# MEANINGFUL THEOREMS: NONPARAMETRIC ANALYSIS OF REFERENCE-DEPENDENT PREFERENCES

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**Overview**

Inspired by evidence on perception, Kahneman and Tversky (1979) and Tversky and Kahneman (1991; "TK") introduce a model of individual decisions in which people have preferences over gains and losses relative to a reference point.

Two visual-perception examples from Kahneman's (2003) Nobel lecture:

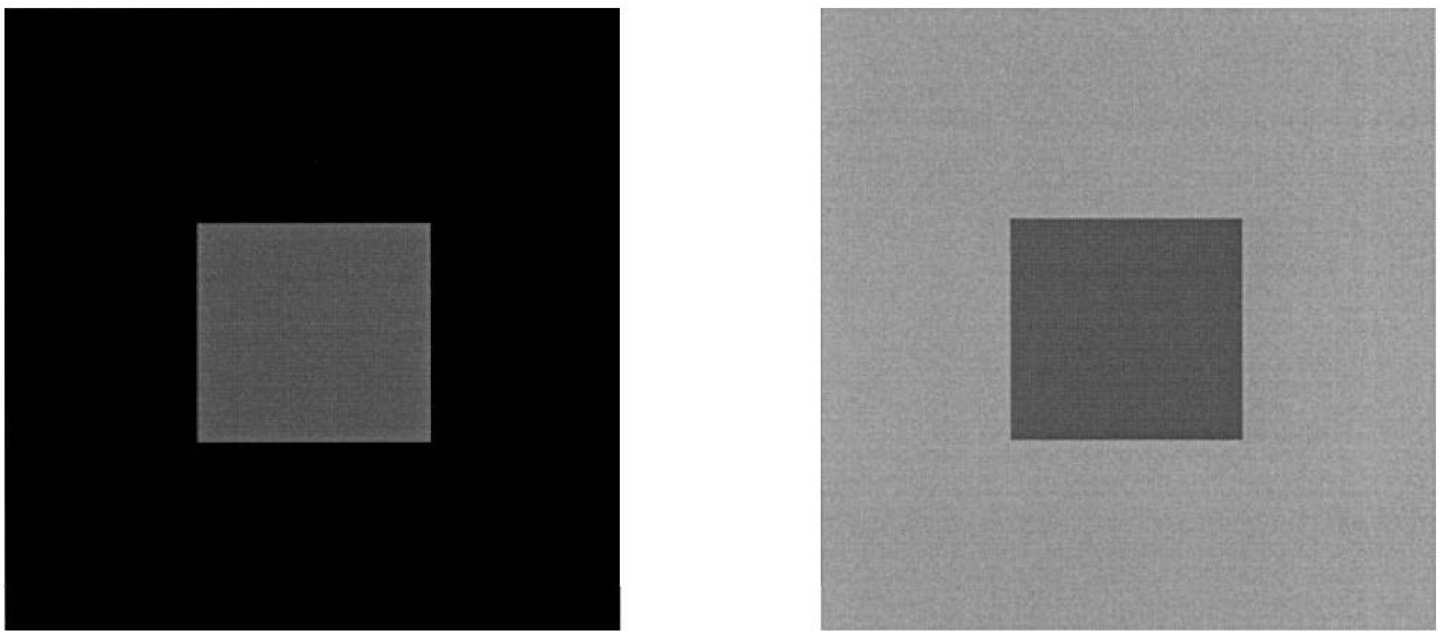


FIGURE 5. REFERENCE-DEPENDENCE IN THE PERCEPTION OF BRIGHTNESS

FIGURE 7. AN ILLUSION OF ATTRIBUTE SUBSTITUTION

In each case the visual illusion persists even after being explained.

(There are also compelling examples for other senses.)

Kahneman and Tversky suggest that we are hard-wired to see changes in what happens relative to reference points at least as clearly as levels of what happens.

Given this kind of evidence, how likely is it that the traditional preferences over levels alone will adequately explain choice behavior?

Kahneman and Tversky's (1979) reference-dependence alters the domain of preferences from levels of outcomes to changes in outcomes; but it remains consistent with a complete and transitive preference ordering over changes, thus not inherently irrational.

Kőszegi and Rabin (2006; "KR") and most recent analyses allow both preferences over levels of consumption, as in neoclassical theory, and preferences over changes in consumption, which is also not inherently irrational.

This paper derives nonparametric conditions for the existence of reference-dependent preferences that can rationalize some patterns of consumer demand behavior that do not allow a neoclassical rationalization.

Reference-dependent consumer theory plays a central role in landmark theory papers such as TK’s and KR’s.

It is also the basis of many structural econometric applications, in finance, housing, labor supply, and consumer demand.

As in KR's analysis, we assume rationality but expand the domain of preferences, allowing them to respond to changes in consumption, relative to a reference point, as well as the levels of a neoclassical analysis.

Expanding the domain of preferences is a slippery slope, but the slippage here is disciplined by the idea of reference-dependence and well supported by evidence.

Focusing on consumer demand, we first identify cases where reference-dependence can*not* be empirically useful:

Unless TK's and KR's notion of sensitivity is constant (TK's "sign-dependence"; KR's assumption A3′) and reference points are precisely modelable or observable (henceforth "modelable"), the hypothesis of reference-dependent preferences can rationalize any data, with a minor exception when constant sensitivity fails.

That is the grain of truth in the common belief that allowing preferences to be reference-dependent destroys the parsimony of neoclassical consumer theory.

(Constant sensitivity, defined more precisely below:

Specifying a reference point divides consumption space into gain-loss regimes, such as "earnings loss, hours gain" in the labor supply models discussed next. With constant sensitivity preferences over consumption bundles must be the same throughout a regime but may vary freely across regimes.)

Next, with constant sensitivity and modelable reference points—so that reference-dependence *can* be empirically useful—we characterize the refutable implications of continuous reference-dependent preferences.

Our characterization relaxes unnecessary functional-structure assumptions that KR made, which with constant sensitivity imply very strong restrictions:

- The preferences that determine demand are additively separable across goods

- Those preferences' marginal rates of substitution satisfy particular knife-edge constraints (specified below) on how they vary with the reference point

Those restrictions have been maintained without testing in all (to our knowledge) theoretical and empirical studies of reference-dependent consumer demand.

Our characterization also relaxes the conventional and untested functional-form assumptions commonly maintained in structural econometric applications.

Our characterization shows how to conduct more general structural econometric or nonparametric analyses in applications, as we plan in companion papers.

In a preliminary nonparametric analysis, Blow and Crawford (2024) revisit Farber's (2005, 2008) and Crawford and Meng's (2011; "CM") econometric analyses of cabdrivers' labor supply, re-analyzing Farber's data driver by driver.

Blow and Crawford control for reference-dependent models' additional flexibility using Beatty and Crawford's (2011, pp. 2786-87) proximity-based variant of Selten and Krischker's (1983) nonparametric measure of predictive success, which judges a model's flexibility by how likely random data are consistent with it.

Their analysis suggests that our generalizations are empirically important:

● Relaxing KR's functional-structure assumptions greatly increases predictive success, for both neoclassical and reference-dependent models.

● For a substantial number of Farber's drivers, a relaxed reference-dependent model has higher predictive success than a comparably relaxed neoclassical model.

**Empirical background**

Reference-dependent consumer theory has played an important role in empirical analyses of workers', consumers', and investors' choice behavior since Camerer et al.'s (1997) analysis of the daily labor supply of New York City cabdrivers.

A neoclassical model of labor supply is analogous to a model of consumer demand for earnings (black-boxing the goods it can buy) and leisure, in this case taking a driver's earnings per hour as analogous to a wage.

Such a model predicts a positive elasticity of hours worked with respect to the wage unless there are very large income effects.

But Camerer et al. estimate a strongly negative elasticity.

To explain the negative elasticity, Camerer et al. propose a model in which drivers have daily earnings targets, analogous to KT's reference points.

They proxy the targets via natural sample analogues, in effect modeling them as rational expectations, assuming stationarity.

Experiments and observational studies suggest that most people are loss-averse—more sensitive to changes below their reference points ("losses") than above them ("gains").

This creates kinks in preferences that make a driver's optimal earnings tend to bunch around his earnings target, working less on days when the "wage" is high.

Depending on the details, such bunching can yield a negative overall wage elasticity of hours, despite the positive incentive effect of anticipated higher wages.

CM's Figure 1, with hours as well as earnings targets:

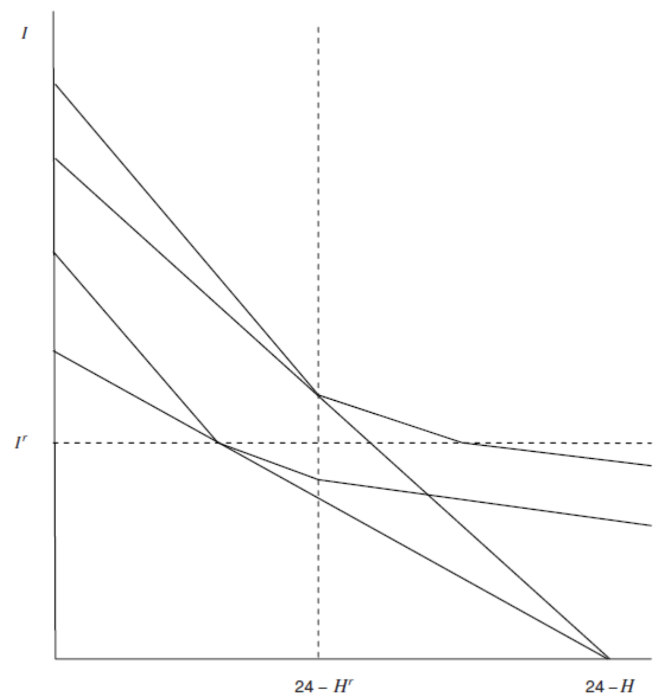



FIGURE 1. A REFERENCE-DEPENDENT DRIVER STOPS AT THE SECOND TARGET HE REACHES: INCOME ON A BAD DAY (*lower budget line*), HOURS ON A GOOD DAY (*upper budget line*)

Farber (2005, 2008) analyzes a newer dataset on New York City cabdrivers, following Camerer et al. in allowing earnings targeting, but with the targets instead treated econometrically as latent variables.

