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JOB MATCHING WITH HETEROGENEOUS FIRMS AND WORKERS

BY VINCENT P. CRAWFORD AND ELSIE MARIE KNOER¹

Competitive adjustment processes in labor markets where firms and workers are heterogeneous but well informed are studied. A natural notion of equilibrium for such markets is defined, and a plausible adjustment process is shown under reasonable assumptions always to converge to an equilibrium; this allows a generalization of several existence results in the literature. Finally, the relationship between market institutions (such as who makes offers) and which of the range of equilibria that heterogeneity makes possible arises, is studied. Generalizing results of Gale and Shapley and Shapley and Shubik, it is shown that all agents on a given side of the market agree on which is the best equilibrium, and that the equilibrium that emerges is the one most favored by the agents on the side of the market that makes offers in the adjustment process. The process can also be viewed as an algorithm for transportation and optimal assignment problems.

1. INTRODUCTION

IT IS WIDELY BELIEVED that under conditions of perfect information, competitive labor markets will match workers and jobs efficiently even when productivity and job satisfaction vary interactively across firms and workers. According to Mortensen [9, p. 572], for example:

"in a world of heterogeneous workers and jobs, the problem of matching the two in some best way exists. A centralized competitive market sorts workers among jobs in a manner that maximizes aggregate output, appropriately defined, when information about technology and the abilities of individual workers is perfect."

In our view, this presumption is premature.² The purpose of this paper is to examine more closely the operation of competitive job markets with heterogeneous firms and workers. We find that with perfect information, there is a sense in which the presumption that such markets will operate efficiently is correct, but possibly only under nonvacuous assumptions about the dynamic process by which salaries and other endogenous job characteristics are adjusted to clear the market. We also find that in such markets, there is a striking relationship between market institutions (such as who makes offers) and which of the multiple equilibrium outcomes made possible by heterogeneity actually occurs.

Any analysis of competitive labor markets with perfect information and heterogeneous firms and workers must deal with several issues: the appropriate choice of equilibrium concept and its internal consistency; its stability; and the determination of which of the equilibria that exist, if there are more than one, will arise. The present paper attempts to resolve all three of these issues by constructing an adjustment process that, we hope, mimics the behavior of real competitive

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² For a similar view expressed in a different but completely analogous context, see Koopmans and Beckmann [7, p. 60].

labor markets, and analyzing its behavior in detail. The model developed here is quite general in almost all respects but its strong perfect information assumption. While we feel that imperfect information is an essential characteristic of real labor markets, we also believe that an analysis of the certainty case can yield useful predictions about markets where conditions are sufficiently stable that agents have had enough time to learn about their environments, and that such an analysis is, at any rate, a necessary preliminary to the study of the effects of imperfect information.³

Gale and Shapley [4] provide the first analysis of the implications of heterogeneity in a market-like setting. Considering the processes by which applicants are sorted into colleges and by which people find mates, they assume perfect information, that agents can pair only with agents on the opposite side of the "market," and that the preferences of agents on each side of the market over agents on the opposite side are fixed; thus, for example, colleges cannot vary their scholarship offers and suitors cannot adjust their promises to prospective mates. They further assume that agents' *capacities*—the number of agents on the opposite side of the market with which they can pair—are exogenous. Gale and Shapley take as their notion of equilibrium the natural one that no pair of agents on opposite sides of the market prefer each other to any of their current partners. As Shapley and Shubik [13, pp. 114–118] point out, this notion of equilibrium is equivalent to the core in such markets because the two-agent coalitions made up of agents on opposite sides of the market can accomplish anything for themselves that they could as members of larger coalitions, and more than they could acting separately. As they also suggest [13, p. 127], it is equivalent to the competitive equilibrium, appropriately defined; see Koopmans and Beckmann [7, pp. 58ff.] for the details. It is, for example, easy to see that this notion of equilibrium implies Pareto efficiency, a necessary condition for core allocations: were there a matching that everyone preferred to the current one, any pair matched in the preferred matching could unilaterally carry out their part of the adjustment, upsetting the current matching and showing that it could not have been an equilibrium.

