set of workers combination (j, C) and (integer) salaries $r^j \equiv (r_{1j}, \ldots, r_{mj})$ that satisfy

$$(3) \qquad u^i(j; r_{ij}) \geqq u^i[f(i); s_{if(i)}] \qquad \text{for all} \quad i \in C, \quad \text{and}$$

$$(4) \qquad \pi^j(C; r^j) \geqq \pi^j(C^j; s^j),$$

with strict inequality holding for at least one member of $C \cup \{j\}$. (Such a coalition will be said to be capable of *improving upon* the allocation.)

D3: A (*discrete*) *core allocation* is defined in the same way as a (discrete) strict core allocation, except that it is required instead that there be no firm-set of workers combination and (integer) salaries that satisfy both (3) and (4) with strict inequality. (Such a coalition will be said to be capable of *strictly improving upon* the allocation.)

3. THE SALARY-ADJUSTMENT PROCESS

This section describes the rules of the salary-adjustment process.

R1. Firms begin facing a set of permitted salaries $s_{ij}(0) = \sigma_{ij}$. Permitted salaries at round t, $s_{ij}(t)$, remain constant, except as noted below. In round zero, each firm makes offers to all workers; this is costless by (MP).

R2. On each round, each firm makes offers to the members of one of its favorite sets of workers, given the schedule of permitted salaries $s^j(t) \equiv [s_{1j}(t), \ldots, s_{mj}(t)]$. That is, firm j makes offers to the members of $C^j[s^j(t)]$, where $C^j[s^j(t)]$ maximizes $\pi^j[C; s^j(t)]$. Firms may break ties between sets of workers however they like, with the following exception: Any offer made by firm j in round $t - 1$ that was not rejected must be repeated in round t. By (GS), the firm sacrifices no profits in doing this, since (by R4) other workers' permitted salaries cannot have fallen, and the salary of a worker who did not reject an offer remains constant.

R3. Each worker who receives one or more offers rejects all but his or her favorite (taking salaries into account), which he or she tentatively accepts. Workers may break ties at any time however they like.

R4. Offers not rejected in previous periods remain in force. If worker i rejected an offer from firm j in round $t-1$, $s_{ij}(t) = s_{ij}(t-1) + 1$; otherwise $s_{ij}(t) = s_{ij}(t-1)$. Firms continue to make offers to their favorite sets of workers, taking into account their permitted salaries.

R5. The process stops when no rejections are issued in some period. Workers then accept the offers that remain in force from the firms they have not rejected.

4. THE EXISTENCE OF CORE ALLOCATIONS

We can now establish the following theorem, whose proof consists of a series of lemmas:

THEOREM 1: *The salary-adjustment process* R1–R5 *converges in finite time to a discrete core allocation in the discrete market for which it is defined.*

LEMMA 1: *Every worker has at least one offer in every period.*

PROOF: In round 0, every firm makes offers to all workers by R1. Since each worker tentatively accepts some offer in each round, by R4 his or her permitted salary at that firm remains constant. By (GS) and R2, that offer must therefore be repeated in the next round. *Q.E.D.*

LEMMA 2: *After a finite number of rounds, every worker has exactly one offer and the process stops.*

PROOF: By Lemma 1, each worker always has at least one offer. By R2, R3, and R4, if a worker has multiple offers, his or her permitted salary must rise for all bidders but one. Since the numbers $y^j(C)$ are finite, each worker eventually loses all but one offer. *Q.E.D.*

LEMMA 3: *The process converges to an individually rational allocation.*

PROOF: Let t^* be the round at which the process stops, and let ϕ and C^j_ϕ denote the assignment to which it converges. That $s_{i\phi(i)}(t^*) \geqq \sigma_{i\phi(i)}$ is immediate from R1 and R4. That $\pi^j[C^j_\phi; s^j(t^*)] \geqq 0$ follows from R2 and the fact that the firm is not required to hire any workers, since $\pi^j[\varnothing; s^j(t^*)] = y^j(\varnothing) = 0$ by (NFL). *Q.E.D.*

LEMMA 4: *The process converges to a discrete core allocation in the discrete market for which it is defined.*

PROOF: By Lemma 2, the process converges to an equilibrium. Denote it $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$, and let C_ϕ^j be the set of workers assigned to firm j by ϕ. Suppose, by way of contradiction, that this is not a discrete core allocation. Then since $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$ is individually rational by Lemma 3, there must exist a firm-set of workers combination (j, C) and integer salaries r^j such that

$$u^i(j; r_{ij}) > u^i[\phi(i); s_{i\phi(i)}(t^*)] \qquad \text{for all} \quad i \in C, \quad \text{and} \tag{5}$$

$$\pi^j(C; r^j) > \pi^j[C_\phi^j; s^j(t^*)]. \tag{6}$$

By (5) and R3, worker i must never have received (and therefore never have rejected) an offer from firm j at a salary r_{ij} or greater. Since permitted salaries never fall, the salaries firm j is permitted to offer the members of C, $s^j(t^*)$, satisfy $s^j(t^*) \leqq r^j$. But then

$$\pi^j[C; s^j(t^*)] \geqq \pi^j(C; r^j) > \pi^j[C_\phi^j; s^j(t^*)]. \tag{7}$$

By R2 and (7), $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$ cannot be an equilibrium, contrary to hypothesis. This establishes the lemma and, therefore, the theorem. *Q.E.D.*

A major difference between this result and Theorem 1 of Crawford and Knoer [**1**] is that, here, firm size is endogenous. At equilibrium, a firm will hire all those workers who make a positive contribution to profits at their current "asking" prices. For the workers the firm hires, the asking price will be the salary that, ignoring the discreteness, just makes it too expensive for any other firm to match the worker's utility level. The size of firms is determined by production opportunities in the rest of the economy. By allowing the set of firms to include potential firms as well, the model determines which firms will actually be in operation.