In his data, as in Camerer et al.'s, a reference-dependent model fits better than a neoclassical model; and his estimates of the wage elasticity are again negative.

But Farber's estimates of the earnings targets are unstable, which he argues precludes a useful reference-dependent model of drivers' labor supply.

Farber (2015) analyzes a much larger dataset on New York City cabdrivers, again concluding that reference-dependence is not useful in explaining labor supply.

In each case Farber concludes that most of his drivers are irrational. E.g. Farber (2008): "This [earnings-targeting] is clearly nonoptimal from a neoclassical perspective, since it implies quitting early on days when it is easy to make money and working longer on days when it is harder to make money. Utility would be higher by allocating time in precisely the opposite manner."

**Theoretical background**

Inspired by Camerer et al.'s and Farber's analyses, KR bring the theory of reference-dependent preferences closer to economic applications.

- The utility function has separate, additively separable components of neoclassical utility of consumption levels and reference-dependent "gain-loss" utility of consumption changes

- Unlike Camerer et al. and Farber, there is reference-dependence for all goods

- Utility is continuous, with gain-loss utility determined by the good-by-good differences between realized and reference consumption utilities

- In the spirit of Camerer et al.'s use of sample averages to proxy earnings targets, KR close the model by setting reference points equal to their rational expectations

**Empirical background continued**

Crawford and Meng (2011; "CM") use KR's model to reconsider Farber's econometric analyses, using Farber's (2005, 2008) data.

As in KR's analysis and previous empirical work on this topic, CM assume:

- The utility function is additively separable across consumption and gain-loss utility

- Preferences have constant sensitivity (TK's sign-dependence; KR's A3'):

  Recall that a reference point divides consumption space into gain-loss regimes, such as "earnings loss, hours gain" in labor supply. With constant sensitivity preferences over consumption bundles must be the same throughout a regime but may vary freely across regimes.

- Utility is continuous, with gain-loss utility determined by the good-by-good differences between realized and reference consumption utilities

CM model KR's rational-expectations reference points via natural sample proxies, instead of treating them as latent variables as Farber did.

This avoids the instability of Farber's estimated latent earnings targets and appears to yield a useful reference-dependent model of drivers' labor supply.

For anticipated changes in earnings and hours gain-loss utility drops out of the model, which then coincides with a neoclassical model, in which higher wages increase labor supply.

But for unanticipated changes loss aversion creates kinks in preferences that allow a rationality-based explanation of negative wage elasticities.

## Theoretical background continued

Despite reference-dependent models' successes, several factors have limited their appeal.

Because they expand the domain of preferences, some researchers doubt that they yield any testable implications—Samuelson's (1947) "meaningful theorems".

Such doubts are exacerbated when reference points are not modeled or observed.

And empirical implementations have relied on parametric structural assumptions that are not directly supported by theory or evidence, and are not entirely natural:

- KR's assumption that the sum of consumption and gain-loss utility that determines consumer demand is additively separable across goods

- KR's constant-sensitivity restrictions on how that sum's marginal rates of substitution vary across gain-loss regimes; CM's Table 1 for two goods:

TABLE 1—MARGINAL RATES OF SUBSTITUTION WITH REFERENCE-DEPENDENT PREFERENCES BY DOMAIN

| | Hours gain ($H < H^r$) | Hours loss ($H > H^r$) |
|---|---|---|
| Income gain ($I > I^r$) | $-U_2'(H)/U_1'(I)$ | $-[U_2'(H)/U_1'(I)][1-\eta+\eta\lambda]$ |
| Income loss ($I < I^r$) | $-[U_2'(H)/U_1'(I)]/[1-\eta+\eta\lambda]$ | $-U_2'(H)/U_1'(I)$ |

- Strong (though standard) assumptions regarding the forms of utility functions

**Reference-dependent preferences**

Our model of reference-dependent preferences follows KR's and CM's models and encompasses TK's, Camerer et al.'s, and Farber's, but without imposing KR's functional-structure or Farber's and CM's functional form assumptions.

Like KR we maintain rationality, while expanding the domain of preferences to include gain-loss as well as consumption utility.

Like KR we assume that there are additively separable components of consumption and gain-loss utility, but with no functional-structure links.

Our theory applies to a single consumer or (as in most studies) a group assumed to have homogeneous preferences, but we'll speak of a single consumer.

The consumer is a price-taker, who chooses among consumption bundles $\boldsymbol{q} \in \mathbb{R}_{+}^{K}$, where goods are indexed $k = 1,\ldots, K$.

Preferences are represented by a family of utility functions $u(\boldsymbol{q}, \boldsymbol{r})$, where $\boldsymbol{r} \in \mathbb{R}_{+}^{K}$ is an exogenous reference point, conformable to a $K$–good consumption bundle.

$u(\boldsymbol{q}, \boldsymbol{r})$ is continuous, increasing in $\boldsymbol{q}$, and decreasing in $\boldsymbol{r}$.

This is at least as flexible as a general increasing function of levels $\boldsymbol{q}$ and changes $\boldsymbol{q} - \boldsymbol{r}$.

It nests the neoclassical case where preferences respond only to levels of consumption; TK's where preferences respond only to changes; and Camerer et al.'s, Farber's, KR's, and CM's where preferences respond to both.

Index observations $t = 1,\ldots,T$.

When reference points are unmodelable, the data are prices and quantities $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=1,\ldots,T}$ and hypothetical reference points are denoted $\{\boldsymbol{r}_t\}_{t=1,\ldots,T}$.

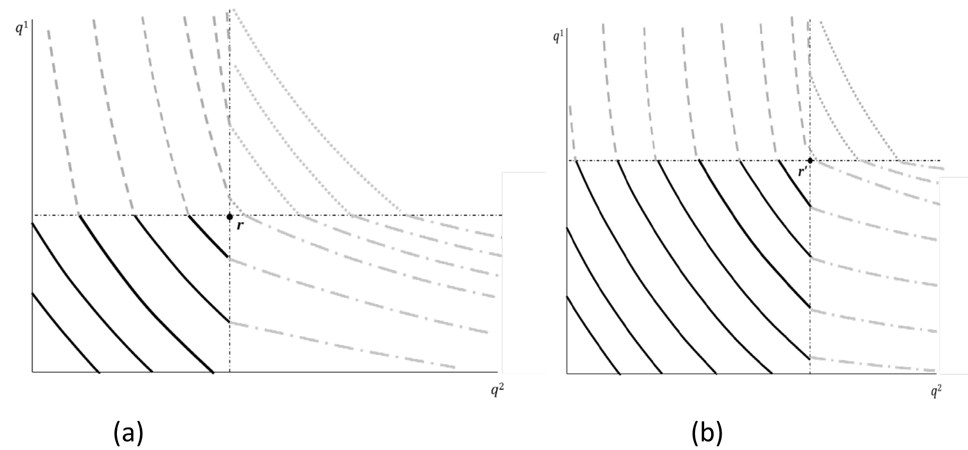
When reference points are modelable, the data are prices, quantities, and reference points $\{\boldsymbol{p}_t, \boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\ldots,T}$.

Goods are sometimes denoted as scalars indexed by subscripts: for $k = 1,\ldots, K$, $\boldsymbol{q} \equiv (q^1, \ldots, q^K)$; and for $t = 1,\ldots, T$, $\boldsymbol{q_t} \equiv (q_t^1, \ldots, q_t^K)$, with analogous notation for $\boldsymbol{p}$, $\boldsymbol{p_t}$, $\boldsymbol{r}$, $\boldsymbol{r_t}$.

Specifying a reference point divides commodity space into gain-loss regimes, such as “earnings-loss and hours-gain” in labor supply.

Constant sensitivity (TK’s 1991 sign-dependence; KR’s A3') requires preferences over consumption bundles to be the same for all bundles in a gain-loss regime but leaves them free to vary across regimes. (General case: variable sensitivity.)

Figure 1. A set of regime maps with constant sensitivity
and the associated global map for alternative reference points

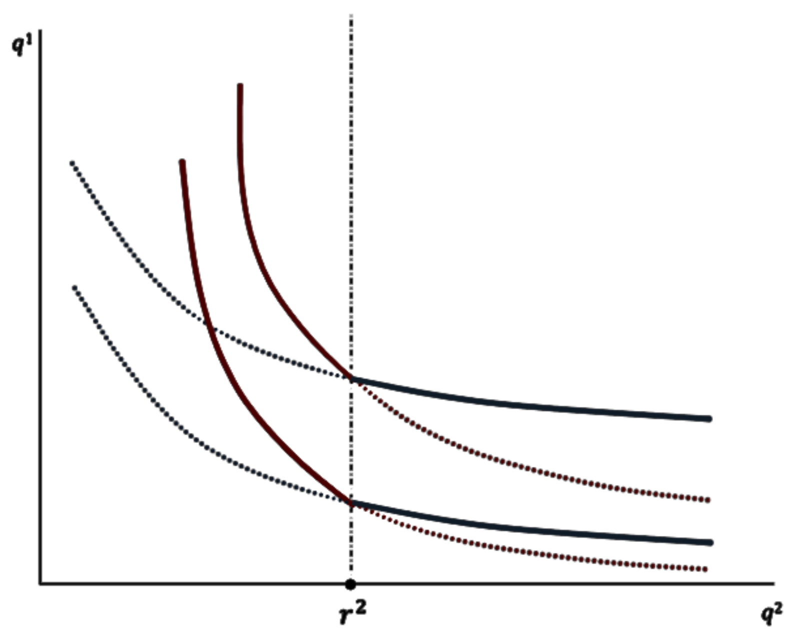



Each gain-loss regime map is defined for the entire space: Different $\boldsymbol{r}$’s “switch on” different maps.