Gale and Shapley [4] construct an adjustment process in which agents on one side of the market (firms, for example) make offers, up to their capacities, to their favorite agents on the other side of the market (workers, for example), who either reject them in favor of preferred offers from other firms or tentatively accept them, up to their capacities. Rejected firms then make offers to their next-most-preferred workers, and the process continues until no more rejections are issued. Gale and Shapley prove that the process just described informally always converges after a finite number of rounds to a matching in the core. This, of course, also demonstrates that the core exists, a result they emphasize is nontrivial by giving a simple four-person example (the "problem of the roommates") in which agents are not constrained to pair with agents on the opposite side of the market,

³ We do not mean to imply that there are no useful analyses of the effects of uncertainty in markets with heterogeneous agents in the literature; see, for example, Diamond and Maskin [1, 2], Jovanovic [6], and Mortensen [9, 10]. But ultimately, a full understanding of the operation of such markets must build on an analysis of the certainty case.

and in which the core is empty. Gale and Shapley close by proving a remarkable theorem: if agents are never indifferent between potential partners, all agents on a given side of the market agree on which is the best matching in the core, and the Gale-Shapley adjustment process always converges to the matching most favored by the firms (or, more generally, the agents who are permitted to make offers).

Gale and Shapley's [4] path-breaking analysis not only deals explicitly with the problem of heterogeneity in markets, but also raises the intriguing possibility that the range of equilibria that heterogeneity makes possible may allow market institutions to influence equilibrium outcomes in a systematic and perhaps unsuspected way. But the fact that jobs have salaries and other characteristics that affect job satisfaction and productivity, and that are normally subject to negotiation between employers and prospective employees, is an important obstacle to viewing Gale and Shapley's model as a model of real labor markets.

Shapley and Shubik [13] take an important step toward overcoming this obstacle, although they do not couch their results in terms of labor markets. They modify Gale and Shapley's model by incorporating a transferable utility good in which salaries can be paid. Since this good enters every agent's utility function in a linear, additively separable way, any core allocation, because it is Pareto efficient, must solve the optimal assignment problem of maximizing, over all possible matchings, the sum of productivities and job satisfactions, measured so as to make utility "transferable" by adjusting salaries. (Actually, if capacities exceed one, the problem is, formally speaking, a transportation problem; but the analysis is essentially the same in either case.) They then use the linear programming duality theory that is available for the assignment problem to prove, in the case of one-to-one matching, the existence of a core allocation—a matching and a salary schedule with the property that no firm and no worker can negotiate a salary at which they prefer each other to their current partners at their current salaries. As they remark [13, p. 127, n. 3], this result is not implied by standard competitive equilibrium or core existence results; compare, for example, the approximate results on the existence of the core and the competitive equilibrium obtained by Henry [5] and Dierker [3], respectively.

Shapley and Shubik [13, pp. 120–122] continue by showing "that the core tends to be *elongated*, with its long axis oriented in the direction of market-wide [salary] trends" (emphasis in original). This generalizes Gale and Shapley's [4] proof that when all job characteristics are exogenous, all agents on a given side of the market agree on the best core allocation. But Shapley and Shubik do not attempt to define an adjustment process for this model like that of Gale and Shapley, or to show that such a process necessarily converges to the end of the core favored by agents on the offer-making side of the market.

Section 2 of this paper presents the model. Section 3 describes a salary adjustment process and shows that for arbitrary capacities, when ties are ruled out it always converges to the end of the core expected from Gale and Shapley's result. This process also provides computational alternatives to the Hungarian method developed by Kuhn [8] for the simple assignment problem and to the standard methods for solving transportation problems. Then, it is shown that these results

can be reinterpreted to show that the core exists and is stable in a class of models in which there can be any number of job characteristics that enter firms' and workers' preferences in any way that is compatible with a continuous, downward-sloping utility possibility frontier for each firm-worker pair. (Firms' and workers' preferences are, however, assumed to be separable across different pairs.) These models are significantly more general than those considered by Shapley and Shubik [13], and cannot be analyzed by their methods because there is no longer a simple problem that the market "solves" in these cases. Thus, the existence and stability of the core, and Gale and Shapley's [4] remarkable result relating market institutions to the core allocation that is realized, can be extended to a model of labor markets that is quite general, except for its perfect information and separability assumptions. Section 4 concludes with a discussion of how closely the salary adjustment process is likely to mimic the operation of real labor markets under conditions of perfect information.

2. THE MODEL

This section studies the sorting process in labor markets where each firm plans to hire a given number of workers, each worker plans to accept a given number of jobs, and job satisfactions and productivities are separable across firm-worker pairs. It is most natural to assume that firms hire many workers while workers accept only one job, but our results do not require this; we shall, however, assume for notational simplicity that each firm hires exactly one worker and each worker accepts exactly one job. Our arguments extend almost without modification to the general case.