As before, the proof that every discrete market has a core, and that the salary-adjustment process for that market converges to a point in the core, can be used to prove the existence of a strict core of the market with continuous salary adjustment.

THEOREM 2: *Every continuous market has a strict core allocation.*[2]

PROOF: Suppose by way of contradiction that there is a continuous market with no strict core allocation. We shall argue that for sufficiently small choices of the unit of measurement, this implies that the corresponding discrete market has

[2] At first glance, this result may appear to be in conflict with Jacques Drèze and Joseph Greenberg's [**2**] demonstration that efficiency may require transfers across coalitions that seem to violate coalition "rationality." There is no contradiction, since Drèze and Greenberg are examining problems that do not arise at core allocations [**2**, Proposition 1]; we are establishing sufficient conditions for the existence of a core.

no core allocation either. The proof proceeds by showing that in a continuous market with no strict core, the potential gains to the firm-set of workers coalition that would gain most by improving upon a given allocation are bounded above zero for all individually rational allocations. Thus, choosing the unit of measurement sufficiently smaller than this bound insures that any individually rational allocation can be improved upon by at least one firm-set of workers coalition in the corresponding discrete market as well, contradicting Theorem 1.

Consider any individually rational allocation $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$ that is not in the core of the continuous market, and call the set of workers assigned to firm j by this allocation C^j_ϕ. Let ρ_{ij} be defined by the equation $u^i(j;\rho_{ij}) \equiv u^i[\phi(i); s_{i\phi(i)}]$, and define

$$(8)\qquad D[(j,C);\phi;s_{1\phi(1)},\ldots,s_{m\phi(m)}] \equiv y^j(C) - \sum_{i\in C}\rho_{ij} - y^j(C^j_\phi) + \sum_{i\in C^j_\phi} s_{ij}.$$

In words, $D(\cdot)$ is the (possibly negative) total gain realizeable by the coalition of workers C and firm j by upsetting the allocation $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$. Define

$$(9)\qquad F(\phi;s_{1\phi(1)},\ldots,s_{m\phi(m)}) \equiv \max_{(j,C)} D[(j,C);\phi;s_{1\phi(1)},\ldots,s_{m\phi(m)}].$$

By hypothesis, $F(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)}) > 0$ as long as $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$ is individually rational, since otherwise no firm-set of workers coalition could improve upon that allocation in the continuous market. We shall now show that $F(\cdot)$ is bounded above zero for all individually rational allocations.

To see this, note that F is continuous in $(s_{1\phi(1)}, \ldots, s_{m\phi(m)})$ for any given assignment ϕ because the maximum of continuous functions is continuous, and let

$$(10)\qquad G(\phi) \equiv \min_{(s_{1\phi(1)},\ldots,s_{m\phi(m)})} F(\phi;s_{1\phi(1)},\ldots,s_{m\phi(m)})$$

subject to

$$s_{i\phi(i)} \geqq \sigma_{i\phi(i)} \qquad \text{for all } i, \quad \text{and}$$

$$\pi^j(C^j_\phi;s^j) \equiv y^j(C^j_\phi) - \sum_{i\in C^j_\phi} s_{ij} \geqq 0 \qquad \text{for all } j.$$

Finally, define

$$(11)\qquad H \equiv \min_{\phi\in\Phi} G(\phi),$$

where Φ is the set of all functions $\phi: \{1, \ldots, m\} \to \{1, \ldots, n\}$. G is well-defined because for any given $\phi \in \Phi$, F is continuous and the feasible region of the problem on the right-hand side of (10) is nonempty (by our assumption (MP)) and compact. Further, $G(\phi) > 0$ for all $\phi \in \Phi$ because, as noted above, $F(\cdot) > 0$ everywhere in the feasible region for all $\phi \in \Phi$. Finally, H is well-defined and strictly positive because Φ is a finite set. Thus choosing the unit of measurement smaller than $H/(m+1)$ will suffice for the validity of the above arguments, and the proof is complete. *Q.E.D.*

It is possible to reinterpret our model, like Crawford and Knoer's [1], to allow any number of endogenous job characteristics (rather than just salaries) that enter firms' and workers' preferences in any way that is compatible with a "well-behaved" utility-possibility frontier. This follows from the fact that all our arguments are completely ordinal, so that the "salaries" can simply be viewed as parameterizations of the relevant utility-possibility frontiers associated with the firms and sets of workers who are matched in the market. Thus, if endogenous job characteristics are negotiated efficiently, all the results go through unchanged.

Given this interpretation, what is the relation between our assumption (GS) and the more customary gross-substitutes assumption applied to firms' demands for labor hours? Since hours are negotiated efficiently, but behind the scenes, in our model, a firm's response to increases in the required utility levels of some of its workers might cause other workers' hours to decline. By (GS), the firm will still be willing to hire these workers at their required utility levels; but the efficient combination of job characteristics that meets those requirements may shift. This differs from what the usual form of the gross-substitutes assumption requires. So, while the analogy with the usual gross-substitutes assumption is suggestive, it is by no means exact.

Theorem 2 can be used to establish sufficient conditions for the existence of core allocations in some markets in which it is not natural to impose the two-sided structure. Such markets can be viewed as ones in which there are no firms and output is the product of workers' coalitions.