With $u(\boldsymbol{q},\boldsymbol{r})$ decreasing in $\boldsymbol{r}$, its *level* varies with $\boldsymbol{r}$ even though a regime’s map is constant.

With constant sensitivity, a collection of gain-loss regime preferences over consumption bundles satisfies loss aversion if and only if, for any observation $\{\boldsymbol{p}_t, \boldsymbol{q}_t, \boldsymbol{r}_t\}$, given $\boldsymbol{r}_t$, the preference ordering's global better-than-$\boldsymbol{q}_t$ set is weakly contained in each regime preference ordering's local better-than-$\boldsymbol{q}_t$ set.

Figure 2. Loss aversion with one active reference point (solid curves for active parts of indifference maps, dashed for inactive parts)



Because the definition’s nesting of local and global better-than sets holds globally, loss aversion is equivalent to requiring the gain-loss regimes’ indifference maps to satisfy a global single-crossing property:

For any observation, across regimes that differ only in the gain-loss status of good *i*, the loss-side marginal rate of substitution between good *i* and any other good (generalized as needed for non-differentiable preferences) must be weakly more favorable to good *i* than the gain-side marginal rate of substitution.

It is this single-crossing property, not the kinks in global indifference maps that it creates, that shapes loss aversion’s nonparametric implications, which are testable with finite data.

Loss aversion precludes nonconvex kinks, so if the regime maps all have convex better-than sets, then so do the associated global maps.

## Nonparametric implications of neoclassical preferences

Our analysis builds on Afriat's (1967), Diewert's (1973), and Varian's (1982) nonparametric analyses of the neoclassical case.

They show that a price-taking consumer's demand behavior can be rationalized by the maximization of a nonsatiated neoclassical utility function if and only if the data satisfy the Generalized Axiom of Revealed Preference ("GARP").

They also construct rationalizing neoclassical utility functions in a way that is useful in our analysis.

*[Rationalization with neoclassical preferences.] Preferences and an associated utility function* $u(\boldsymbol{q})$ *rationalize the data* $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=1,\dots,T}$ *if and only if* $u(\boldsymbol{q}_t) \geq u(\boldsymbol{q})$ *for all* $\boldsymbol{q}$ *and* $\boldsymbol{t}$ *such that* $\boldsymbol{p}_t \cdot \boldsymbol{q} \leq \boldsymbol{p}_t \cdot \boldsymbol{q}_t$.

*[Generalized Axiom of Revealed Preference ("GARP").]* $\boldsymbol{q_s} R \boldsymbol{q_t}$ *implies* $\boldsymbol{p}_t \cdot \boldsymbol{q}_t \leq \boldsymbol{p}_t \cdot \boldsymbol{q}_s$*, where R indicates that there is some sequence of observations* $\boldsymbol{q}_h, \boldsymbol{q}_i, \boldsymbol{q}_j, \dots, \boldsymbol{q}_t$ *such that* $\boldsymbol{p}_h \cdot \boldsymbol{q}_h \geq \boldsymbol{p}_h \cdot \boldsymbol{q}_i, \boldsymbol{p}_i \cdot \boldsymbol{q}_i \geq \boldsymbol{p}_i \cdot \boldsymbol{q}_j, \dots, \boldsymbol{p}_s \cdot \boldsymbol{q}_s \geq \boldsymbol{p}_s \cdot \boldsymbol{q}_t$.

*AFRIAT'S THEOREM: [Afriat 1967, Diewert 1973, Varian 1982.] The following statements are equivalent:*

*[A] There exists a utility function* $u(\boldsymbol{q})$ *that is continuous, non-satiated, and concave, and that rationalizes the data* $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=1,\ldots,T}$.

*[B] There exist numbers* $\{\boldsymbol{U}_t, \lambda_t > 0\}_{t=1,\ldots,T}$ *such that*

(1) $$U_s \leq U_t + \lambda_t \boldsymbol{p}_t \cdot (\boldsymbol{q}_s - \boldsymbol{q}_t) \text{ for all } s, t \in \{1, \ldots, T\}$$

*[C] The data* $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=1,\ldots,T}$ *satisfy GARP.*

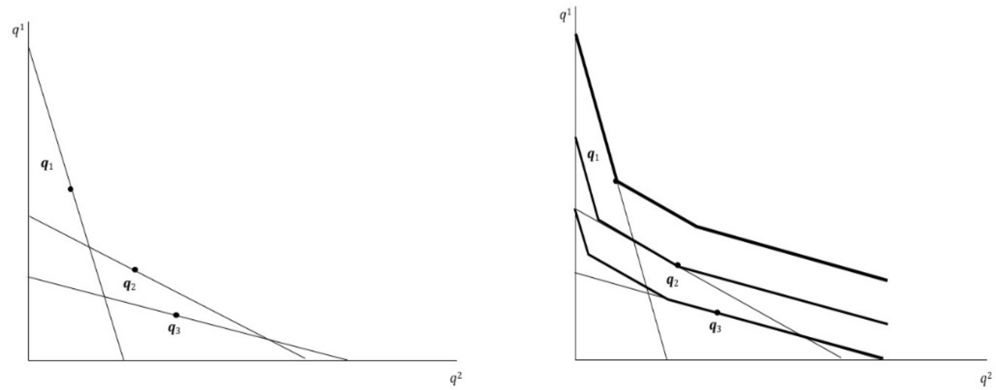
*[D] There exists a non-satiated utility function* $u(\boldsymbol{q})$ *that rationalizes the data* $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=1,\ldots,T}$.

In the proof, [B]'s inequalities (1) hold with equality for at least one $s \neq t$, which yields canonical "Afriat" rationalizing preferences and utility function.

*[Afriat preferences and utility function.] For data* $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=1,\dots,T}$ *that satisfy GARP, or equivalently condition [B] of Afriat's Theorem, the Afriat preferences are those represented by the Afriat utility function* $u(\boldsymbol{q}) = min_{t \in \{1,\dots,T\}}\{U_t + \lambda_t \boldsymbol{p}_t \cdot (\boldsymbol{q} - \boldsymbol{q}_t)\}$*, where the* $U_t$ *and* $\lambda_t$ *are those that satisfy the binding condition [B] inequalities (1) in Afriat's Theorem.*

Figure 3 illustrates Afriat preferences for a three-observation dataset that satisfies GARP. Figure 3b shows the Afriat indifference map, whose marginal rates of substitution are determined by the budget lines. The Afriat utility function is piecewise linear, continuous, non-satiated, and concave.

Figure 3. Afriat preferences for data that satisfy GARP



(a) (b)

With finite data the Afriat preferences are only one of many possibilities for a rationalization (Varian 1982, Fact 4, Figure 3).

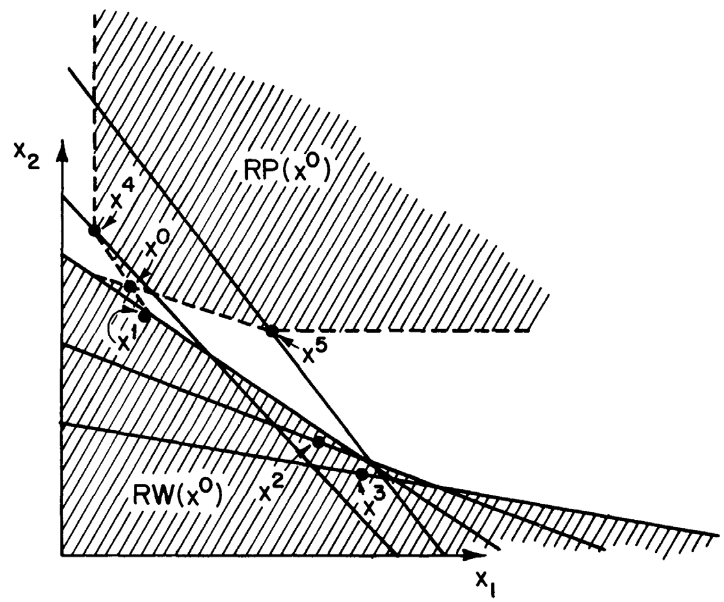



FIGURE 3.

However, their reference-dependent generalization plays a central role in our analysis.

**Nonparametric implications of reference-dependent preferences**

A neoclassical nonparametric analysis makes essential use of rationality.

We can adapt its methods because we maintain rationality in a larger domain.

Even so, our analysis raises new issues because the consumer chooses levels and changes bundled and priced together, and her/his choices can influence reference-dependent preferences by changing how consumption relates to the reference point.

The existence of a reference-dependent rationalization depends on two factors:

- Whether reference points are unmodelable (as Farber assumes) or modelable (as Camerer et al., KR, and CM assume)

- Whether sensitivity is constant (as Farber, usually KR, and CM assume) or variable

We now consider these cases in turn.