There are m workers and n firms, indexed $i = 1, \dots, m$ and $j = 1, \dots, n$ respectively. The i th worker's job satisfaction, productivity, and salary at the j th firm are denoted a_{ij} , b_{ij} , and s_{ij} respectively; $s_{ij}(t)$ is the salary the j th firm will be permitted to offer the i th worker at time t in the adjustment process described below. If $m \neq n$, dummy firms or workers whose satisfactions and productivities are always equal to zero can be created to make $m = n$; the adjustment process will ensure that the corresponding dummy salaries are zero as well. Henceforth, it will be assumed that this has been done, so that $m = n$; we shall denote their common value by n .

To avoid technical difficulties as much as possible, we begin by assuming that the a_{ij} , b_{ij} , s_{ij} , and $s_{ij}(t)$ are all integers; the continuous case will be analyzed by using results from the discrete case. Salaries are paid in a transferable-utility good whose units of measurement are the same as those of the a_{ij} and b_{ij} . Thus, worker i 's total satisfaction at the j th firm is $a_{ij} + s_{ij}$; his net productivity at the j th firm is $b_{ij} - s_{ij}$. The transferable-utility assumption will be relaxed below. In the general case when matching is not one-to-one, preferences are assumed to be *separable across pairs*, in the sense that the a_{ij} and the b_{ij} do not depend on which other workers are hired by the j th firm or which other firms employ the i th worker. It is further assumed that $a_{ij} + b_{ij} \geq 0$ for all (i, j) . This is a natural restriction, since any firm-worker pair for which it were not valid could simply choose to do nothing if

they found themselves matched; and letting zero represent the utility of doing nothing is simply a normalization. No further restrictions on the a_{ij} , b_{ij} , s_{ij} , and $s_{ij}(t)$ are needed. Except for the integer restrictions, this model is identical but for notation and interpretation to that of Shapley and Shubik [13].

Under perfect certainty, the natural notion of equilibrium in a labor market with heterogeneous firms and workers and fixed capacities requires that no firm and worker should be able to negotiate an agreement that is mutually beneficial when compared to (any of) those in force with their current partners. As mentioned above, in such markets this notion coincides with the competitive equilibrium, properly construed, and the core. To preserve the distinction between this kind of equilibrium and the equilibria of the dynamic salary adjustment process studied below, we shall refer to the present notion as the "core."

Actually, we shall find it useful to distinguish between two different notions of the core, and further to distinguish between the core in discrete markets and in the continuous market, with no integer restrictions on salaries, to which the discrete markets converge. These considerations motivate the following definitions, which are all readily extended, with some notational effort, to the case of arbitrary capacities:

DEFINITION 1: An *individually rational allocation* is an assignment of workers to firms together with a salary schedule such that, if $f: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is the one-to-one function that represents the assignment (so that $f(i)$ is the firm the i th worker is assigned to and $f^{-1}(j) \equiv g(j)$ is the worker assigned to the j th firm),

$$(1) \quad a_{if(i)} + s_{if(i)} \geq 0 \quad \text{and}$$

$$(2) \quad b_{g(i)f} - s_{g(i)f} \geq 0.$$

DEFINITION 2: A (*discrete*) *strict core allocation* is an individually rational allocation $(f; s_{1f(1)}, \dots, s_{nf(n)})$ such that there is no worker-firm combination (i, j) and (integer) salary s that satisfy:

$$(3) \quad a_{ij} + s \geq a_{if(i)} + s_{if(i)} \quad \text{and}$$

$$(4) \quad b_{ij} - s \geq b_{g(i)f} - s_{g(i)f}$$

with strict inequality holding in at least one of (3) and (4). (Such a coalition will be said to be capable of *improving upon* the allocation.)

DEFINITION 3: A (*discrete*) *core allocation* is defined in the same way as a (*discrete*) *strict core allocation*, except that it is required instead that there be no worker-firm combination and (integer) salary s that satisfy both (3) and (4) with strict inequality. (Such a coalition will be said to be capable of *strictly improving upon* the allocation.)

