To be worked and handed in for the Week 8 class:

5. Consider a two-firm Cournot model in which the firms have constant unit costs but the costs differ across firms. Let c_j be firm j's unit cost, j=1,2, and assume that $c_1 > c_2$. The firms' products are perfect substitutes, and if $q = q_1 + q_2$ is total output in the market, the inverse demand function is $p(q) = a – bq$, with $a > c_1 > c_2$ and $b > 0$. The structure is common knowledge.

(a) Derive the Nash equilibrium of the Cournot game in which firms choose their quantities simultaneously. For what values of c_1, c_2, a, and b does this equilibrium involve only one firm producing? Which firm will this be?

(a) $\max_{q_i \geq 0}$ $(a - c_i – b(q_i+q_j))\, q_i$ yields Kuhn-Tucker first-order condition $a - c_i – 2bq_i - bq_j \leq 0$, $= 0$ if $q_i > 0$. If $q_i > 0$ and $q_j > 0$, we get $q_i = (a - c_i)/2b - q_j/2$ and $q_j = (a – c_j)/2b – q_i/2$, which solve out to $q_i = (a - 2c_i + c_j)/3b$ and $q_j = (a - 2c_j + c_i)/3b$. This works whenever $(a - 2c_i + c_j) > 0$ and $(a - 2c_j + c_i) > 0$, which given $c_1 > c_2$ is true whenever $(a - 2c_1 + c_2) > 0$. Otherwise $q_1 = 0$ and $q_2 = (a - c_2)/2b > 0$, so firm 2 never shuts down as long as $a > c_1 > c_2$.

(b) When the equilibrium in (a) involves both firms producing, how do their equilibrium outputs and profits vary when c_1 increases? Explain your answer for firm 2, using the notion of strategic substitutes.

(b) When c_1 increases, q_1 decreases, and q_2 increases because firms' quantities are strategic substitutes in the Cournot model. π_1 goes down because c_1 increases, the effect of the response in q_1 can be ignored by the Envelope Theorem, and q_2 increases, which hurts firm 1 in the Cournot model. π_2 goes up because c_2 is unchanged and q_1 decreases, which helps firm 2 in the Cournot model.

6. Consider the following two-person zero-sum betting game with private information. Each of two players, 1 and 2, is independently given correct but possibly imprecise information about which of three ex ante equally likely states has occurred, A, B, or C. As indicated by the borders in the table below, player 1 learns either that the state is A or that it is {B or C}; and player 2 learns either that the state is {A or B} or that it is C. The rules of the game, the information structure, and players' rationality are common knowledge. Once informed, the players choose simultaneously between two decisions: Bet or Pass. A player who chooses Pass earns 10 no matter what the state is. If one player chooses Bet while the other chooses Pass, both earn 10 no matter what the state is. But if both players choose Bet, then they get the payoffs listed for them in the table, for the state that actually occurs.

player/state	A	B	C
1	0	30	5
2	25	5	20

(a) What are the feasible pure strategies for player 1? For player 2? Explain any notation clearly.

(a) A feasible pure strategy for player 1 must specify whether he chooses Bet or Pass in information set {A}, and independently in {B or C}. Thus his feasible pure

strategies are (Bet, Bet) (meaning (Bet in {A}, Bet in {B or C})), (Bet, Pass) (Pass, Bet), and (Pass, Pass). A feasible pure strategy for player 2 must specify whether he chooses Bet or Pass in information set {A or B}, and independently in {C}. Thus his feasible pure strategies are (Bet, Bet) (now meaning (Bet in {A or B}, Bet in {C})), (Bet, Pass) (Pass, Bet), and (Pass, Pass).

(b) Show that the betting game has a trivial Bayesian Nash equilibrium in which each player Passes in each of his information sets.
(b) If each player Passes in each of his information sets, neither player can unilaterally change the outcome, so their strategies are in equilibrium.

(c) Identify a nontrivial Bayesian Nash equilibrium. (Hint: Try using iterated weak dominance.) Can betting ever take place in your equilibrium? Explain.
(c) Pass is weakly dominated by Bet for player 2 in {C}, because 20 > 10. Bet is weakly dominated by Pass for player 1 in {A}, because 0 < 10. Given this, Bet is weakly dominated by Pass for player 2 in {A or B}, because 1 will Pass in {A}, so Betting in {A or B} yields player 2 at most 5 < 10. Given this, Bet is weakly dominated by Pass for player 1 in {B or C}, because 2 will Pass in {A or B}, so Betting in {B or C} yields 1 at most 5 < 10. This covers all the contingencies. Thus (Pass, Pass) for player 1 and (Pass, Bet) for player 2 is an equilibrium. In no state do both players Bet, so in equilibrium no betting ever takes place.