*A. Reference-dependent rationalization with unmodelable reference points*

*[Rationalization with unmodelable reference points.] Reference-dependent preferences, an associated utility function* $u(\boldsymbol{q}, \boldsymbol{r})$*, and hypothetical reference points* $\{\boldsymbol{r}_t\}_{t=1,\dots,T}$*, rationalize the data* $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=1,\dots,T}$ *if and only if* $u(\boldsymbol{q}_t, \boldsymbol{r}_t) \geq u(\boldsymbol{q}, \boldsymbol{r}_t)$ *for all* $\boldsymbol{q}$ *and* $\boldsymbol{t}$ *such that* $\boldsymbol{p}_t \cdot \boldsymbol{q} \leq \boldsymbol{p}_t \cdot \boldsymbol{q}_t$*.*

*PROPOSITION 1: [Rationalization with unmodelable reference points via preferences with variable or constant sensitivity.] For any data* $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=1,\dots,T}$ *with unmodelable reference points, there exist reference-dependent preferences and an associated utility function* $u(\boldsymbol{q}, \boldsymbol{r})$ *that are continuous, increasing in* $\boldsymbol{q}$*, and decreasing in* $\boldsymbol{r}$*, and a sequence of hypothetical reference points* $\{\boldsymbol{r}_t\}_{t=1,\dots,T}$*, that rationalize the data.*

The proof hypothesizes a reference point for each observation that coincides with its consumption bundle, and preferences that with those reference points put each observation's bundle at the kink of an approximately Leontief indifference curve.

That the rationalization works entirely by varying reference points across observations shows that the parsimony of reference-dependent consumer theory depends on modeling (or observing) reference points.

Analyses that treat reference points as latent variables may be as heavily influenced by the constraints they impose in estimating reference points as by reference-dependence per se.

*Proof:* Recall that $\boldsymbol{q} \equiv (q^1, \ldots, q^K)$, $\boldsymbol{q_t} \equiv (q_t^1, \ldots, q_t^K)$, and so on. Let $a^k \equiv min_{t=1,\ldots,T}\{p_t^k\} > 0$ for each *k* and *t* s.t. $q_t^k \geq r_t^k$; and $a^k \equiv max_{t=1,\ldots,T}\{p_t^k\} > 0$ for each *k* and *t* s.t. $q_t^k < r_t^k$. Define utility function $u(\boldsymbol{q}, \boldsymbol{r}) \equiv \sum_k a^k q^k + \sum_k a^k(q^k - r^k)$, which is strictly increasing in $\boldsymbol{q}$, strictly decreasing in $\boldsymbol{r}$, and satisfies constant sensitivity and Proposition 1's conditions for continuity. For observation *t*, set $\boldsymbol{r}_t = \boldsymbol{q}_t$ and consider any bundle $\boldsymbol{q} \neq \boldsymbol{q}_t = \boldsymbol{r}_t$ that exactly satisfies *t*'s budget constraint. For such bundles, $\sum_k p_t^k (q^k - q_t^k) = 0$ and, by the definition of the $a^k$,

(2) $\sum_k (a^k - p_t^k)\big(q^k - q_t^k\big) = \sum_k \big(a^k - p_t^k\big)(q^k - r_t^k) < 0$ and $\sum_k a^k(q^k - r_t^k) < 0$

and

(3) $u(\boldsymbol{q}, \boldsymbol{r}_t) - u(\boldsymbol{q}_t, \boldsymbol{r}_t) = 2\sum_k a^k(q^k - q_t^k) = 2\sum_k a^k(q^k - r_t^k) < 0,$

so $u(\boldsymbol{q}, \boldsymbol{r})$ rationalizes the choice of $\boldsymbol{q}_t$. Similarly for variable sensitivity. ■

*B. Reference-dependent rationalization with modelable reference points and variable sensitivity*

*[Rationalization with modelable reference points.] Reference-dependent preferences and an associated utility function $u(\boldsymbol{q},\boldsymbol{r})$ rationalize the data $\{\boldsymbol{p}_t,\boldsymbol{q}_t,\boldsymbol{r}_t\}_{t=1,\ldots,T}$ with modelable reference points if and only if $u(\boldsymbol{q}_t,\boldsymbol{r}_t) \geq u(\boldsymbol{q},\boldsymbol{r}_t)$ for all $\boldsymbol{q}$ and $\boldsymbol{t}$ such that $\boldsymbol{p}_t \cdot \boldsymbol{q} \leq \boldsymbol{p}_t \cdot \boldsymbol{q}_t$.*

*PROPOSITION 2: [Rationalization with modelable reference points via preferences with variable sensitivity.] For any data $\{\boldsymbol{p}_t,\boldsymbol{q}_t,\boldsymbol{r}_t\}_{t=1,\ldots,T}$ with modelable reference points, there exist reference-dependent preferences and an associated utility function $u(\boldsymbol{q},\boldsymbol{r})$ that for each observation t and reference point $\boldsymbol{r}_t$, are continuous and strictly increasing in $\boldsymbol{q}$ and that rationalize the data, if and only if every subset of the data whose observations share exactly the same reference point satisfies GARP.*

Proposition 2 shows that with modelable reference points and variable sensitivity, the hypothesis of reference-dependent preferences is nonparametrically refutable only via violations of GARP within subsets of observations that share the same reference point.

The proof adapts the standard proof of Afriat's Theorem, showing that variable sensitivity allows preferences that rationalize choices in any subsets of the data whose observations share exactly the same reference point can be extended to rationalize the entire dataset.

Thus, reference-dependence with variable sensitivity adds nothing to the neoclassical model in the way of refutable implications.

*Proof:* Partition the observations into subsets $\tau^j, j = 1, \ldots, J$, such that if and only if two observations $\{\boldsymbol{p}_s, \boldsymbol{q}_s, \boldsymbol{r}_s\}$ and $\{\boldsymbol{p}_t, \boldsymbol{q}_t, \boldsymbol{r}_t\}$ have the same reference point $\boldsymbol{r}_s = \boldsymbol{r}_t$, they are in the same subset. If there exists a reference-dependent utility function with the stated properties that rationalizes the data, then the data must satisfy GARP within any such subset, by Afriat's Theorem. Conversely, suppose the data within each such subset satisfies GARP. Let $b^k \equiv min_{t=1,\ldots,T}\{p_t^k\}$, so that $0 < b^k \leq p_t^k$, and let $\boldsymbol{b} \equiv (b^1, \ldots, b^K)$. For any subset $\tau^j$and observation $t \in \tau^j$, let the indicator function $I_{\tau^j}(t) = 1$ if the observation $t \in \tau^j$ and $I_{\tau^j}(t) = 0$ otherwise, and let $u(\boldsymbol{q}, \boldsymbol{r}) \equiv \sum_j I_{\tau^j}(t) U^j(\boldsymbol{q}, \boldsymbol{r}_t)$, where $U^j(\boldsymbol{q}, \boldsymbol{r}_t) \equiv min_{\rho\in\tau^j}\{U_\rho^j + \lambda_\rho^j \boldsymbol{p}_\rho \cdot (\boldsymbol{q} - \boldsymbol{q}_\rho)\} - \boldsymbol{b} \cdot \boldsymbol{r}_t$, which is Definition 5's Afriat utility function for observations in $\tau^j$, with the $U_\rho^j$ and $\lambda_\rho^j$ taken from $\tau^j$'s binding condition B) inequalities (1) in Afriat's Theorem. If $\tau^j$ is a singleton subset, the terms in $U^j(\boldsymbol{q}, \boldsymbol{r}_t)$ follow observation *t*'s budget line. If not, those terms follow the minimum of $\tau^j$'s observations' budget lines, as in Figure 3b. Either way, $\boldsymbol{r}_t$ completely determines the $\boldsymbol{p}_\rho$ and $\boldsymbol{q}_\rho$ for all $\rho \in \tau^j$, as required to determine $U^j(\boldsymbol{q}, \boldsymbol{r}_t)$. For each $\boldsymbol{r}_t$, $u(\boldsymbol{q}, \boldsymbol{r}_t)$ and $U^j(\boldsymbol{q}, \boldsymbol{r}_t)$ are continuous and increasing in $\boldsymbol{q}$. For any subset $\tau^j$and observation $t \in \tau^j$ and any $\boldsymbol{q}$ with $\boldsymbol{p}_t \cdot \boldsymbol{q} \leq \boldsymbol{p}_t \cdot \boldsymbol{q}_t$, using $\tau^j$'s binding condition B) inequalities (1) for the preferences in that subset,

$$(4) \qquad U^j(\boldsymbol{q}, \boldsymbol{r}_t) \equiv min_{\rho\in\tau^j}\{U_\rho^j + \lambda_\rho^j \boldsymbol{p}_\rho \cdot (\boldsymbol{q} - \boldsymbol{q}_\rho)\} - \boldsymbol{b} \cdot \boldsymbol{r}_t$$
$$\leq U_t^j + \lambda_t^j \boldsymbol{p}_t \cdot (\boldsymbol{q}_t - \boldsymbol{q}_t) - \boldsymbol{b} \cdot \boldsymbol{r}_t = U_t^j - \boldsymbol{b} \cdot \boldsymbol{r}_t \equiv U^j(\boldsymbol{q}_t, \boldsymbol{r}_t). \blacksquare$$

**Characterizing reference-dependent preferences with constant sensitivity and continuity**

Let $G(\boldsymbol{q}, \boldsymbol{r})$ be a vector of binary numbers of length $K$ with $k$th component 1 if $q^k \geq r^k$ and 0 otherwise.

The gain-loss regime indicator $I_{\boldsymbol{g}}(\boldsymbol{q}, \boldsymbol{r}) = 1$ if $\boldsymbol{g} = G(\boldsymbol{q}, \boldsymbol{r})$ and 0 otherwise; and the gain-loss indicators $G_+^k(\boldsymbol{q}, \boldsymbol{r}) = 1$ if $q_t^k \geq r_t^k$ and 0 otherwise and $G_-^k(\boldsymbol{q}, \boldsymbol{r}) = 1$ if $q_t^k < r_t^k$and 0 otherwise.