The distinction between the strict core and the core is frequently important in models with indivisibilities; see, for example, Shapley and Scarf [12] and Roth and

Postlewaite [11]. The two notions are equivalent in models like our continuous market, in which agents have continuous preferences and there is a desirable, perfectly divisible good; they are also equivalent in models like our discrete markets when agents can never be indifferent between potential partners. (Gale and Shapley [4], for example, avoid this whole issue by ruling out ties a priori.) In models where the two notions of core are not equivalent, it is immediately clear from the definitions that the strict core is always contained in the core. Further, it is easy to see that an allocation in the (strict) core of a continuous market at which salaries satisfy the appropriate integer restrictions is also in the (strict) core of the corresponding discrete market.

3. THE SALARY-ADJUSTMENT PROCESS

Consider the following adjustment process, defined for a discrete market on $[s_{ij}(t)]$, the matrix of integer salary offers that firms are *permitted to make* to workers at time t . Time is measured discretely.

R1. $s_{ij}(0) = -a_{ij}$ for all (i, j) . Unless otherwise noted below, $s_{ij}(t)$ is constant.

R2. Each firm initially makes an offer to its favorite worker, net of salary, given the schedule of permitted salaries $[s_{ij}(0)]$. Firm j , for example, makes an offer to worker i , where i is a solution of the problem $\max_k [b_{ki} - s_{kj}(0)]$. Firms may break ties at any time however they like.

R3. Each worker who receives one or more offers rejects all but his or her favorite (taking salaries into account), which he or she tentatively accepts. Workers may break ties at any time however they like.

R4. Offers not rejected in previous periods remain in force. If worker i rejected an offer from firm j in period $t-1$, $s_{ij}(t) = s_{ij}(t-1) + 1$; if not, $s_{ij}(t) = s_{ij}(t-1)$. Rejected firms continue to make offers to their favorite workers, taking into account their current permitted salaries.

R5. The process stops when no rejections are issued in some period. Workers then accept the offers that remain in force from the firms they have not rejected.

(When matching is not one-to-one, the only modifications required are that each firm makes offers in R2 and R4 up to its capacity and workers tentatively accept offers in R3 up to their capacities.)

We shall now establish the following theorem by proving a series of lemmas; straightforward modifications of the proofs extend them to the case where firms and workers have arbitrary capacities.

THEOREM 1: *The salary adjustment process R1–R5 converges in finite time to a discrete core allocation in the discrete market for which it is defined.*

LEMMA 1: *If worker i has at least one offer at time t , he or she always has at least one offer at any time $t' > t$.*

The proof of Lemma 1 is immediate from R3 and R4.

LEMMA 2: *After a finite number of periods, no rejections are issued, every worker gets exactly one offer, and the process stops.*

PROOF: Suppose worker i has no offers at some point in the process. As long as i has no offers, the salaries at which firms are permitted to make offers to him or her remain constant by R4, for i can issue no rejections. On the other hand, as long as i remains without an offer, at least one other worker, say, k , must be issuing rejections. Since rejections by k cause at least one of his or her permitted salaries to rise one unit per period, and at least one other worker's permitted salaries must remain constant in any period in which he issues a rejection, k cannot issue rejections indefinitely without reducing the number of his or her offers to one, at which point k can no longer issue rejections. As this argument is equally valid for any worker who has more than one offer, worker i 's competitive disadvantage must fall, in finite time, to the point where he or she gets at least one offer. When every worker who was initially without an offer gets one, as must happen in finite time by the above argument and Lemma 1, the process stops by R5. *Q.E.D.*

LEMMA 3: *The process converges to an individually rational allocation.*

PROOF: The proof that $a_{ij} + s_{ij}(t^*) \geq 0$ for all (i, j) , where t^* is the time at which the process stops, is immediate, since $s_{ij}(0) = -a_{ij}$ for all (i, j) and the $s_{ij}(t)$ never fall by R1 and R4.

To prove that $b_{ij} - s_{ij}(t^*) \geq 0$ if firm j hires worker i when the process converges, suppose to the contrary that $b_{ij} - s_{ij}(t^*) < 0$. By R4, firm j could not have made an offer at salary $s_{ij}(t^*)$ to worker i unless $b_{kj} - s_{kj}(t^*) < 0$ were true for all $k \neq i$ as well. But since $b_{ij} - s_{ij}(0) = a_{ij} + b_{ij} \geq 0$ for all (i, j) by R1 and our assumption that $a_{ij} + b_{ij} \geq 0$, it follows in this case from R1 and R4 that firm j was rejected at some salary at least once at some time strictly before t^* by every worker. But this is impossible, since before t^* at least one worker (the last one to acquire an offer) was always without an offer by Lemma 1 and R5; by R3, that worker could not have rejected firm j before t^* .⁴ This contradiction completes the proof. *Q.E.D.*

