Supplementary Handout on Zero-Sum Two-Person Games **Vincent Crawford**
Economics 172A **Winter 2008**

Example: Pure-strategy security levels (floors on left, floor and ceiling on right)

	L (-1*)	C (-2)	R (-5)
T (1*)	-1 1	-2 2	-4 4
M (0)	0 0	0 0	-5 5
B (0)	0 0	-1 1	0 0

Both Players' Payoffs Shown

	L (1*)	C (2)	R (5)
T (1*)	1	2	4
M (0)	0	0	5
B (0)	0	1	0

Only Row's Payoffs Shown

Because Row's and Column's pure-strategy security levels are consistent (*s mark maximal floors 1 and -1 on left; maximal floor 1 and minimal ceiling 1 on right), there is no role for mixed strategies in this game.

Example: Mixed-strategy security levels (floors on left, floor and ceiling on right)

	L (-1)	C (-1)	R (-2)
T (-1)	-1 1	1 -1	-2 2
M (-1)	1 -1	-1 1	-2 2
B (-2)	2 -2	2 -2	2 -2

Both Players' Payoffs Shown

	L (1)	C (1)	R (2)
T (-1)	1	-1	2
M (-1)	-1	1	2
B (-2)	-2	-2	-2

Only Row's Payoffs Shown

Because Row's and Column's pure-strategy security levels are inconsistent (maximal floors -1 for Row + -1 for Column on left add to < 0; maximal floor for Row -1 < minimal ceiling for Column on right), mixed strategies play a role.

On the left, Row's security level for Pr{T} = Pr{M} = ½ is an expected payoff of 0; and Column's for Pr{L} = Pr{C} = ½ is an expected payoff of 0. Equivalently on the right, Row's maximal floor is 0 and Column's minimal ceiling is 0. Thus Row's and Column's mixed-strategy security levels are consistent.

Using linear programming to find security-level maximizing strategies

	L (-1*)	C (-2)	R (-5)
T (1*)	-1 1	-2 2	-4 4
M (0)	0 0	0 0	-5 5
B (0)	0 0	-1 1	0 0

Both Players' Payoffs Shown

	L (1*)	C (2)	R (5)
T (1*)	1	2	4
M (0)	0	0	5
B (0)	0	1	0

Only Row's Payoffs Shown

In this game one can easily find security-level maximizing strategies by iterated elimination of dominated strategies, which never eliminates all security-level maximizing strategies: Iterated elimination first eliminates R for Column and B for Row, then M for Row and C for Column, leaving only T for Row and L for Column, which we have already seen are the security-level maximizing strategies.

But often it's not so easy, as in the game with mixed-strategy security-level maximizing strategies, and we need a systematic method: linear programming.

Consider this game again:

	L (-1)	**C (-1)**	**R (-2)**
T (-1)	1, -1	-1, 1	2, -2
M (-1)	-1, 1	1, -1	2, -2
B (-2)	-2, 2	-2, 2	-2, 2

Both Players' Payoffs Shown

	L (1)	**C (1)**	**R (2)**
T (-1)	1	-1	2
M (-1)	-1	1	2
B (-2)	-2	-2	-2

Only Row's Payoffs Shown

Row's linear programming problem would be:

Choose x_1, x_2, x_3, and v to maximize v s.t.

$$v \leq 1x_1 - 1x_2 - 2x_3 \ (y_1)$$
$$v \leq -1x_1 + 1x_2 - 2x_3 \ (y_2)$$
$$v \leq 2x_1 + 2x_2 - 2x_3 \ (y_3)$$
$$x_1 + x_2 + x_3 = 1 \ (w)$$
$$x_1, x_2, x_3 \geq 0.$$

(Ignore the dual shadow prices y_1, y_2, y_3, and w for now. A more standard way to write the first constraint would be $v - 1x_1 + 1x_2 + 2x_3 \leq 0$.)

The constraints define v as a floor under Row's expected payoff, saying that v is no greater than Row's expected payoff if Column plays L, C, or R respectively.

(What if Column plays a mixed strategy? v is still a floor, because Row's expected payoff is a weighted average of his payoffs when Column plays pure strategies.)

Thus the problem finds the x_1^*, x_2^*, x_3^* that put the highest possible floor, v^*, under Row's payoff: Row's security-level maximizing strategy.

If we note that B can never be played with positive probability in a security-level-maximizing strategy for Row (why?), we can solve Row's problem graphically.