Let the set of workers be indexed $i = 1, \ldots, m$, as before. There are no firms. The technological possibilities of the one-sided economy are represented by a function v, which gives the output that can be produced by any coalition $C \subseteq \{1, \ldots, m\}$. Thus, $v(C)$ is the product of group of workers C. While v is analogous to a transferable-utility characteristic function, we shall use the somewhat weaker assumption that each worker has a utility function $\mu^i(s_i)$ which depends only on his or her consumption of the product, and which is continuous and strictly increasing in s_i.

The definitions of an individually rational allocation, a strict core allocation, and our (MP) assumption must be modified to fit the one-sided market:

D1′: An *individually rational allocation in the one-sided market* is a partition of the set of workers $(C_1, \ldots, C_Z)$, such that every worker i is a member of just one coalition—$i \in \cup_{z=1}^{Z} C_z$, and $C_z \cap C_{z'} = \emptyset$ if $z \neq z'$—together with a salary schedule $s \equiv (s_1, \ldots, s_m)$ that satisfies

(12) $$s_i \geqq 0 \quad \text{for} \quad i = 1, \ldots, m, \quad \text{and}$$

(13) $$\sum_{i=1}^{m} s_i \leqq \sum_{z=1}^{Z} v(C_z).$$

D2′: A *strict core allocation in the one-sided market* is an individually rational allocation in the one-sided market, denoted $(C_1, \ldots, C_Z; s)$, for which there is no set of workers C and salaries $r \equiv (r_1, \ldots, r_m)$ that satisfy

(14) $\quad r_i \geqq s_i \quad$ for all $\quad i \in C, \quad$ and

(15) $\quad \sum_{i \in C} r_i \leqq v(C),$

with strict inequality of (14) holding for at least one member of C. Such a set of workers will be said to be capable of *improving upon* the allocation *in the one-sided market*.

(MP′) $\quad v(C \cup \{i\}) - v(C) \geqq 0 \qquad$ for all i and C.

Create $m + 1$ dummy firms and endow each of them with the production technology $y^j(C) \equiv v(C)$. Workers are supposed to be indifferent between the firms, so $u^i(j; s_{ij}) \equiv \mu^i(s_{ij})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m + 1$.

THEOREM 3:[3] *If the production technology v obeys assumptions (MP'), (NFL), and (GS) applied to the $m + 1$ dummy firms, there is a strict core allocation in the one-sided market.*

PROOF: The method of proof is to use the dummy firms to provide a second side for the market. The resulting fictitious market meets the requirements of Theorem 2, so it has a strict core allocation in the sense of D2. We shall argue that in the fictitious market, there are enough opportunities for workers that its strict core allocations are strict core allocations of the one-sided market.

Since the firms play no other role in this proof than to embody the technology, v, the firms' indices can be used as the names of the workers' coalitions in the one-sided market. The set of workers' coalitions associated with the strict core allocation in the fictitious market, $\{C_1, \ldots, C_{m+1}\}$, is a partition of the set of workers. To see that this partition together with the salaries awarded to the workers at that strict core allocation comprise a strict core allocation of the one-sided market in the sense of D2′, note that dummy firms' profits must all be zero at a strict core allocation of the fictitious market. (This is so since there are not enough workers for all firms to employ one, and firms with no workers receive $y^j(\emptyset) = 0$, by (NFL). Because firms are identical, a coalition involving a zero-profit firm could always be formed to improve upon any allocation that yielded any firm positive profits.) It follows that an allocation that can be improved upon in the one-sided market can also be improved upon in the fictitious, two-sided market. Thus, a strict core allocation in the two-sided market

[3]This result is similar in spirit to, but logically independent of, the core existence result established in Shapley [5]. The difference between our result and Shapley's is due to our different definition of substitutability. We define it as a property of firms' input demands; Shapley defines it in terms of the second differences of the characteristic function.

always yields a strict core allocation in the one-sided market in the sense of D2′, completing the proof. *Q.E.D.*

5. CHARACTERISTICS OF THE SALARY-ADJUSTMENT PROCESS EQUILIBRIUM

In models like those considered here, heterogeneity and indivisibility usually result in a range of equilibria; this raises the possibility that which equilibrium arises will depend on the choice of adjustment process. The salary-adjustment process of this paper shares this property; it creates a systematic bias in firms' favor, because they make offers.[4] The extent of this bias depends on the degree to which firms and workers are substitutable for other members of their sides of the market.

It is convenient to prove this for discrete markets that do not have ties between the utilities that arise in core allocations and those that arise from other assignments. The no-ties requirement is that no worker (or firm) is indifferent between an assignment and salary (or profit) that forms part of a core allocation and any other assignment and salary (or profit) that is permitted in the discrete market.

If there is any discrete (strict) core allocation in a given discrete market that assigns worker i to firm j at salary s, we say i is (*strictly*) *s-possible* for j. Suppose worker i is s_{ij}-possible for firm j. Then for any other firm k, and any salary s_{ik} firm k is permitted to offer worker i in this market,

$$\text{(NTW)} \quad u^i(j; s_{ij}) \neq u^i(k; s_{ik}).$$

Similarly, if there is any discrete core allocation of a given discrete market that assigns set of workers C^j to firm j at salaries $s^j \equiv (s_{1j}, \ldots, s_{mj})$, then for any other non-empty set of workers $\tilde{C}$ and salaries $\tilde{s}^j \equiv (\tilde{s}_{1j}, \ldots, \tilde{s}_{mj})$ that firm j is permitted to offer the members of $\tilde{C}$ in this market,

$$\text{(NTF)} \quad y^j(C^j) - \sum_{i \in C^j} s_{ij} \neq y^j(\tilde{C}) - \sum_{i \in \tilde{C}} \tilde{s}_{ij}.$$

When the productivities $y^j(C)$ are randomly chosen real numbers, and the utility function $u^i(\cdot\,;\cdot)$ randomly chosen from an appropriate class of functions, (NTW) and (NTF) will "almost never" fail to hold in discrete markets, except at "the origin": R1 of the salary-adjustment process requires that all workers be indifferent between the offers they receive from firms in round 0. However, for any discrete market where one of these assumptions fails, a "similar" discrete market can be found where (NTW) and (NTF) hold. This is accomplished by perturbing the starting salaries σ_{ij} independently of the unemployment utility levels $u^i(0; 0)$ in any way that preserves (MP).