LEMMA 4: *The process converges to a discrete core allocation in the discrete market for which it is defined.*

PROOF: By Lemma 2, the process converges to an equilibrium; denote it $(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)})$, and let ψ stand for ϕ^{-1} . Suppose by way of contradiction that $(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)})$ is not a discrete core allocation. Then since $(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)})$ is individually rational by Lemma 3, there must exist a

⁴ In the case of a firm whose capacity is greater than one, this argument becomes: Suppose that $b_{ij} - s_{ij}(t^*) < 0$ for some worker i hired by firm j . Then firm j must not, by R4, have been able to make offers up to its capacity to workers k for which $b_{kj} - s_{kj}(t^*) \geq 0$. This implies that firm j was rejected at some salary by a number of workers that exceeds the total capacity of all other firms; but this contradicts Lemma 1 and R5, since firms never make more than one offer at a time to a given worker.

worker i , a firm j , and an integer salary s such that:

$$(5) \quad a_{ij} + s > a_{i\phi(i)} + s_{i\phi(i)} \quad \text{and}$$

$$(6) \quad b_{ij} - s > b_{\psi(j)i} - s_{\psi(j)i}$$

For any integer s satisfying (5) and (6), firm j must at one time have made an offer to worker i at salary s by R4. (By (5) and Lemma 3, $a_{ij} + s > 0$, so this is consistent with the rules of the adjustment process.) Again by R4, this offer must have been rejected, or firm j could not later have made an offer to worker $\psi(j)$. As workers never lose offers except by rejecting them for ones at least as good, worker i 's equilibrium offer from firm $\phi(i)$ at salary $s_{i\phi(i)}$ must satisfy

$$(7) \quad a_{i\phi(i)} + s_{i\phi(i)} \geq a_{ij} + s.$$

But this contradicts (5), establishing the lemma. Q.E.D.

Lemma 4 completes the proof of Theorem 1. It is interesting to note that in our adjustment process, the market is cleared by a simultaneous lowering of the "expectations" of firms and raising of the expectations of workers. This is in contrast with the symmetry across sides of the market of market-clearing adjustments in the more familiar general equilibrium adjustment processes like the Hahn process and the Edgeworth process. As our colleague Joel Sobel has pointed out, the asymmetry in our model is made possible by its assignment/transportation problem structure, which makes the sides of the market identifiable independent of salaries.

We have confined our discussion so far to discrete markets for mathematical convenience. Because small changes in other workers' permitted salaries can induce large changes in the adjustments a worker's permitted salaries must undergo, the continuous analog of our adjustment process appears to lead to systems of differential equations like

$$(8) \quad \dot{x} \equiv \begin{cases} 1, & x \leq 0, \\ 0, & x > 0, \end{cases}$$

which do not have exact formal solutions because of discontinuities. While Theorem 1 shows that a discrete core exists in any discrete market, however small the unit of measurement, and the fact that salaries in our model must be rounded to the nearest dollar, penny, or mill has little practical import, it would be of interest to know, for the sake of comparison with Shapley and Shubik's [13] results, what our results imply about the existence of cores in continuous markets. Perhaps somewhat surprisingly, the existence of a core in every discrete market allows a proof of the existence of a strict core in every continuous market, as we shall now argue.

THEOREM 2: Every continuous market has a strict core allocation.

PROOF: Suppose by way of contradiction that there is a continuous market with no strict core allocation. We shall argue that for sufficiently small choices of the

unit of measurement, this implies that the corresponding discrete market has no core allocation either. The proof proceeds by showing that in a continuous market with no strict core, the potential gains to the firm-worker coalition that would gain most by improving upon a given allocation are bounded above zero for all individually rational allocations. Thus, choosing the unit of measurement smaller than this bound ensures that any individually rational allocation can be improved upon by at least one firm-worker coalition in the corresponding discrete market as well, contradicting Theorem 1.