*PROPOSITION 3: [Preferences and utility functions with continuity and constant sensitivity.] Suppose there are K ≥ 2 goods, with reference-dependence active for all K goods, and that a reference-dependent preference ordering and an associated utility function have additively separable consumption utility and gain-loss utility components. Then the ordering satisfies constant sensitivity if and only if an associated utility function* $u(\boldsymbol{q},\boldsymbol{r})$ *can be written, for some consumption utility function* $U(\cdot)$ *and gain-loss regime utility functions* $V_{\boldsymbol{g}}(\cdot,\cdot)$ *and* $v_{\boldsymbol{g}}(\cdot)$*, as*

(5) $$u(\boldsymbol{q},\boldsymbol{r}) \equiv U(\boldsymbol{q}) + \sum_{\boldsymbol{g}} I_{\boldsymbol{g}}(\boldsymbol{q},\boldsymbol{r}) V_{\boldsymbol{g}}(v_{\boldsymbol{g}}(\boldsymbol{q}),\boldsymbol{r}).$$

*Suppose further that the induced preferences over* $\boldsymbol{q}$ *are differentiable in the interior of each regime, with marginal rates of substitution that differ across regimes throughout commodity space. Then the ordering satisfies constant sensitivity and continuity if and only if it is representable by a utility function* $u(\boldsymbol{q},\boldsymbol{r})$ *that can be written, for some consumption utility function* $U(\cdot)$ *and gain-loss component utility functions* $v_+^k(\cdot)$ *and* $v_-^k(\cdot)$ *(with the indicator functions* $G_+^k(\cdot,\cdot)$ *and* $G_-^k(\cdot,\cdot)$ *doing the work of the indicator* $I_{\boldsymbol{g}}(\cdot,\cdot)$*), as*

(6) $$u(\boldsymbol{q},\boldsymbol{r}) \equiv U(\boldsymbol{q}) + \sum_k [G_+^k(\boldsymbol{q},\boldsymbol{r})\{v_+^k(q^k) - v_+^k(r^k)\} + G_-^k(\boldsymbol{q},\boldsymbol{r})\{v_-^k(q^k) - v_-^k(r^k)\}].$$

*Conversely, any combination of induced regime preferences over* $\boldsymbol{q}$ *is consistent with continuity and constant sensitivity for some gain-loss utility functions.*

Proposition 3 derives, from continuity, KR's functional-structure assumption that gain-loss utility is determined, additively separably across goods, by the good-by-good differences between realized and reference consumption utilities.

Informally,

(5) $$u(\boldsymbol{q},\boldsymbol{r}) \equiv U(\boldsymbol{q}) + \textstyle\sum_{\boldsymbol{g}} I_{\boldsymbol{g}}(\boldsymbol{q},\boldsymbol{r}) V_{\boldsymbol{g}}(v_{\boldsymbol{g}}(\boldsymbol{q}),\boldsymbol{r}).$$

is continuous if and only if for any $\boldsymbol{q}$, $\boldsymbol{r}$, and *i* with $q^i = r^i$ and any gain-loss regimes $\boldsymbol{g}$ and $\boldsymbol{g}'$ that differ in the gain-loss status of good $i$

(7) $$V_{\boldsymbol{g}}\big(v_{\boldsymbol{g}}(\boldsymbol{q}),\boldsymbol{r}\big) = V_{\boldsymbol{g}'}\big(v_{\boldsymbol{g}'}(\boldsymbol{q}),\boldsymbol{r}\big).$$

A change in one good's consumption can change the gain-loss regime, which unless each regime's $V_{\boldsymbol{g}}\big(v_{\boldsymbol{g}}(\boldsymbol{q}),\boldsymbol{r}\big)$ is additively separable in the components of $\boldsymbol{q}$, can violate (7).

Unless gain-loss utility is determined by the good-by-good differences between realized and reference consumption utilities as in

(6) $$u(\boldsymbol{q},\boldsymbol{r}) \equiv U(\boldsymbol{q}) + \textstyle\sum_k [G_+^k(\boldsymbol{q},\boldsymbol{r})\{v_+^k(q^k) - v_+^k(r^k)\} + G_-^k(\boldsymbol{q},\boldsymbol{r})\{v_-^k(q^k) - v_-^k(r^k)\}]$$

changing $q^k$ and $r^k$ with $r^k = q^k$ can violate (7).

Proposition 3 allows consumption utility, and thus the sum of consumption and gain-loss utility that determines consumer demand, *not* to be additively separable across goods.

Proposition 3 also allows the preferences over consumption bundles induced by consumption plus gain-loss utility to vary as freely as possible across gain-loss regimes while preserving continuity.

It thereby relaxes the knife-edge cross-regime links between marginal rates of substitution implied by KR's assumption that consumption and gain-loss utility have the same additively-separable-across-goods functional form.

CM's Table 1:

TABLE 1—MARGINAL RATES OF SUBSTITUTION WITH REFERENCE-DEPENDENT PREFERENCES BY DOMAIN

| | Hours gain ($H < H^r$) | Hours loss ($H > H^r$) |
|---|---|---|
| Income gain ($I > I^r$) | $-U_2'(H)/U_1'(I)$ | $-[U_2'(H)/U_1'(I)][1 - \eta + \eta\lambda]$ |
| Income loss ($I < I^r$) | $-[U_2'(H)/U_1'(I)]/[1 - \eta + \eta\lambda]$ | $-U_2'(H)/U_1'(I)$ |

Recall that with $u(\boldsymbol{q},\boldsymbol{r})$ decreasing in $\boldsymbol{r}$, its *level* varies with $\boldsymbol{r}$ even within a gain-loss regime. Proposition 3's equation (6)

(6) $u(\boldsymbol{q},\boldsymbol{r}) \equiv U(\boldsymbol{q}) + \sum_k [G_+^k(\boldsymbol{q},\boldsymbol{r})\{v_+^k(q^k) - v_+^k(r^k)\} + G_-^k(\boldsymbol{q},\boldsymbol{r})\{v_-^k(q^k) - v_-^k(r^k)\}]$.

assigns each gain-loss regime *g* a "loss cost" (the parts of (6) that depends on $\boldsymbol{r}$), which is incurred whenever any bundle $\boldsymbol{q}$ in regime *g* is chosen, but which is otherwise independent of $\boldsymbol{q}$ within the regime.

Verifying a full rationalization with observable reference points and constant sensitivity depends on specifying loss costs as a function of $\boldsymbol{q}$ and $\boldsymbol{r}$, because they determine a consumer's incentive to "defect" from an observation's consumption bundle to *some* bundle in its budget set in another gain-loss regime, with possibly different preferences.

Although a consumer's choices do not reveal loss costs directly, Propositions 4 and 5 use Proposition 3's characterization to show that they can be inferred enough to verify a full rationalization from the sum of consumption and gain-loss utility that rationalize his choices within each gain-loss regime.

Proposition 3's characterization shows how to conduct a general structural econometric analysis of data like Farber's (2005, 2008, 2015), CM's, or those of more recent studies, using sample proxies like CM's for KR's rational-expectations targets.

Such an analysis should reveal the extent to which unnecessarily restrictive functional-structure assumptions bias the results of previous econometric analyses.

We now show how to use Proposition 3's characterization to conduct a general nonparametric analysis.

**Rationalization with modelable reference points and constant sensitivity**

We now use Proposition 3's characterization of reference-dependent preferences that satisfy constant sensitivity and continuity to derive nonparametric sufficient conditions for a reference-dependent rationalization in that case.

It seems clear from the literature and our own efforts that there are no simple combinatoric conditions that are necessary and sufficient for a reference-dependent rationalization, as GARP is in the neoclassical case.

With modelable reference points, the observations' consumption bundles can be objectively sorted into gain-loss regimes.

By Afriat's Theorem, GARP for each regime's observations is necessary for a rationalization, because it is required for the existence of preferences that preclude defections from an observation's bundle to affordable bundles within the same regime.

However, GARP for each regime's observations is not sufficient for a rationalization, for two reasons.

- The gain-loss regime utility functions that rationalize the consumer's choices within each regime must satisfy Proposition 3's restrictions that their component utility functions must be the same across all regimes, and GARP does not ensure that that is possible.

- The rationalizing regime utility functions must also prevent defections from an observation's bundle to affordable bundles in other regimes, in which preferences may differ. This involves Section V's loss costs, which are determined by the rationalizing regime utility functions.

Another difficulty in deriving conditions for a rationalization is that there is normally a range of rationalizing gain-loss regime utility functions, as in the neoclassical case (Varian (1982, Fact 4, Figure 3).

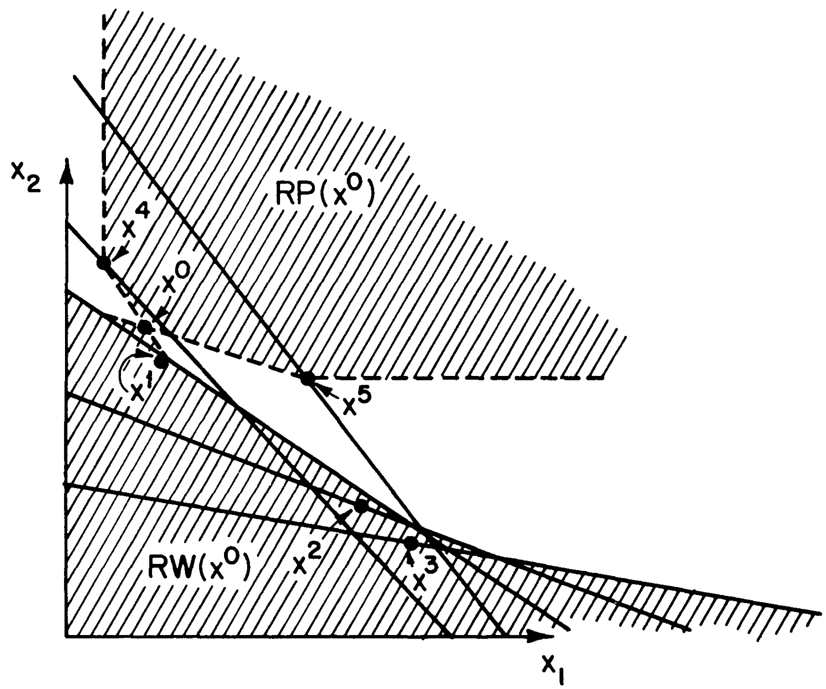



FIGURE 3.