[4] We have not considered the effects of reversing the roles of firms and workers in the adjustment process, as was done by Crawford and Knoer [1]. Because of the use made of our assumption that workers are indifferent about which other workers their firms hire, this seems to involve significantly greater difficulties in the non-separable case studied here.

THEOREM 4: *Consider a discrete market in which (NTW) and (NTF) hold. In such a market, the salary-adjustment process converges to a discrete strict core allocation that is at least as good for every firm as any other strict core allocation.*

PROOF: The proof of Theorem 4 follows a proof by Crawford and Knoer [**1**, Theorem 3], which is itself based on a proof by Gale and Shapley [**3**]. First, convergence to a discrete strict core allocation will be proved, then the bias in the firms' favor.

If an equilibrium of the salary-adjustment process $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$ were not a discrete strict core allocation, there would be a firm j and a set of workers C who could improve upon it. Since any equilibrium is a discrete core allocation by Theorem 1, assumptions (NTW) and (NTF) apply, so firm j and those workers in C who were not assigned to firm j by ϕ will strictly prefer the improved configuration to $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$. The salary-adjustment process guarantees that the salaries firm j would be permitted to offer the members of C are not larger than those they would have to receive to prefer the improved configuration, so firm j must have violated R2, and $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$ could not be an equilibrium of the salary adjustment process.

The proof that the equilibrium allocation is at least as good for every firm as any other strict core allocation consists of showing that no worker will ever reject any firm at salary s when the worker is strictly s-possible for the firm. Since permitted salaries can only rise, this implies that every set of workers who are strictly possible for a firm in a strict core allocation is available at every round of the salary-adjustment process at permitted salaries that are no bigger than those at which they are strictly possible. By R2 and the fact that firms prefer lower salaries, *ceteris paribus*, a simple revealed-preference argument establishes the result.

To show that no worker will ever reject a firm at salary s if the worker is strictly s-possible for the firm, it suffices to show that no worker will ever be the first to do so. Suppose that, until some round of the salary-adjustment process, no worker has rejected any firm at a salary at which the worker is strictly possible for the firm. At that round, worker i rejects an offer from firm j at salary s_{ij} in favor of an offer from firm k at salary s_{ik}. Then i cannot be strictly s_{ij}-possible for j.

If worker i were indifferent between the offers from firm j and firm k, he or she would not be possible for either of them at the offered salaries, by (NTW). If worker i is not indifferent, the choice means that $u^i(k; s_{ik}) > u^i(j; s_{ij})$. On firm k's side, note that at any strict core allocation, the salaries facing firm k are at least as large as the salaries permitted to k when it made the offer to i at s_{ik}. This is so since, by hypothesis, k must not yet have been rejected by any worker at a salary at which the worker is strictly possible for k. Thus, by assumption (GS), if i were available at s_{ik}, firm k would prefer to add worker i to any set of workers assigned to k in a strict core allocation, if that set did not already include i. Now consider any allocation that assigns worker i to firm j at salary s_{ij}, and all other workers to firms at salaries at which they are strictly possible for the firms. Then worker i, firm k, and the other workers assigned to k can improve upon this assignment; hence, it is not a strict core assignment and i is not strictly s_{ij}-possible for j. *Q.E.D.*

COROLLARY: *In a discrete market in which (NTW) and (NTF) hold, the equilibrium of the salary-adjustment process is unique.*

PROOF: By Theorem 4, the salary-adjustment process converges to a discrete strict core allocation that is at least as good for every firm as any other discrete strict core allocation. The proof of this corollary consists in showing that the allocation with this property is unique.

Theorem 4 implies that if there were multiple equilibria of the salary-adjustment process in a discrete market, each firm would be indifferent among them. Call one such equilibrium $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$. That the assignment ϕ is unique follows directly from the firms' indifference and the requirement that there be no ties between the profits that accrue to firms in core allocations and the profits from hiring other sets of workers. This, in turn, implies that the sum of the salaries of the workers that are assigned by ϕ to each firm is the same in all equilibria, but no worker's salary can be higher in one equilibrium than in another. If it were, the worker would have rejected the firm at a salary for which he or she is possible for the firm; this contradicts the proof of Theorem 4. We conclude that every worker receives the same salary, and that the equilibrium is unique. *Q.E.D.*

Theorem 4 is the analog to Gale and Shapley's [**3**] result that, of the core assignments of men and women in the marriage game, the "deferred-acceptance" procedure will choose the one most favorable to the side that makes the offers. This shows that the salary-adjustment process (both in its present incarnation and in Crawford and Knoer's [**1**]) will reach the firms' end of Shapley and Shubik's elongated core (suitably restricted for discrete salaries). What happens to this advantage if the players change? Intuitively, we expect that increasing the number of firms will make workers better off, and that the reverse situation, increasing the number of workers, will leave firms better off. This intuition is correct as a comparison of equilibria of the salary-adjustment process. Indivisibility and heterogeneity notwithstanding, adding a firm results in an equilibrium of the salary-adjustment process which is at least as good for every worker as the original equilibrium, and *vice versa*.