Consider any individually rational allocation $(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)})$ that is not in the core of the continuous market. Let

$$(9) \quad \hat{s}[(i, j); \phi; s_{1\phi(1)}, \dots, s_{n\phi(n)}] \equiv a_{ij} + b_{ij} - (a_{i\phi(i)} + s_{i\phi(i)}) - (b_{\psi(i)j} - s_{\psi(i)j}).$$

In words, $\hat{s}[\cdot]$ is the (possibly negative) total gain realizable by the coalition of worker i and firm j by upsetting the allocation $(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)})$. Define

$$(10) \quad F(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)}) \equiv \max_{(i, j)} \hat{s}[(i, j); \phi; s_{1\phi(1)}, \dots, s_{n\phi(n)}].$$

By hypothesis, $F(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)}) > 0$ as long as $(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)})$ is individually rational, since otherwise no firm-worker coalition could improve upon that allocation in the continuous market. We shall now show that $F(\cdot)$ is bounded above zero for all individually rational allocations.

To see this, note that F is continuous in $(s_{1\phi(1)}, \dots, s_{n\phi(n)})$ for any given ϕ because the maximum of continuous functions is continuous, and let

$$(11) \quad G(\phi) \equiv \min_{(s_{1\phi(1)}, \dots, s_{n\phi(n)})} F(\phi; s_{1\phi(1)}, \dots, s_{n\phi(n)})$$

subject to $a_{i\phi(i)} + s_{i\phi(i)} \geq 0$ for all i ,

$b_{\psi(i)j} - s_{\psi(i)j} \geq 0$ for all j .

Finally, define

$$(12) \quad H \equiv \min_{\phi \in \Phi} G(\phi),$$

where Φ is the set of all one-to-one functions $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$. G is well defined because for any given $\phi \in \Phi$, F is continuous and the feasible region of the problem on the right-hand side of (11) is nonempty (by our assumption that $a_{ij} + b_{ij} \geq 0$ for all (i, j)) and compact. Further, $G(\phi) > 0$ for all $\phi \in \Phi$ because, as noted above, $F(\cdot) > 0$ everywhere in the feasible region for all $\phi \in \Phi$. Finally, H is well defined and strictly positive because Φ is a finite set. Thus, choosing the unit of measurement smaller than $H/2$ will suffice for the validity of the above arguments, and the proof is complete. Straightforward modifications of the proof show that Theorem 2 remains valid in the case of arbitrary capacities. *Q.E.D.*

REMARKS: We conjecture, but have not constructed an example to show, that in spite of Theorems 1 and 2, the salary adjustment process may fail to converge to the core of the corresponding continuous market even when that core is non-

degenerate, however small the unit of measurement. Paradoxically, the method of proof used to establish Theorem 2 shows that for sufficiently small choices of the unit of measurement, the existence of a discrete core implies the existence of a strict core in the continuous market, without also showing that any discrete core allocation in a market with a small unit of measurement is also in the continuous core.

Nevertheless, the salary adjustment process can still be viewed as an algorithm for computing fully optimal assignments and the optimal solutions of transportation problems. This is so in spite of the above observation because for small choices of the unit of measurement (calculable in advance if, for example, the a_{ij} and b_{ij} are known to be integers), the salary adjustment process is easily shown to converge in finite time to an allocation with the (possibly nonunique) assignment ϕ that is associated with strict core allocations in the continuous market. While the salaries in the equilibrium allocation may prevent it from being in the strict continuous core, that allocation includes a strictly Pareto efficient assignment of workers to firms. Thus, the process can be run with a unit of measurement calculable in advance for a given problem, and it will converge in finite time to an allocation that includes a solution of the assignment or transportation problem associated with the market.

We now turn to the question of whether Gale and Shapley's [4] result relating to market institutions the core allocation to which the process converges, remains valid in the present model. The answer is yes, at least for discrete markets in which there can be no ties, as shown by the following theorem.⁵

THEOREM 3: *Consider a discrete market in which no firm or worker is ever indifferent between potential partners at any permitted salary. In such a market, the salary adjustment process converges to a discrete strict core allocation that is at least as good for every firm as any other allocation in the discrete strict core of that market.*

PROOF: The argument is based on Gale and Shapley's [4] proof of the analogous proposition in a model without salary adjustment. Note that if ties cannot occur (see the remarks below), the discrete strict core coincides with the discrete core; it therefore suffices to consider the latter. Call worker i s -possible for firm j if there is some discrete core allocation in that market that assigns worker i to firm j at salary s . The proof is inductive. Suppose that up to some point in the adjustment process, no firm has yet been rejected by a worker at salary s when the worker is s -possible for the firm. Further suppose that worker i then rejects an

⁵ Ties invalidate Gale and Shapley's [4] original result as well. Joel Sobel has constructed an example with two "men" and two "women" (men make offers), where man 1 prefers woman 1 and all other agents are indifferent between prospective mates. If men break ties in favor of woman 1 and women break ties in favor of man 2, man 2 is matched with woman 1. But the other match is better for man 1, and no worse for man 2; and both matches are in the core. The example also shows that the Gale-Shapley process need not converge to the strict core. All these conclusions hold *a fortiori* for our model.

offer from firm j at salary s^i in favor of an offer from firm k at salary s^k . We shall argue that worker i is not s^i -possible for firm j .