Setting $x_3 = 0$ and $x_2 = 1 - x_1$, the problem

Choose x_1, x_2, x_3, and v to maximize v s.t.

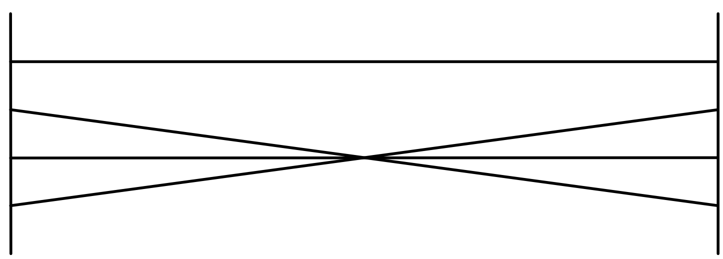
$$\begin{aligned} &v \le 1x_1 - 1x_2 - 2x_3 \\ &v \le -1x_1 + 1x_2 - 2x_3 \\ &v \le 2x_1 + 2x_2 - 2x_3 \\ &x_1 + x_2 + x_3 = 1 \\ &x_1, x_2, x_3 \ge 0. \end{aligned}$$

becomes

Choose x_1, x_2, and v to maximize v s.t.

$$\begin{aligned} &v \le 1x_1 - 1x_2 = 2x_1 - 1 \\ &v \le -1x_1 + 1x_2 = 1 - 2x_1 \\ &v \le 2x_1 + 2x_2 = 2 \\ &x_1 + x_2 + x_3 = 1 \\ &x_1, x_2, x_3 \ge 0. \end{aligned}$$

With x_1 on the horizontal axis and v on the vertical axis we get the graph::



Not drawn to scale.

The top horizontal line is at height 2 and the bottom horizontal line is at height 0; the other lines intersect the v axis at 1 and -1. The constraint $v \le 2$ is slack at the solution. The solution is at the intersection of $v = 2x_1 - 1$ and $v = 1 - 2x_1$, so $x_1^* = x_2^* = ½$, $x_3^* = 0$, and $v^* = 0$.

The optimal mixed strategy yields a security level higher than any pure strategy's.

Linear Programming Duality and Zero-Sum Two-Person Games

Row's security-level-maximizing problem with constraints put into standard form:

Choose x_1, x_2, x_3, and v to maximize v s.t.

$$v - 1x_1 + 1x_2 + 2x_3 \leq 0 \ (y_1)$$
$$v + 1x_1 - 1x_2 + 2x_3 \leq 0 \ (y_2)$$
$$v - 2x_1 - 2x_2 + 2x_3 \leq 0 \ (y_3)$$
$$x_1 + x_2 + x_3 = 1 \ (w)$$
$$x_1, x_2, x_3 \geq 0.$$

The linear programming dual of Row's security-level-maximizing problem:

Choose y_1, y_2, y_3, and w to minimize w s.t.

$$w - 1y_1 + 1y_2 - 2y_3 \geq 0 \ (x_1)$$
$$w + 1y_1 - 1y_2 - 2y_3 \geq 0 \ (x_2)$$
$$w + 2y_1 + 2y_2 + 2y_3 \geq 0 \ (x_3)$$
$$y_1 + y_2 + y_3 = 1 \ (v)$$
$$y_1, y_2, y_3 \geq 0.$$

In constructing the dual, and checking the relationship between the primal and the dual, note that v and w are variables just like the x_i and y_i; that all constraint constants are 0 but those on the equality constraints $x_1 + x_2 + x_3 = 1$ and $y_1 + y_2 + y_3 = 1$, whose shadow prices are v and w. This is why v and w are unrestricted but the x_i and y_i must be ≥ 0, and why the x_i and y_i don't appear in the objective functions.

Finally, note that the first three dual constraints, whose coefficients are the transposes of those of the first three primal constraints, define w as a ceiling over Row's expected payoff (as we did in the right-hand examples above), saying that w is no less than Row's expected payoff if Row plays T, M, or B respectively.

Because minimizing the height of the ceiling is equivalent to maximizing the height of the floor –w under Column's expected payoff, the dual determines Column's security-level maximizing strategy.

Duality and complementary slackness yields useful conclusions about the optimal strategies: All pure strategies played with strictly positive probability must yield a player exactly his security level. And slack constraints in the primal (dual) must be associated with strategies played with zero probability in the dual (primal).