7. Consider the Battle of the Sexes game with payoffs as indicated below. Assume, here and below, that the structure is common knowledge. For each of the variations of timing and information described below, write the game tree and payoff matrix, and then find the game's subgame-perfect equilibrium or equilibria, and its equilibria (subgame-perfect or not):

	Fights	Ballet
Fights	1 3	0 0
Ballet	0 0	3 1

(a) The original simultaneous-move game is a complete model of the players' situation.
(a) The game tree can start with a node for Row with two branches labeled F and B, each leading to a node for Column with two branches labeled F and B. Column's two nodes are in one information set. Payoffs are easy to derive from the matrix above. (Can reverse roles.) Row and Column each have two pure strategies, F and B; the matrix is the 2x2 matrix above.
There are three equilibria, each of which is trivially subgame-perfect: (F, B), (B, F), and a mixed-strategy equilibrium, in Row (Column) chooses Fights (Ballet) with probability ¼.

(b) The game is modified so that Row chooses her/his strategy first and Column gets to observe her/his choice (including the realization of randomization) before choosing her/his own strategy.

(b) The game tree is as in (a), but without the information set for Column. The payoff matrix has two pure strategies for Row, F and B; but now four for Column: F if (Row chose) F, F if B; B if F, F if B; B if F, B if B; and F if F, B if B. There is a unique subgame-perfect equilibrium, in Row chooses Fights and Column plays the strategy (F if F, B if B), and so also chooses Fights on the equilibrium path. There are also imperfect equilibria in which Row chooses Ballet and Column plays the strategy (B if F, B if B); or in which Row chooses Fights and Column plays the strategy (F if F, F if B).

(c) The game is modified so that Row chooses her/his strategy first but Column does NOT get to observe her/his choice before choosing her/his own strategy.
(c) The game tree, payoff matrix, equilibria, and subgame-perfect equilibria are all exactly as in (a).

(d) The game is modified so that Row chooses her/his strategy first and Column gets to observe her/his choice (including the realization of randomization) before choosing her/his own strategy, but then Row gets to observe Column's choice and (costlessly) revise her/his own choice if s/he wishes, and this decision ends the game (so Column cannot revise her/his choice). In this part, you are not asked to completely describe players' strategies or the entire set of equilibria, just describe the subgame-perfect equilibrium or equilibria. Hint: Does Row's initial choice have any effect on the subgames that follow?
(d) Row's initial choice is cheap talk, because it can be costlessly changed; thus it cannot affect the subgames that follow, or the set of subgame-perfect equilibrium strategies as they pertain to the rest of the game. The subgame-perfect equilibrium outcomes in the rest of the game are as in (b), but with Row's and Column's roles reversed (because Column now makes the first real move). Thus there is a unique subgame-perfect equilibrium outcome, in which Column chooses B and (ignoring Row's initial choice as irrelevant, Row plays the strategy F if F, B if B, with the outcome (B, B).

8. (a) Give a clear and concise definition of the term "strategy".
(a) A strategy is a complete contingent plan of action for a player in a game, specifying the decision he would make at every decision node (or information set) where he has one (whether these nodes are on or off the equilibrium path of play).

(b) Explain why a strategy must specify what the player does off as well as on the equilibrium path.
(b) To evaluate the rationality of a strategy, a player must compare the consequences of going off the path to staying on it. In general, this requires a complete contingent plan for each player.
(c) Give a clear and concise definition of the term "information set".
(c) An information set is a set of decision nodes for a player among which the player cannot distinguish.
(d) Explain why a player must make the same decision at each node in one of his information sets.

(d) If he cannot tell which node he is at, he does not have enough information to make his decision depend on which node.

For each of the following statements, say whether it is true or false. If it is true, explain why (or if you prefer, give a proof). If it is false, give an example of a game for which it is false.

(e) A subgame-perfect equilibrium in a sequential-move game must be a Nash equilibrium.
(e) True. If neither player can do better by changing his decision at any of the information sets along the equilibrium path, then neither player can do better by changing his strategy.

(f) A Nash equilibrium in a sequential-move game must be a subgame-perfect equilibrium.
(f) False. Nash equilibrium requires that the decisions specified by players' strategies are optimal only at information sets on the equilibrium path, but rollback equilibrium requires this off as well as on the equilibrium path. The contracting game with observable proposal is an example.