A discrete market in this paper is defined by a set of firms $\{1, \ldots, n\}$, a set of workers $\{1, \ldots, m\}$, a set of productivities $\{y^j(C)\}$, the workers' utility functions $\{u^i(\cdot;\cdot)\}$, and a unit size by which firms' permitted salaries can increase. Call the equilibrium of the salary-adjustment process in this market $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$. We shall consider the *firm-augmented* market, which differs from the original market only by including firm $n+1$, along with its production technology $y^{n+1}(C)$ and workers' preferences with regard to it, $u^i(n+1; \cdot)$, and the *worker-diminished* market, which differs from the original market only by the absence of worker m. In the firm-augmented market, the new technology and workers' preferences must satisfy the same restrictions as in the original market. The equilibrium $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$ will be compared to the equilibrium of the salary-adjustment process in the firm-augmented market, denoted $(\psi; s_{1\psi(1)}, \ldots, s_{m\psi(m)})$, and to the equilibrium of the salary-adjustment process in the worker-diminished market, $(\theta; s_{1\theta(1)}, \ldots, s_{m-1,\theta(m-1)})$.

THEOREM 5: *When the conditions of Theorem* 4 *hold for the discrete market, its firm-augmented form, and its worker-diminished form*: (FA) *each of the workers is at least as well-off at the equilibrium of the salary-adjustment process in the firm-augmented market as it is at the equilibrium of the salary-adjustment process in the original market*; *and* (WD) *each of the firms is at least as well-off at the equilibrium of the salary-adjustment process in the original market as it is at the equilibrium of the salary-adjustment process in the worker-diminished market.*

REMARK: It is revealing to see why Theorem 5 (FA) is true in the case of transferable utility and separable production functions. In this case, $u^i(j; s_{ij}) \equiv a_{ij} + s_{ij}$, and worker i's productivity at firm j is fixed at b_{ij} independent of who his or her co-workers are, so $y^j(C) \equiv \sum_{i\in C} b_{ij}$. Suppose worker i is assigned to firm j at salary s_{ij} at the equilibrium of the salary-adjustment process in the original market. Call the second-best firm for worker i to work with firm k, and let $\tilde{s}_{ik}$ be the maximum salary firm k is both permitted and willing to offer worker i; $\tilde{s}_{ik}$ is the permitted salary with the property that

(16) $$b_{ik} - 1 < \tilde{s}_{ik} < b_{ik}.$$

Due to the discreteness of salary offers and the absence of ties, worker i's equilibrium utility $a_{ij} + s_{ij}$ is completely determined by the inequalities

(17) $$a_{ij} + s_{ij} - 1 < a_{ik} + \tilde{s}_{ik} < a_{ij} + s_{ij},$$

given the parameters of the market. The firm worker i is assigned to must at least match his or her second-best offer; but it need not do more than that (within the limits of discreteness). Since a worker's equilibrium utility in the original market is specified by his or her job satisfaction a_{ik} and productivity b_{ik} at the second-best firm, adding a new firm will change that utility only if it changes the second-best firm. Doing so can only raise the bounds that determine the worker's utility. This is the sense in which opportunity costs rather than marginal products determine workers' utilities at the equilibrium of the salary-adjustment process. The proof of Theorem 5 generalizes this result, showing that if a new firm enters or another worker leaves, worker i's opportunity cost cannot fall.

PROOF OF THEOREM 5: The proof uses the Corollary to Theorem 4. Since the equilibrium of the salary-adjustment process is unique, *any* process which attains an allocation in the strict core of a market that is (weakly) preferred by all firms must have attained the allocation that is the equilibrium of the salary-adjustment process in that market. We construct two processes, one for the firm-augmented case and one for the worker-diminished case, that modify the equilibrium of the original market $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$, changing it to strict core allocations (weakly) preferred by the firms in the two modified markets. Since these allocations are unique, they must be $(\psi; s_{1\psi(1)}, \ldots, s_{m\psi(m)})$ and $(\theta; s_{1\theta(1)}, \ldots, s_{m-1,\theta(m-1)})$ respectively. By inspection of the two modification processes, the theorem, which is a statement comparing equilibria, follows immediately.

Consider the salary-adjustment process to have taken place in the original market, and to have reached an equilibrium $(\phi; s_{1\phi(1)}, \ldots, s_{m\phi(m)})$. The *new-firm-on-the-block process* (NFB) is defined as follows:

NFB1: Firm $j (j = 1, \ldots, n)$ has starting permitted salaries $s^j(t^*) \equiv [s_{1j}(t^*), \ldots, s_{mj}(t^*)]$ from the final round, t^*, of the salary-adjustment process. The new firm, $n+1$, has starting permitted salaries $(\sigma_{1,n+1}, \ldots, \sigma_{m,n+1})$.

NFB2–NFB5: These are the same as R2–R5 of the salary-adjustment process, *mutatis mutandis* (Section 3).

The *take-my-marbles process* (TMM) is defined as follows:

TMM1: Firm j has starting permitted salaries $[s_{1j}(t^*), \ldots, s_{m-1,j}(t^*), \bar{s}_{mj}]$, where the first $m-1$ of these are the final-round permitted salaries of the salary-adjustment process, and $\bar{s}_{mj} \equiv [\max_C y^j(C)] + 1$, the maximum taken over all sets of workers $C \subseteq \{1, \ldots, m\}$.

TMM2–TMM5: These are the same as R2–R5 of the salary-adjustment process, *mutatis mutandis* (Section 3).