Choosing among them involves complex trade-offs, because a choice that lowers the gain from defecting *from* bundles in a regime raises the gain from defecting *to* them.

Propositions 4 and 5 approach these difficulties in two steps.

Proposition 4 translates the requirements for a rationalization with modelable reference points and constant sensitivity into the language of Proposition 3, showing that necessary and sufficient conditions for a rationalization are the existence of continuous, strictly increasing consumption utility function and gain-loss component utility functions that preclude, for any observation and bundle:

● defections from its bundle to any bundle in the same gain-loss regime in its budget set (these conditions parallel Afriat's Theorem's inequalities (1), imposing constancy of component utility functions across gain-loss regimes)

● defections from its bundle to any bundle in another regime in its budget set

Because those conditions are conditional, they are not directly applicable.

Proposition 5 then derives directly applicable sufficient conditions based on rationalizing regime utility functions like Definition 3's Afriat utility functions.

Because other rationalizing regime utility functions usually exist, those conditions are not necessary; but with rich enough data they should be asymptotically necessary, as explained below.

Let $\Gamma(g;\boldsymbol{r})\}$ be the set of $\boldsymbol{q}$ in regime *g* for $\boldsymbol{r}$. Let $\Theta\left(\{\boldsymbol{q}_t,\boldsymbol{r}_t\}_{t=1,\dots,T};g\right) \equiv \{t \in \{1,\dots,T\} \,\middle|\, \boldsymbol{q}_t \in \Gamma(g;\boldsymbol{r}_t)\}$ be the set of $t$ with $\boldsymbol{q}_t$ in regime *g* for $\boldsymbol{r}_t$.

*PROPOSITION 4: [Rationalization with modelable reference points via preferences and utility functions with constant sensitivity.] Suppose that reference-dependent preferences and an associated utility function are defined over K ≥ 2 goods, that reference-dependence is active for all K goods, that the preferences satisfy constant sensitivity and are continuous, and that the utility function satisfies Proposition 3's (6). Consider data* $\{\boldsymbol{p}_t,\boldsymbol{q}_t,\boldsymbol{r}_t\}_{t=1,\dots,T}$ *with modelable reference points. Then the statements [A] and [B] are equivalent:*

*[A] There exists a continuous reference-dependent utility function* $u(\boldsymbol{q},\boldsymbol{r})$ *that satisfies constant sensitivity; is strictly increasing in* $\boldsymbol{q}$ *and strictly decreasing in* $\boldsymbol{r}$*; and that rationalizes the data* $\{\boldsymbol{p}_t,\boldsymbol{q}_t,\boldsymbol{r}_t\}_{t=1,\dots,T}$.

*[B] Each gain-loss regime's data satisfy GARP within the regime; and there is some combination of preferences over consumption bundles, with continuous, strictly increasing consumption utility function $U(\cdot)$ and gain-loss component utility functions $v_+^k(\cdot)$ and $v_-^k(\cdot)$, such that, for any regime $g$ and any pair of observations $\sigma, \tau \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g)$ (with the indicator functions $G_+^k(\cdot,\cdot)$ and $G_-^k(\cdot,\cdot)$ again doing the work of $I_{\boldsymbol{g}}(\cdot,\cdot)$),*

(10)
$$U(\boldsymbol{q}_\sigma) + \sum_k [G_+^k(\boldsymbol{q}_\sigma, \boldsymbol{r}_\tau) v_+^k(q_\sigma^k) + G_-^k(\boldsymbol{q}_\sigma, \boldsymbol{r}_\tau) v_-^k(q_\sigma^k)]$$
$$\leq \mathrm{U}(\boldsymbol{q}_\tau) + \sum_k [G_+^k(\boldsymbol{q}_\tau, \boldsymbol{r}_\tau) v_+^k(q_\tau^k) + G_-^k(\boldsymbol{q}_\tau, \boldsymbol{r}_\tau) v_-^k(q_\tau^k)] + \lambda_\tau \boldsymbol{p}_\tau \cdot (\boldsymbol{q}_\sigma - \boldsymbol{q}_\tau)$$

*and for each observation $\{\boldsymbol{p}_\tau, \boldsymbol{q}_\tau, \boldsymbol{r}_\tau\}_{t=1,\dots,T}$ with $\tau \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g)$ and each $\boldsymbol{q} \in \Gamma(g'; \boldsymbol{r}_\tau)$ with $g' \neq g$ for which $\boldsymbol{p}_\tau \cdot \boldsymbol{q} \leq \boldsymbol{p}_\tau \cdot \boldsymbol{q}_\tau$,*

(11)
$$U(\boldsymbol{q}) + \sum_k [G_+^k(\boldsymbol{q}, \boldsymbol{r}_\tau)\{v_+^k(q^k) - v_+^k(r_\tau^k)\} + G_-^k(\boldsymbol{q}, \boldsymbol{r}_\tau)\{v_-^k(q^k) - v_-^k(r_\tau^k)\}]$$
$$\leq U(\boldsymbol{q}_{\boldsymbol{\tau}}) + \sum_k [G_+^k(\boldsymbol{q}_{\boldsymbol{\tau}}, \boldsymbol{r}_{\boldsymbol{\tau}})\{v_+^k(q_\tau^k) - v_+^k(r_\tau^k)\} + G_-^k(\boldsymbol{q}_{\boldsymbol{\tau}}, \boldsymbol{r}_{\boldsymbol{\tau}})\{v_-^k(q_\tau^k) - v_-^k(r_\tau^k)\}].$$

Proving Proposition 4 requires linking loss costs to things that can be estimated from the data, not only at given points but as functions of $\boldsymbol{r}$.

The proof operationalizes conditions (11) by taking the rationalizing regime preferences represented by $U(\cdot)$ and the $v_+^k(\cdot)$ and $v_-^k(\cdot)$, which satisfy (10), and using them to write the condition preventing defections from the bundle of observation $\tau \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g)$ in regime $g$ to a bundle $\boldsymbol{q} \in \Gamma(g'; \boldsymbol{r}_\tau)$ in regime $g' \neq g$ for $\boldsymbol{r}_\tau$ *with* $\boldsymbol{p}_\tau \cdot \boldsymbol{q} \leq \boldsymbol{p}_\tau \cdot \boldsymbol{q}_\tau$:

$$u(\boldsymbol{q}, \boldsymbol{r}_\tau) - U(\boldsymbol{r}_\tau) \equiv U(\boldsymbol{q}) + \textstyle\sum_k [G_+^k(\boldsymbol{q}, \boldsymbol{r}_\tau)\{v_+^k(q^k) - v_+^k(r_\tau^k)\} + G_-^k(\boldsymbol{q}, \boldsymbol{r}_{\tau_\tau})\{v_-^k(q^k) - v_-^k(r_\tau^k)\}] - U(\boldsymbol{r}_\tau)$$

(12) $$\equiv \{U(\boldsymbol{q}) + \textstyle\sum_k [G_+^k(\boldsymbol{q}, \boldsymbol{r}_{\boldsymbol{\tau}}) v_+^k(q^k) + G_-^k(\boldsymbol{q}, \boldsymbol{r}_{\boldsymbol{\tau}}) v_-^k(q^k)]\} - \{U(\boldsymbol{r}_\tau) + \textstyle\sum_k [G_+^k(\boldsymbol{q}, \boldsymbol{r}_{\boldsymbol{\tau}})\{v_+^k(r_\tau^k) + G_-^k(\boldsymbol{q}, \boldsymbol{r}_{\boldsymbol{\tau}}) v_-^k(r_\tau^k)]\}$$

$$\leq \{U(\boldsymbol{q}_\tau) + \textstyle\sum_k [G_+^k(\boldsymbol{q}_{\boldsymbol{\tau}}, \boldsymbol{r}_{\boldsymbol{\tau}}) v_+^k(q_\tau^k) + G_-^k(\boldsymbol{q}_{\boldsymbol{\tau}}, \boldsymbol{r}_{\boldsymbol{\tau}}) v_-^k(q_\tau^k)]\} - \{U(\boldsymbol{r}_{\boldsymbol{\tau}}) + \textstyle\sum_k [G_+^k(\boldsymbol{q}_{\boldsymbol{\tau}}, \boldsymbol{r}_{\boldsymbol{\tau}})\{v_+^k(r_\tau^k) + G_-^k(\boldsymbol{q}_{\boldsymbol{\tau}}, \boldsymbol{r}_{\boldsymbol{\tau}}) v_-^k(r_\tau^k)]\}$$

(12)'s central inequality can then be rearranged to yield (11).

Figures 4 and 5 illustrate Proposition 4.

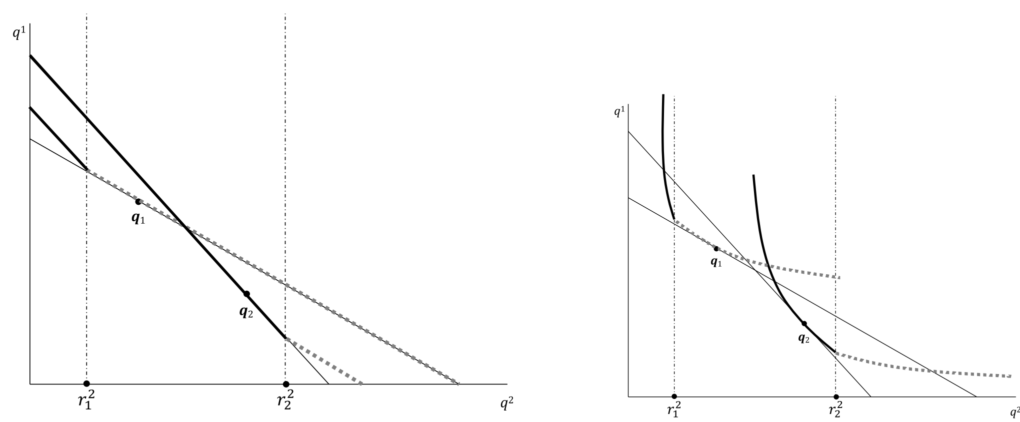
In each case the entire dataset violates GARP, with observation 1's consumption bundle chosen in 1's budget set over observation 2's bundle, and vice versa.