Morra

	12	13	23	24
12	0	2	-3	0
13	-2	0	0	3
23	3	0	0	-4
24	0	-3	4	0

Recall the rules of Morra: Players simultaneously hold up either one or two fingers (i) and call out a number (j); call this "ij". If your number equals the total of your and the other's fingers, you win that amount from the other player. (Both can win.)

If Row's mixed-strategy probabilities of playing 12, 13, 23, 24, are x_1, x_2, x_3, x_4 and Column's mixed-strategy probabilities of playing 12, 13, 23, 24 are y_1, y_2, y_3, y_4, then any $(x_1, x_2, x_3, x_4) = (0, \rho, 1- \rho, 0)$ where $4/7 \leq \rho \leq 3/5$ (and only those) is security-level maximizing for Row; and any $(y_1, y_2, y_3, y_4) = (0, \sigma, 1- \sigma, 0)$ where $4/7 \leq \sigma \leq 3/5$ (and only those) is security-level maximizing for Column.

You can verify directly that those strategies yield Row and Column security levels of zero and no other strategy yields a security level as high as zero. (In a symmetric zero-sum game, the players can't have positive or negative security levels.)

For example, suppose Row plays $(x_1, x_2, x_3, x_4) = (0, \rho, 1- \rho, 0)$ with $4/7 \leq \rho \leq 3/5$. Then if Column plays 12 Row's expected payoff is $-2\rho + 3(1- \rho) \geq 0$ (as long as $\rho \leq 3/5$); if Column plays 13 or 23 Row's expected payoff is 0; and if Column plays 24 Row's expected payoff is $3\rho - 4(1- \rho) \geq 0$ (as long as $4/7 \leq \rho$).

If both players play security-level-maximizing strategies, the game is boring because they always tie. But unlike in matching pennies, the security-level-maximizing strategy yields a chance of gain against an opponent's 12 or 24.

(Note the typo on this in Prof. Sobel's notes IX. Two-Person Zero-Sum Game Theory, p. 12: It is not correct that "Game-theoretic analysis recommends that you mix between your first three strategies (there are mixed strategies that guarantee an expected payoff of zero that use 12 with positive probability)." The only optimal strategies are those that mix in the stated proportions between 13 and 23.)

It's a little surprising (though not truly surprising) that 12 and 24 must be played with zero probability even though they are not dominated by 23 or 24. They are, however, dominated by any optimal mixture of 23 and 24.

	12	13	23	24
12	0	2	-3	0
13	-2	0	0	3
23	3	0	0	-4
24	0	-3	4	0

You can also verify the security-level-maximizing strategies from the primal and dual security-level-maximizing linear programs:
Row's problem is:
Choose x_1, x_2, x_3, x_4, and v to maximize v s.t.

$v \leq -2x_2 + 3x_3$ (y_1)
$v \leq 2x_1 - 3x_4$ (y_2)
$v \leq -3x_1 + 4x_4$ (y_3)
$v \leq 3x_2 - 4x_3$ (y_4)
$x_1 + x_2 + x_3 + x_4 = 1$ (w)
$x_1, x_2, x_3, x_4 \geq 0$.

Column's problem is:
Choose y_1, y_2, y_3, y_4, and w to minimize w s.t.

$w \geq 2y_2 - 3y_3$ (x_1)
$w \geq -2y_1 + 3y_4$ (x_2)
$w \geq 3y_1 - 4y_4$ (x_3)
$w \geq -3y_2 + 4y_3$ (x_4)
$y_1 + y_2 + y_3 + y_4 = 1$ (v)
$y_1, y_2, y_3, y_4 \geq 0$.

You can solve one or both of these using Solver. (They're way too big to graph.)

It's not hard to check that $(x_1, x_2, x_3, x_4) = (0, \rho, 1-\rho, 0)$ where $4/7 \leq \rho \leq 3/5$ and $(y_1, y_2, y_3, y_4) = (0, \sigma, 1-\sigma, 0)$ where $4/7 \leq \sigma \leq 3/5$ are feasible for the primal and the dual, and yield the same objective function value, 0.

Or you can check that they are feasible and satisfy complementary slackness.

Thus either way, by the Duality Theorem, both are optimal.

Unlike for Matching Pennies, it's hard to imagine finding the optimal strategies for Morra by intuition or luck. (Even if you know it is wrong to play 12 or 24 with positive probability, ρ (or σ) must be between 0.587 and 0.6, a very narrow range.)

Thus knowing the solution (and being immune to boredom) may allow you to make some money.