The process NFB1–NFB5 converges in finite time to a discrete strict core allocation of the firm-augmented market that is at least as good for every firm $j = 1, \ldots, n+1$ as any other strict core allocation of the firm-augmented market. The proof of this statement is the same as the proof of Theorems 1 and 4 applied to the sequence of the salary-adjustment process for the original market and then NFB. Since that allocation is unique, it is $(\psi; s_{1\psi(1)}, \ldots, s_{m\psi(m)})$, the equilibrium of the salary-adjustment process in the firm-augmented market.

The process TMM1–TMM5 converges in finite time to a discrete strict core allocation of the worker-diminished market that is at least as good for every firm as any other strict core allocation of the worker-diminished market. The proof of this statement is the same as the proofs of Theorems 1 and 4 applied to the sequence of the salary-adjustment process for the original market and then TMM. When the proof of Theorem 1 refers to "every worker," this should be interpreted as every worker in the worker-diminished market, $\{1, \ldots, m-1\}$. At every round of TMM, worker m has no offers, and so does not appear in the equilibrium allocation. We can identify that equilibrium with the equilibrium of the salary-adjustment process in the worker-diminished market, $(\theta; s_{1\theta(1)}, \ldots, s_{m-1,\theta(m-1)})$.

To complete the proof of Theorem 5, note that in both NFB and TMM, salaries to workers can only rise. Workers are no worse off when there are more firms; they will (weakly) prefer the firm-augmented market equilibrium to the original one. Firms are no worse off when there are more workers; they will (weakly) prefer the original market equilibrium to the worker-diminished one. *Q.E.D.*

6. THE GROSS-SUBSTITUTES CONDITION

This section examines the meaning of the gross-substitutes condition. When workers are all alike in production, the gross-substitutes condition is equivalent to the requirement that technologies show nonincreasing returns to workers. When workers are identical, the product of a group of workers with a firm depends only on the number of workers it hires. Thus, $y^j(C) \equiv \tilde{y}^j(|C|)$, where $|C|$ is the cardinality of C. In this environment, nonincreasing returns to workers means that

$$\text{(DR)} \qquad \tilde{y}^j(w+1) - \tilde{y}^j(w) \leqq \tilde{y}^j(w) - \tilde{y}^j(w-1)$$

for integer values of $w, 1 \leqq w \leqq m-1$, where w is the number of (identical) workers firm j hires.

THEOREM 6: *If workers are alike in production, (DR) and (GS) are equivalent.*

PROOF: To see that (DR) implies (GS), suppose that group of workers $C^j(s^j)$ maximizes firm j's profits $\pi^j(C; s^j)$ at the salaries s^j facing firm j, and let $|C^j(s^j)| = w^*$. Order the workers by their permitted salaries, so that $s_{1j} \leqq s_{2j} \leqq \cdots \leqq s_{mj}$. $C^j(s^j)$ is the set of workers with indices $i \leqq w^*$. Since the production technology exhibits (DR), and salaries rise with rank in the ordering above,

$$\text{(18)} \qquad s_{ij} \leqq \tilde{y}^j(i) - \tilde{y}^j(i-1) \qquad \text{if and only if} \quad i \leqq w^*.$$

If the salary firm j is permitted to offer worker i remains the same when the permitted salaries of some workers rise and none fall, worker i's rank in the salary ordering above can only fall, so the production standard he or she must meet can only rise. If $i \leqq w^*$, worker i will remain part of the maximizing set of workers for firm j, and (GS) is satisfied.

To see that (GS) implies (DR), we show that if (DR) is not true, it is possible to construct a pair of salary vectors that leads to the violation of (GS). If (DR) does not hold everywhere, there is some integer w, $1 \leqq w \leqq m-1$, for which

$$\text{(19)} \qquad \tilde{y}^j(w+1) - \tilde{y}^j(w) > \tilde{y}^j(w) - \tilde{y}^j(w-1).$$

If permitted salaries are less than $\min_{i=1,\dots,w}[\tilde{y}^j(i) - \tilde{y}^j(i-1)]$ for w low-salaried workers, greater than $\max_{i=w+1,\dots,m}[\tilde{y}^j(i) - \tilde{y}^j(i-1)]$ for $m-w-1$ high-salaried workers, and between the two sides of inequality (19) for worker h, then worker h will be included in the maximizing set of workers for firm j. Consider another set of permitted salaries that are the same for all workers except worker l. At the original permitted salaries, worker l was one of the low-salaried workers; in the new set of permitted salaries, he or she is high-salaried. At the new permitted salaries, worker h is no longer a member of any profit-maximizing coalition of workers. Thus, (GS) does not hold. *Q.E.D.*

Since the restriction of (GS) to markets with workers who are identical (in production) is nonincreasing returns to workers, (GS) can be regarded as a generalization of this familiar condition to the case of heterogeneous workers. This notion of (GS) makes more intuitive the reasons that something like it is required to prove the existence of a competitive equilibrium; it has the effect of a convexity condition on firms' production technologies.

In the heterogeneous case (GS) is equivalent to subadditivity of the production function when there are only two workers; but subadditivity does not imply (GS) when there are three or more workers. The latter point will be established by an example, which is also used to show that without (GS), the core and, equivalently, the competitive equilibrium may fail to exist in the model of this paper.

More precisely, define subadditivity of a firm's production technology to mean that unions of disjoint sets of workers can produce no more with that firm's technology than the sum of the products of those sets taken separately. Superadditivity is defined analogously; and we shall use these terms, which are used in the same sense in which they are applied to characteristic functions in game theory, to refer to the "weak" notions just defined.