In each case each regime's single observation trivially satisfies GARP within its regime.

And in each case the observations' reference points put their bundles in different gain-loss regimes, so constant sensitivity allows different preferences for each observation.

Figure 4a depicts Afriat rationalizing regime preferences and Figure 4b depicts non-Afriat rationalizing regime preferences. With condition (11) satisfied, a rationalization is possible.

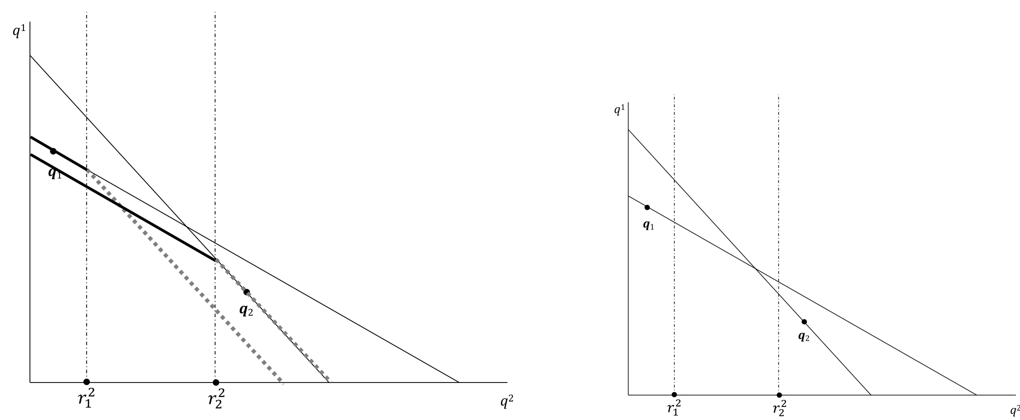
Figure 4. Rationalizing data that violate GARP via reference-dependent preferences with constant sensitivity (solid lines for loss maps, dashed lines for gains maps)




(a) (b)

In Figure 5a Afriat rationalizing regime preferences do not satisfy condition (11). Figure 5b shows more generally that there can be no choice of rationalizing regime preferences for which (11) is satisfied, so a rationalization, Afriat or not, is not possible.

Figure 5. Failing to rationalize data that violate GARP via reference-dependent preferences with constant sensitivity (solid lines for loss maps, dashed lines for gains maps)




(a) (b)

The difference between Figure 4's and Figure 5's examples can be understood in terms of loss aversion. The change in Afriat preferences across regimes in Figure 4a is consistent with loss aversion, but not the change in Figure 5a.

A Corollary shows that if the rationalizing regime preferences (Afriat or not) satisfy loss aversion, Proposition 4's conditions (11) are automatically satisfied; but that loss aversion is not quite necessary for a rationalization.

Recall that the gain-loss indicator functions $G_{+}^{k}(\boldsymbol{q},\boldsymbol{r}) = 1$ if $q_t^k \geq r_t^k$ and 0 otherwise and $G_{-}^{k}(\boldsymbol{q},\boldsymbol{r}) = 1$ if $q_t^k < r_t^k$ and 0 otherwise; and that $\Theta\left(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g\right) \equiv \{t \in \{1, \dots, T\} \,\middle|\, \boldsymbol{q}_t \in \Gamma(g; \boldsymbol{r}_t)\}$ is the set of observation indicators $t$ for which $\boldsymbol{q}_t$ is in regime *g* for $\boldsymbol{r}_t$.

COROLLARY*: [Rationalization with modelable reference points via preferences and utility functions with constant sensitivity that satisfy a condition weaker than loss aversion.] Suppose that reference-dependent preferences and an associated utility function are defined over K ≥ 2 goods, that reference-dependence is active for all K goods, that the preferences satisfy constant sensitivity and are continuous, and that the utility function therefore satisfies Proposition 3's (6). Consider data* $\{\boldsymbol{p}_t, \boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}$ *with modelable reference points. If each gain-loss regime's data satisfy GARP within the regime; and there is some combination of preferences over consumption bundles, with continuous, strictly increasing consumption utility function* $U(\cdot)$ *and gain-loss component utility functions* $v_+^k(\cdot)$ *and* $v_-^k(\cdot)$*, such that, for any regime* $g$ *and any pair of observations* $\sigma, \tau \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g)$ *for which* $\boldsymbol{p}_\tau \cdot \boldsymbol{q}_\sigma \leq \boldsymbol{p}_\tau \cdot \boldsymbol{q}_\tau$ *(with the indicator functions* $G_+^k(\cdot,\cdot)$ *and* $G_-^k(\cdot,\cdot)$ *doing the work of a regime indicator function* $I_{\boldsymbol{g}}(\cdot,\cdot)$*),*

(13) $U(\boldsymbol{q}_\sigma) + \sum_k [G_+^k(\boldsymbol{q}_\sigma, \boldsymbol{r}_\tau) v_+^k(q_\sigma^k) + G_-^k(\boldsymbol{q}_\sigma, \boldsymbol{r}_\tau) v_-^k(q_\sigma^k)]$
$\leq \mathrm{U}(\boldsymbol{q}_\tau) + \sum_k [G_+^k(\boldsymbol{q}_\tau, \boldsymbol{r}_\tau) v_+^k(q_\tau^k) + G_-^k(\boldsymbol{q}_\tau, \boldsymbol{r}_\tau) v_-^k(q_\tau^k)] + \lambda_\tau \boldsymbol{p}_\tau \cdot (\boldsymbol{q}_\sigma - \boldsymbol{q}_\tau),$

*and there are no observations for which* $\boldsymbol{q}_t$ *is* not *on the boundary of the convex hull of* $\boldsymbol{q}_t$*'s upper contour set for the associated candidate global preference ordering for* $\boldsymbol{r}_t$*, then the consumption utility function* $U(\cdot)$ *and gain-loss component utility functions* $v_+^k(\cdot)$ *and* $v_-^k(\cdot)$ *rationalize the data.*

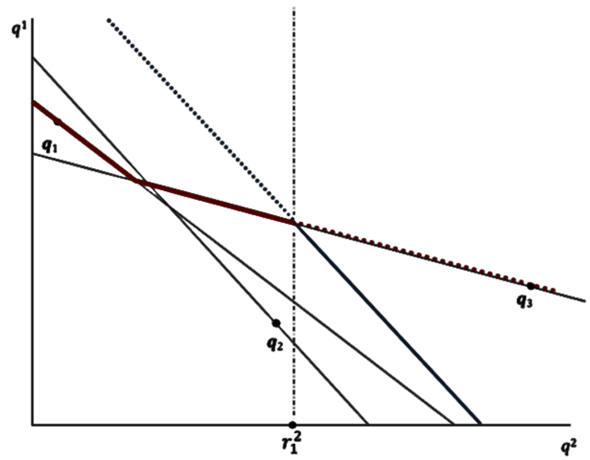
Loss aversion is an empirically well-supported assumption known to have important implications, but to our knowledge it has not previously been linked to the *existence* of a reference-dependent rationalization.

And in the Corollary it appears as part of conditions that are only sufficient, not necessary.

As the proof suggests, loss aversion's testability is limited for the same reason that the convexity of neoclassical preferences is not nonparametrically testable.

The Corollary's final "no observations for which $\boldsymbol{q}_t$ is *not* on the boundary" condition rules out bunching of consumption bundles in regions of commodity space where the rationalizing regime preferences violate loss aversion and is vacuously satisfied for preferences that satisfy loss aversion. Such restrictions on bunching are unusual in a nonparametric analysis.

Figure 6. Rationalizing data that violate GARP when preferences violate loss aversion but satisfy the Corollary's sufficient conditions for a rationalization (solid curves for active parts of indifference maps, dashed for inactive parts)




In Figure 6 the entire dataset violates GARP, the Afriat gain-loss regime preferences violate loss aversion, but the data satisfy the Corollary's final conditions, thus allowing a rationalization.

Only reference point $\boldsymbol{r}_1$ is shown and observation 1 is in the good-2 loss regime.

Assume that $\boldsymbol{r}_2 = [0, 0]$, so that observation 2's budget set is entirely in the good-2 gain regime; and that $\boldsymbol{r}_3 = [0, m]$, where *m* is large enough that observation 3's budget set is entirely in the good-2 loss regime.

The Afriat regime preferences yield a candidate for global preferences that make all three observations' consumption bundles optimal:

Observations 2's and 3's budget sets are entirely in their regimes (good-2 gain and good-2 loss, respectively), so their bundles' optimality in their regimes suffices for global optimality.

Observation 1's bundle is optimal for its good-2 loss regime preferences and the Corollary ensures that its bundle's optimality extends to its entire budget set.

As already noted, Proposition 4's necessary and sufficient conditions for a rationalization are not directly applicable because with finite data there is normally a range of preferences that rationalize a gain-loss regime's data (Varian 1982, Fact 4) and Proposition 4's condition [B] rests on an unspecified choice among those rationalizing regime preferences.

Finding a choice that precludes beneficial cross-regime defections involves complex trade-offs, because preferences that reduce the gain from defecting *from* bundles in a gain-loss regime increase the gain from defecting *to* bundles in the regime.

Proposition 5 uses Proposition 4's conditions to derive directly applicable sufficient conditions by specializing the choice of rationalizing gain-loss regime utilities to a reference-dependent generalization of the Afriat regime utilities.