9. Imagine a market setting with three firms. Firms 2 and 3 are already operating as monopolists in two different industries (they are not competitors). Firm 1 must decide whether to enter Firm 2's industry and compete with Firm 2, or enter Firm 3's industry and thus compete with Firm 3. Production in Firm 2's industry occurs at zero cost, while the cost of production in Firm 3's industry is 2 per unit. Demand in Firm 2's industry is given by $p = 9 - Q$, while demand in Firm 3's industry is given by $p' = 14 - Q'$, where p and Q denote price and total quantity in Firm 2's industry and p' and Q' denote price and total quantity in Firm 3's industry.

The firms interact as follows. First, Firm 1 chooses between E^2 and E^3, where E^2 means "enter Firm 2's industry" and E^3 means "enter Firm 3's industry." This choice is observed by Firms 2 and 3. Then, if Firm 1 chose E^2, Firms 1 and 2 compete as Cournot duopolists, where they select quantities q_1 and q_2. In this case, Firm 3 automatically gets the monopoly profit of 36 in its own industry. On the other hand, if Firm 1 chose E^3, then Firms 1 and 3 compete as Cournot duopolists, where they select quantities q_1' and q_3'. In this case, Firm 2 automatically gets the monopoly profit of 20¼ in its own industry.

(a) Calculate the subgame-perfect Nash equilibrium of this game and report the subgame-perfect equilibrium quantities. In the equilibrium, does Firm 1 enter Firm 2's industry or Firm 3's industry?
(a) tree has choice between E1 and E2 for Firm 1, then Cournot subgame with whichever of firms 2 or 3 is entered. Compute Cournot payoffs in the subgames, then backward induct to see which industry gets entered. E2 yields subgame with Cournot equilibrium $q_1 = q_2 = 3$ and profit of 9 for each; E3 yields subgame with Cournot equilibrium $q_1 = q_3 = 4$ and profit of 16 for each. Thus Firm 1 enters against Firm 3, and that subgame ensues, with Firm 2 planning to produce 3 if Firm

1 enters against it.

(b) Is there a Nash equilibrium (not necessarily subgame-perfect) in which Firm 1 selects E^2? If so, describe it. If not, briefly explain why.
(b) Yes. Firm 3 could adopt a strategy in which it sets q_3 = 12 if entered, "blowing up" the industry, lowering Firm 1's profits from entering to 0, below those of entering Firm 2. In the equilibrium, this deters Firm 1 from entering against Firm 3, and so the suboptimal behavior following entry does not lower Firm 3's payoff. The equilibrium is not subgame-perfect.

10. This question concerns repeated Prisoner's Dilemma games with one-stage payoff matrix:

		Column	
		Cooperate	Defect
	Cooperate	3 , 3	0 , 5
Row	Defect	5 , 0	1 , 1

The "Tit-for-tat" strategy defined by: Cooperate on the first play and from then on, in each period, play the other player's most recently observed pure strategy.

(a) Is (Tit For Tat, Tit For Tat) a subgame-perfect equilibrium in a finitely repeated Prisoner's Dilemma (with or without discounting)? Is it a Nash equilibrium? (You are not required to prove your answers here, just explain them briefly.)
(a) Neither, because in equilibrium must get Defect, Defect in last period after any history.

Now suppose that you are playing an infinitely repeated Prisoner's Dilemma, with discount factor 0.99 and payoffs as above. The periods are numbered 1,2,3,….

(b) Is (Tit For Tat, Tit For Tat) a subgame-perfect equilibrium in an infinitely repeated Prisoner's Dilemma with discounting? Is it a Nash equilibrium? (You are not required to prove your answers here, just explain them briefly.)
(b) No. Yes. Sketch proof, similar to the one in class.

(c) What is your best response to your partner's strategy, "Cooperate in even-numbered periods no matter what happened before, and Defect in odd-numbered periods no matter what happened before"? (That is, what strategy maximizes your expected discounted payoff, given the stated strategy for your partner? Be sure to specify your best-response strategy completely.)
(c) all-D no matter what, because partner does not respond to your actions, and D is a dominant strategy in each period

(d) What is your best response to your partner's strategy, "Start out Cooperating, and Cooperate in any period in which the other player (that is, you) have not just Defected *twice* in a row" (this strategy is called Tit For Two Tats)? Does your answer depend on the discount factor? Explain.
(d) D,C,D,C,D,C,… when the discount factor is high enough, because your partner will then cooperate every period, yielding you 5,3,5,3,5,3,…, and for high discount factors this is better than D,D,D,D, …. versus C,D,D,D,…, which yields you 5,5,1,1,1,…. But when the discount factor is low enough, all-D is better.

(e) Is (Tit For Two Tats, Tit For Two Tats) a subgame-perfect equilibrium in the infinitely repeated Prisoner's Dilemma with discount factor 0.99? Is it a Nash equilibrium?
(e) Neither, because of the answer to (d).