To show that, in the two-worker case, subadditivity, which includes the separable case analyzed by Crawford and Knoer as a borderline case, is equivalent to (GS), we find the regions in salary space corresponding to a representative firm's demands for workers. For some firm j, let $y^j(\{1\}) \equiv v_1$; $y^j(\{2\}) \equiv v_2$; $y^j(\{1,2\}) \equiv v_{12}$; and recall that $y^j(\emptyset) \equiv 0$. There are four possible demands at any salary pair (s_{1j}, s_{2j}). The firm will want to hire $\{1,2\}$ if (neglecting ties),

$$v_{12} - s_{1j} - s_{2j} > \max\{v_1 - s_{1j}, v_2 - s_{2j}, 0\}. \tag{20}$$

Similar conditions define the regions in salary space where the firm wishes to hire $\{1\}$, $\{2\}$, and $\emptyset$. These regions are depicted in Figure 1 for the subadditive case, which is characterized by the condition $v_1 + v_2 \geqq v_{12}$; and in Figure 2 for the superadditive case, which is characterized by the condition $v_1 + v_2 \leqq v_{12}$; "strict" sub- and super-additivity are illustrated in the figures. As the figures make clear, our (GS) assumption is always satisfied in the subadditive case, where, for example, a rise in s_{1j} never induces the firm to stop demanding worker 2, but never in the strictly supperadditive case. This is a natural extension of what happens in the extreme superadditive case of perfect complements.

When there are three or more workers, the relationship between the production technology and (GS) becomes more complex. Examples can still be constructed

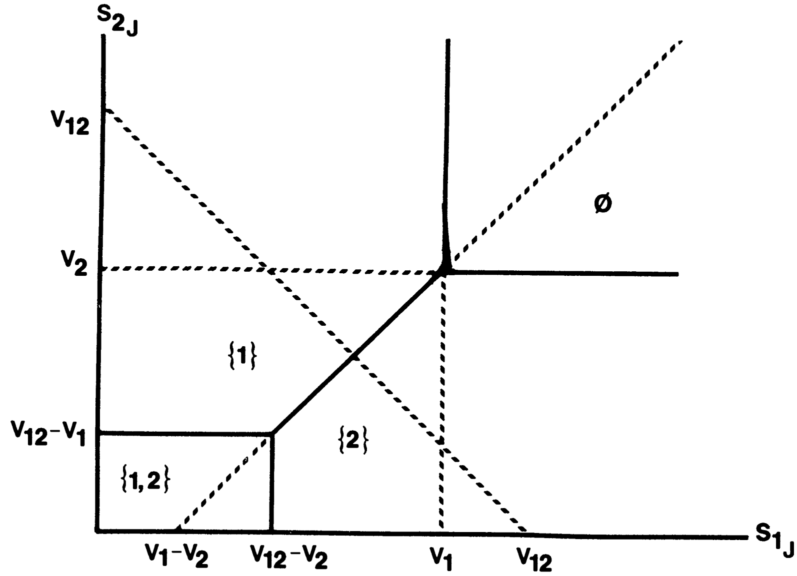

FIGURE 1—Subadditive case.

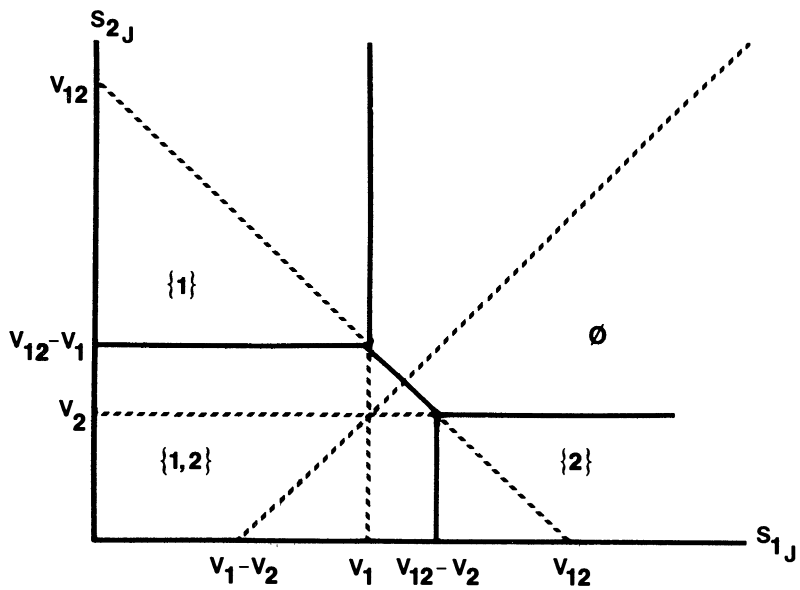

FIGURE 2—Superadditive case.

(by adding workers who enter the technology nearly separably) where (GS) fails due to superadditivity. But in the m-worker case, subadditivity without complete separability no longer implies (GS). Even though the technology may otherwise be regular, if what worker i adds to the firm's product depends on who his or her co-workers are, the possibility arises that whether the firm will want to hire worker i or not will depend on the salaries it is permitted to offer the other workers. A rise in the salary of a worker to whose efforts worker i adds relatively a lot may result in the firm ceasing to want to hire worker i.