Recall that $\Gamma(g;\boldsymbol{r})\}$ is the set of $\boldsymbol{q}$ in regime *g* for $\boldsymbol{r}$. And $\Theta\left(\{\boldsymbol{q}_t,\boldsymbol{r}_t\}_{t=1,\dots,T};g\right) \equiv \{t \in \{1,\dots,T\} \,\middle|\, \boldsymbol{q}_t \in \Gamma(g;\boldsymbol{r}_t)\}$ is the set of $t$ with $\boldsymbol{q}_t$ in regime *g* for $\boldsymbol{r}_t$.

*PROPOSITION 5: [Sufficient conditions for rationalization with modelable reference points, via reference-dependent preferences and utility function with constant sensitivity and continuity.] The following conditions are sufficient for the existence of continuous reference-dependent preferences and utility function with constant sensitivity $u(\boldsymbol{q},\boldsymbol{r})$ that rationalize data with modelable reference points $\{\boldsymbol{p}_t,\boldsymbol{q}_t,\boldsymbol{r}_t\}_{t=1,\dots,T}$: There exist numbers $U_t$, $v_{t+}^k$, $v_{t-}^k$, and $\lambda_t > 0$ for each $k = 1,\dots,K$ and $t = 1,\dots,T$ such that:*

*[A] For any gain-loss regime $g$ and any pair of observations $\sigma,\tau \in \Theta\left(\{\boldsymbol{q}_t,\boldsymbol{r}_t\}_{t=1,\dots,T};g\right)$ (with the indicator functions $G_+^k(\cdot,\cdot)$ and $G_-^k(\cdot,\cdot)$ again doing the work of $I_{\boldsymbol{g}}(\cdot,\cdot)$),*

(13)
$$U_\sigma + \textstyle\sum_k[G_+^k(\boldsymbol{q}_\sigma,\boldsymbol{r}_\tau)v_{\sigma+}^k + G_-^k(\boldsymbol{q}_\sigma,\boldsymbol{r}_\tau)v_{\sigma-}^k]$$
$$\leq U_\tau + \textstyle\sum_k[G_+^k(\boldsymbol{q}_\tau,\boldsymbol{r}_\tau)v_{\tau+}^k + G_-^k(\boldsymbol{q}_\tau,\boldsymbol{r}_\tau)v_{\tau-}^k] + \lambda_\tau\boldsymbol{p}_\tau\cdot(\boldsymbol{q}_\sigma - \boldsymbol{q}_\tau).$$

*[B] For observations $\sigma,\tau$, $q_\sigma^k \geq q_\tau^k$ for k = 1,…, K, $U_\sigma \geq U_\tau$; and for observations $\sigma,\tau$ and any k = 1,…, K, $q_\sigma^k \geq q_\tau^k$, $v_{\sigma+}^k \geq v_{\tau+}^k$, and $v_{\sigma-}^k \geq v_{\tau-}^k$.*

*[C] For any pair of regimes $g$ and $g' \neq g$, observation $\tau \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g)$, and bundle $\boldsymbol{q} \in \Gamma(g'; \boldsymbol{r}_\tau)$ for which $\boldsymbol{p}_\tau \cdot \boldsymbol{q} \leq \boldsymbol{p}_\tau \cdot \boldsymbol{q}_\tau$,*

$$
\begin{aligned}
(14) \qquad & min_{\rho \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g')}\left\{U_\rho + \textstyle\sum_k [G_+^k(\boldsymbol{q}_\rho, \boldsymbol{r}_\tau) v_{\rho+}^k + G_-^k(\boldsymbol{q}_\rho, \boldsymbol{r}_\tau) v_{\rho-}^k] + \lambda_\rho \boldsymbol{p}_\rho \cdot (\boldsymbol{q} - \boldsymbol{q}_\rho)\right\} \\
& - min_{\rho \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g')}\left\{U_\rho + \textstyle\sum_k [G_+^k(\boldsymbol{q}_\rho, \boldsymbol{r}_\tau) v_{\rho+}^k + G_-^k(\boldsymbol{q}_\rho, \boldsymbol{r}_\tau) v_{\rho-}^k] + \lambda_\rho \boldsymbol{p}_\rho \cdot (\boldsymbol{r}_\tau - \boldsymbol{q}_\rho)\right\} \\
& \leq min_{\rho \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g)}\left\{U_\rho + \textstyle\sum_k [G_+^k(\boldsymbol{q}_\rho, \boldsymbol{r}_\tau) v_{\rho+}^k + G_-^k(\boldsymbol{q}_\rho, \boldsymbol{r}_\tau) v_{\rho-}^k] + \lambda_\rho \boldsymbol{p}_\rho \cdot (\boldsymbol{q}_\tau - \boldsymbol{q}_\rho)\right\} \\
& - min_{\rho \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g)}\left\{U_\rho + \textstyle\sum_k [G_+^k(\boldsymbol{q}_\rho, \boldsymbol{r}_\tau) v_{\rho+}^k + G_-^k(\boldsymbol{q}_\rho, \boldsymbol{r}_\tau) v_{\rho-}^k] + \lambda_\rho \boldsymbol{p}_\rho \cdot (\boldsymbol{r}_\tau - \boldsymbol{q}_\rho)\right\}.
\end{aligned}
$$

Proposition 5's conditions (14) precluding beneficial defections across gain-loss regimes requires linking Proposition 3's loss costs to things that can be estimated from the data, not only at particular points but as functions of $\boldsymbol{r}$. This is done just as in Proposition 4, but now using the Afriat rationalizing regime utilities.

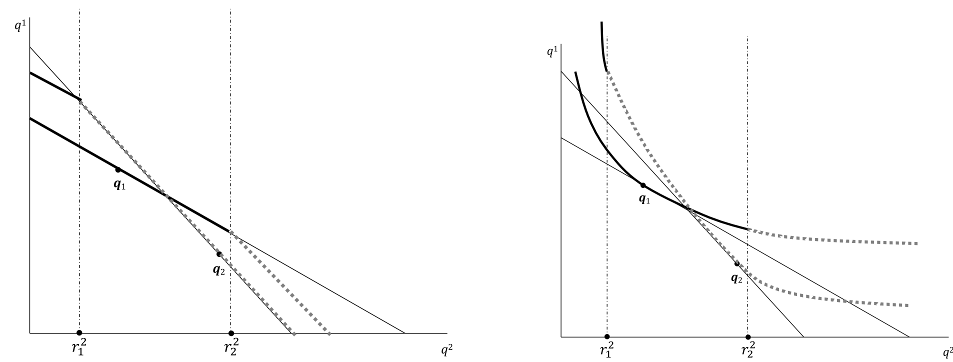
Like Proposition 4, Proposition 5 relaxes KR's constant-sensitivity assumption that the sum of consumption and gain-loss utility that determines consumer demand is additively separable across goods, and KR's cross-regime links between marginal rates of substitution (CM's Table 1).

Both generalizations appear to be empirically important.

Proposition 5's restriction to the Afriat regime preferences means that its sufficient conditions for a rationalization may not be necessary.

For example, the Afriat regime preferences in Figure 7a do not yield a rationalization but the non-Afriat regime preferences in Figure 7b do.

Figure 7. A rationalization may require non-Afriat rationalizing regime preferences (solid lines for the loss map, dashed for the gain map)




(a) (b)

Although Proposition 5's sufficient conditions are not necessary, Mas-Colell's (1978) and Forges and Minelli's (2009) analyses of the neoclassical case suggest that in the limit as the data become rich, so each regime's range of convexified rationalizing regime preferences collapses on its Afriat preferences, those conditions are asymptotically necessary.

Proposition 5 immediately suggest a procedure for nonparametrically estimating a continuous reference-dependent model with constant sensitivity:

(i) Use the observations' modeled reference points to sort their consumption bundles into gain-loss regimes.

(ii) Pooling the data from all regimes, use linear programming to find Afriat numbers $U_t$, $v_{t+}^k$, $v_{t-}^k$, and $\lambda_t > 0$ for each $k = 1, \dots, K$ and $t = 1, \dots, T$ that satisfy [A]'s Afriat inequalities (15).

(iii) Use the fact that for each observation in a regime, (15) can hold with equality for another observation in the regime, to choose numbers so that for observation *t* in regime *g*, the rationalizing Afriat utilities are given as in (17) in the proof of Proposition 5:

$$U_t = u^g(\boldsymbol{q}_t, \boldsymbol{r}_t) \equiv min_{\rho \in \Theta(\{\boldsymbol{q}_t, \boldsymbol{r}_t\}_{t=1,\dots,T}; g)} \left\{ U_\rho + \sum_k [G_+^k(\boldsymbol{q}_\rho, \boldsymbol{r}_t) v_{\rho+}^k + G_-^k(\boldsymbol{q}_\rho, \boldsymbol{r}_t) v_{\rho-}^k] + \lambda_\rho \boldsymbol{p}_\rho \cdot (\boldsymbol{q}_t - \boldsymbol{q}_\rho) \right\}.$$

(iv) Use (ii)'s Afriat numbers $U_t$, $v_{t+}^k$, and $v_{t-}^k$ to check that [B]'s monotonicity restrictions are satisfied.

(v) Use (iii)'s rationalizing Afriat utilities to check, regime by regime and observation by observation, that [C]'s conditions (16) are satisfied by scanning along the budget surface.

Proposition 5's conditions (15) involve linear inequalities in a finite number of variables; and its conditions (16) involve nonlinear inequalities in a continuum of $\boldsymbol{q}$ values. Both sets of inequalities are finitely parameterized by the $U_t$, $v_{t+}^k$, $v_{t-}^k$, and $\lambda_t$ that satisfy [A]'s (15). Thus our procedure satisfies most of the desiderata of and should inherit much of the tractability of Diewert's (1973) and Varian's (1982) methods for the neoclassical case.