As ties have been ruled out, R3 implies that

$$(13) \quad a_{ik} + s^k > a_{ij} + s^i.$$

Further, by R4, worker i at salary s^k is better for firm k than any other worker at his or her current permitted salary. And the other workers are not possible for firm k at lower salaries by the induction hypothesis, since by R1 and R4 their permitted salaries could only have risen to their current levels if they had rejected offers from firm k at all lower salaries. Thus, if $h \neq i$,

$$(14) \quad b_{ik} - s^k > b_{hk} - s',$$

where s' is any salary at which worker h is s' -possible for firm k .

Consider an allocation that assigns worker i to firm j at salary s^i , and all other workers to firms at salaries at which they are possible for the firms. By (13) and (14), worker i and firm k both prefer each other at the integer salary s^k to such an allocation; hence it is not in the discrete core, and worker i is not s^i -possible for firm j .

The inductive argument shows that in the salary adjustment process, no firm is ever rejected by a worker at a salary s at which the worker is s -possible for the firm. From the fact that firms make offers to their favorite (net of salaries) workers first, it therefore follows that firms unanimously prefer the discrete core allocation to which the process converges to any other allocation in the discrete core. (This preference may be weak for some firms, since in general some of the other core allocations differ only in what happens to other agents.) Once again, minor modifications adapt the proof of Theorem 3 to the arbitrary capacities case.

Q.E.D.

REMARKS: The asymmetry between firms and workers arises only because firms make offers in our process. Were firms and workers interchanged everywhere in the model of this section, the analog of Theorem 3 with firms and workers interchanged would also be true. Also, while ties *must* always occur in workers' preferences given R1, "almost any" small perturbation of the $s_{ij}(0)$ away from those specified in R1 will ensure, for any given choice of the unit of measurement, that ties never occur at any permitted salary.

Our results extend those obtained by Gale and Shapley [4] on the stability of core allocations and their sensitivity to the market's institutional structure from a setting in which all job characteristics are completely exogenous to one in which salaries, which enter firms' and workers' utility functions in a linear, separable fashion, are set endogenously. Our results also provide a new proof of the existence of the strict core of the continuous market in the latter model, a result originally established by Shapley and Shubik [13]. Provided only that job satisfactions and productivities are separable across firm-worker pairs, all of these results extend unmodified to the case where firms (or workers) have arbitrary job

capacities.⁶ Existence of the core is not implied by the standard results because of the indivisibilities in our model; see, for example, the example in Shapley and Scarf [12, Section 8] or Gale and Shapley's [4] "roommates" example; or compare the approximate existence results obtained by Henry [5] and Dierker [3]. But the main interest of Theorems 1 and 2 from the standpoint of proving existence of the core is not that they provide an alternative (though perhaps less elegant) proof of Shapley and Shubik's [13] result. It is, rather, that they allow a significant generalization of that result.

Shapley and Shubik's [13, pp. 117–118] argument, because it relies on the duality theory of the simple assignment problem (or, in the case of capacities larger than one, the transportation problem), makes use of the transferable utility assumption in an essential way. When salaries enter firms' and workers' utility functions in a linear, separable fashion, any Pareto efficient allocation, and therefore any strict core allocation, must solve the optimal assignment (or transportation, in the case of larger capacities) problem associated with the market; solving the dual of this problem yields a strict core allocation. When, on the other hand, firms' and workers' utility functions do not satisfy this restriction, no such simple characterization of Pareto efficient or strict core allocations exists. Nevertheless, as we shall now argue, all of our results, *mutatis mutandis*, remain completely valid for the more general model in which firms and workers have essentially arbitrary preferences (except for the separability-across-pairs requirement) and there is any number of endogenous job characteristics.