The following non-pathological example illustrates that (GS) can fail under our (MP) assumption and subadditivity. There are three workers; 1, 2, and 3. Firm j's production technology is

$$y^j(\{1\}) \equiv 4, \quad y^j(\{2\}) \equiv 4, \quad y^j(\{3\}) \equiv 4\tfrac{1}{4},$$

$$y^j(\{1,2\}) \equiv 7\tfrac{1}{2}, \quad y^j(\{1,3\}) \equiv 7, \quad y^j(\{2,3\}) \equiv 7,$$

$$y^j(\{1,2,3\}) \equiv 9, \quad y^j(\varnothing) \equiv 0.$$

This technology is subadditive. Let $\sigma_{ij} = 0$ for $i = 1,2,3$. Then both (MP) and (NFL) hold. But (GS) fails. To see this, consider the two salary vectors $s^j \equiv (s_{1j}, s_{2j}, s_{3j}) \equiv (3,3,3)$, and $\tilde{s}^j \equiv (\tilde{s}_{1j}, \tilde{s}_{2j}, \tilde{s}_{3j}) \equiv (3,4,3)$. The unique preferred set of workers for firm j at salary vector s^j is $\{1,2\}$; this choice gives profits $\pi^j(\{1,2\}; s^j) = 1\frac{1}{2}$. At salary vector $\tilde{s}^j$, the unique preferred set of workers for firm j is $\{3\}$, which gives profits $\pi^j(\{3\}; \tilde{s}^j) = 1\frac{1}{4}$. Firm j does not want to hire worker 1 when permitted salaries are $\tilde{s}^j$, even though it does want to hire worker 1 when permitted salaries are s^j; $\tilde{s}_{1j} = s_{1j}$, and $\tilde{s}^j \geqq s^j$. The problem is that the product worker 1 adds when paired with worker 3 is enough less than what he or she adds to worker 2 that the firm wants to hire worker 1 if worker 2 is available at the right price, but not at prices where it prefers worker 3 to worker 2.

Without assumption (GS) or some other restriction on production technologies, there may be no core allocation and, therefore, no competitive equilibrium in the model of this paper. That there need not be a competitive equilibrium when goods are not perfectly divisible is an established fact. The results of this paper can be looked upon, therefore, as a set of conditions under which an equilibrium will exist, in spite of the indivisibility. A competitive equilibrium of the model of this paper with continuously variable salaries is an assignment of workers to firms and a set of mn salaries, one for each firm-worker pair, with the following properties: for all workers i and firms j, if worker i is assigned to firm j,

$$(21) \qquad u^i(j; s_{ij}) \geqq u^i(k; s_{ik}) \qquad \text{for all firms } k;$$

and if set of workers C^j is assigned to firm j,

$$(22) \qquad \pi^j(C^j; s^j) \geqq \pi^j(C; s^j) \qquad \text{for all sets of workers } C,$$

where $s^j \equiv (s_{1j}, \ldots, s_{mj})$ is the subset of the competitive equilibrium salaries facing firm j. In general, the set of core allocations includes, but may be larger than, the set of competitive equilibria. In this model, however, they are identical. The indivisibility and the two-sided structure of the market allow a competitive equilibrium set of salaries to be constructed to support every core allocation when there are enough prices so that different commodities are not required to sell at the same price.

As an example of an economy without a core in the absence of (GS), append to the example above a second firm, k, with a production technology that is symmetric to that of firm j:

$$y^k(\{1\}) = 4\tfrac{1}{4}, \quad y^k(\{2\}) = 4, \quad y^k(\{3\}) = 4,$$

$$y^k(\{1,2\}) = 7, \quad y^k(\{1,3\}) = 7, \quad y^k(\{2,3\}) = 7\tfrac{1}{2},$$

$$y^k(\{1,2,3\}) = 9, \quad y^k(\varnothing) = 0.$$

Let $\sigma_{ik} = 0$, and suppose that

$$u^i(j; s_{ij}) = s_{ij} \quad \text{and} \quad u^i(k; s_{ik}) = s_{ik} \tag{23}$$

for $i = 1,2,3$. When salaries can vary continuously, this economy does not have a core allocation. To see this, note that because utility is transferable for both firms and workers, and because $\sigma_{ij} = \sigma_{ik} = 0$ for all workers, any core allocation must, by its Pareto efficiency, maximize total product over all feasible assignments. In our example, this requires that either $\{1\}$ be assigned to firm j and $\{2,3\}$ to firm k, or $\{1,2\}$ to firm j and $\{3\}$ to firm k. Both yield a total product of $11\frac{1}{2}$; and they play symmetric roles in our example, so it suffices to show that either assignment cannot be part of a core allocation. Thus, consider the former assignment, and suppose that (s_{1j}, s_{2j}, s_{3j}) and (s_{1k}, s_{2k}, s_{3k}) are the associated salary schedules facing firms j and k. For this to be a core allocation, no firm and set of workers must be able to improve upon it. In particular, the following inequalities must hold as implications of the coalitions in parentheses being unable to improve upon the allocation:

$$4 - s_{1j} + s_{3k} \leqq 4\tfrac{1}{4} \qquad (j; \{3\}), \tag{24}$$

$$4 - s_{1j} + s_{1j} + s_{2k} \geqq 7\tfrac{1}{2} \qquad (j; \{1,2\}), \tag{25}$$

$$7\tfrac{1}{2} - s_{2k} - s_{3k} + s_{1j} \geqq 4\tfrac{1}{4} \qquad (k; \{1\}), \tag{26}$$

$$7\tfrac{1}{2} - s_{2k} - s_{3k} + s_{3k} \geqq 4 \qquad (k; \{3\}). \tag{27}$$

Inequalities (25) and (27) together imply that $s_{2k} = 3\frac{1}{2}$. Using this, (24) and (26) can be rewritten as

$$s_{3k} - s_{1j} \geqq \tfrac{1}{4}, \tag{24$'$}$$

$$s_{3k} - s_{1j} \leqq -\tfrac{1}{4}, \tag{26$'$}$$

a contradiction, completing the proof that there is no core, and thus no competitive equilibrium, in the example.

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