The key to this generalization is the observation that, as will be shown, none of the arguments used to prove Theorems 1–3 use the transferable utility assumption in an essential way. So as not to beg the question of Pareto efficiency that must underlie any discussion of the core, we assume that firms always make offers to workers that involve a Pareto efficient specification of whatever endogenous job characteristics there are. We also require that firms' and workers' preferences can be represented by utility functions, and that preferences and the sets of feasible agreements are sufficiently regular that the utility possibility frontier faced by worker i and firm j , if the latter hires the former, is continuous and strictly downward sloping when all physically feasible specifications of the endogenous job characteristics are allowed. The meanings of these assumptions are well understood.

Now, s_{ij} can simply be viewed as a parameterization of the (i, j) th of the above mentioned utility possibility frontiers. When $n > 2$, it is not generally possible to find order preserving transformations that make all firms' and workers' utility functions separable and linear in some common variable. But inspection of the proofs of Theorems 1–3 reveals that they never actually use this property in an essential way; thus, the $a_{ij} + s_{ij}$ and the $b_{ij} - s_{ij}$ can simply be replaced by utilities

⁶ Since this paper was written, Alexander S. Kelso has shown how to modify our adjustment process to make firms' capacities endogenous and to allow workers to be gross substitutes from the standpoint of each firm. Gross substitutes is implied by our separability assumption in the present context, but allows significantly more general production technologies. These modifications will be described in a subsequent paper.

that represent firms' and workers' preferences. Further, it is clearly possible to choose these representations to satisfy the appropriate integer restrictions; and the minimum amount by which job satisfactions and productivities are required to adjust in discrete markets can be interpreted as a "just noticeable" difference in utility and taken to be as small as desired. Finally, our assumption that $a_{ij} + b_{ij} \geq 0$ for all (i, j) translates into the requirement that for every (i, j) , there exists some specification of endogenous job characteristics that both worker i and firm j weakly prefer to not working or not hiring anyone at all, respectively.

Given the formal analogy just developed, Theorems 1-3 remain completely valid for the more general model just described. Theorems 1 and 2 establish the existence of the strict core and convergence to the core in settings significantly more general than those considered by Shapley and Shubik [13]. And Theorem 2 and the remarks following its proof show that in the transferable utility case, the salary adjustment process provides a new computational alternative to the "Hungarian method" developed by Kuhn [8] to solve the simple optimal assignment problem, and to the standard methods of solving transportation problems as well.⁷ Theorem 3 extends to models much more descriptive of labor markets Gale and Shapley's [4] demonstration that the core allocation actually reached favors firms or workers in a way that depends simply and systematically on the institutional structure of the market, and shows that Shapley and Shubik's [13, p. 120] observation that "the core tends to be *elongated*, with its long axis oriented in the direction of market-wide [salary] trends" (italics in original) is considerably more robust than one might have thought.

4. CONCLUSION

It is natural to ask to what extent the artificial salary adjustment process constructed here can serve as a description of the adjustments that take place in real competitive labor markets with heterogeneous, well informed firms and workers. Aside from the assumption of perfect certainty, the potentially important unrealistic features of our adjustment process are that offers start out too low and adjust too slowly, and that there is recontracting, in the sense that workers are allowed to retain offers indefinitely without committing themselves until equilibrium is reached. But firms that are well informed about the possibilities in the market are unlikely to overbid for workers, so even if actual offers start out higher and adjust in larger steps it is unlikely that the final outcome will differ significantly. And the convention about permitted salaries is just a formalization of the common-sense considerations that normally prevent firms from repeating offers that have already been rejected.

⁷The salary adjustment process for one-to-one matchings is essentially different from the Hungarian method, which despite its great computational efficiency seems to behave significantly less like real markets. For example, in the Hungarian method, the dual variables can easily adjust in such a way that the analog of a firm's salary offer to a worker goes down without a corresponding decrease in the salary received by that worker should he or she be assigned to that firm. Our process also differs in the general, transportation problem case from the stepping stone algorithm.

With regard to recontracting, John Conlisk has pointed out that our model can be reinterpreted as a nontâtonnement model, in which, while the process converges, workers actually work for firms whose offers they have not yet rejected. With this interpretation, however, the behavior assumed for firms is less plausible than in the model with recontracting, where making offers that are later rejected has no real opportunity cost. For example, a firm might easily find it more profitable in the long run to begin by making offers to a worker it expects to be less "popular" than to follow the policy in R2 and R4 of always making offers to its favorite worker, net of permitted salary. But as Conlisk has also pointed out, our convergence results are likely to remain valid in a wide class of adaptive models, which class seems likely to include many of the offer strategies that are most sensible in a market without recontracting.

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