Game Theory and IO Practice Problems for Classes in Weeks 6 and 8
First-Year M. Phil Microeconomics, Michaelmas Term 2011
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To be worked and handed in for the Week 6 class:

1. Consider the following two-person game:

		Player 2 L	R
Player 1	U	1, 2	0, 1
	D	3, 0	x, 1

Assume that both players know the value of x, and both know that they know, and so on.

(a) For what values of x (if any) is there a Nash equilibrium in which Player 2 chooses R with probability one? Explain, and describe the equilibrium or equilibria in different cases.

(b) For what values of x (if any) does decision R for Player 2 survive iterated deletion of strictly dominated strategies? Explain.

2. Two players, Row and Column, are driving toward each other on a one-lane road. Each player chooses simultaneously between going straight (S), swerving left (L), and swerving right (R). If one player goes straight while the other swerves, either right or left, the one who goes straight gets payoff 3 while the other gets –1. If each player swerves to his left, or each swerves to his right, then each gets 0 (remember, they are going in opposite directions). If both go straight, or if one swerves to his left while the other swerves to his right, then the cars crash and each gets payoff –4.

(a) Write the payoff matrix for this game.

(b) Find all of the game's rationalizable strategies for each player.

(c) Find all of the game's Nash equilibria in pure strategies.

(d) Find a Nash equilibrium in which Row uses a pure strategy and Column mixes between two of his strategies. Clearly identify which strategy or strategies have positive probabilities for each player, and what Column's mixing probabilities are. (Hint: Which of Row's pure strategies could make Column willing to put positive probability on two of Column's pure strategies?)

(e) Find a Nash equilibrium in which both Row and Column mix between two of their strategies. Clearly identify which strategies have positive probabilities for each player, and their mixing probabilities are. (Hint: Pick two pure strategies for each player—because the game is symmetric, it's natural to try the same two strategies for each—and figure out what the mixing probabilities would have to be on just those two strategies. Then compare each player's expected payoff with what he could get by switching to his third strategy.)

(f) Find the (unique) Nash equilibrium where each player uses all three of his strategies in a mixture. (Hint: First prove that the probabilities of L and R must be equal in the equilibrium mixture. Then show that for each player the probability of S must be 5/8.)

3. Suppose three identical, risk-neutral firms must decide simultaneously and irreversibly whether to enter a new market which can accommodate only two of them. If all three firms enter, all get payoff 0; otherwise, entrants get 9 and firms that stay out get 8.

(a) Identify the unique mixed-strategy equilibrium and describe the resulting probability distribution of the total ex post number of entrants. (You are not asked to show this, but the game also has three pure-strategy equilibria, in each of which exactly two firms enter; but these equilibria are arguably unattainable in a one-shot game in the absence of prior agreement or precedent. The mixed-strategy equilibrium is symmetric, hence attainable.)

4. In each of the following games, graph players' best-response curves, letting p be the probability that Row plays Up and q be the probability that Column plays Left. Then use your best-response curves to find all the equilibria in each game, whether in pure or mixed strategies. (Games (a) and (b) are zero-sum and only Row's payoffs are shown; games (c) and (d) are non-zero-sum and both players' payoffs are shown.)

(a) Column

		Left	Right
	Up	- 3	1
Row	Down	0	- 5

(b) Column

		Left	Right
	Up	5	- 4
Row	Down	1	2

(c) Bill

		B1	B2
	A1	4 , 2	- 5, 6
Ann	A2	- 1, 5	0 , - 2

(d) Bill

		B1	B2
	A1	3 , 2	0 , 1
Ann	A2	1, 0	4 , 6

To be worked and handed in for the Week 8 class:

5. Consider a two-firm Cournot model in which the firms have constant unit costs but the costs differ across firms. Let c_j be firm j's unit cost, j=1,2, and assume that $c_1 > c_2$. The firms' products are perfect substitutes, and if $q = q_1 + q_2$ is total output in the market, the inverse demand function is $p(q) = a - bq$, with $a > c_1 > c_2$ and $b > 0$. The structure is common knowledge.

(a) Derive the Nash equilibrium of the Cournot game in which firms choose their quantities simultaneously. For what values of c_1, c_2, a, and b does this equilibrium involve only one firm producing? Which firm will this be?

(b) When the equilibrium in (a) involves both firms producing, how do their equilibrium outputs and profits vary when c_1 increases? Explain your answer for firm 2, using the notion of strategic substitutes.

6. Consider the following two-person zero-sum betting game with private information. Each of two players, 1 and 2, is independently given correct but possibly imprecise information about which of three ex ante equally likely states has occurred, A, B, or C. As indicated by the borders in the table below, player 1 learns either that the state is A or that it is {B or C}; and player 2 learns either that the state is {A or B} or that it is C. The rules of the game, the information structure, and players' rationality are common knowledge. Once informed, the players choose simultaneously between two decisions: Bet or Pass. A player who chooses Pass earns 10 no matter what the state is. If one player chooses Bet while the other chooses Pass, both earn 10 no matter what the state is. But if both players choose Bet, then they get the payoffs listed for them in the table, for the state that actually occurs.

player/state	A	B	C
1	0	30	5
2	25	5	20

(a) What are the feasible pure strategies for player 1? For player 2? Explain any notation clearly.

(b) Show that the betting game has a trivial Bayesian Nash equilibrium in which each player Passes in each of his information sets.

(c) Identify a nontrivial Bayesian Nash equilibrium. (Hint: Try using iterated weak dominance.) Can betting ever take place in your equilibrium? Explain.

7. Consider the Battle of the Sexes game with payoffs as indicated below. Assume, here and below, that the structure is common knowledge. For each of the variations of timing and information described below, write the game tree and payoff matrix, and then find the game's subgame-perfect equilibrium or equilibria, and its equilibria (subgame-perfect or not):

	Fights	Ballet
Fights	1 3	0 0
Ballet	0 0	3 1

(a) The original simultaneous-move game is a complete model of the players' situation.
(b) The game is modified so that Row chooses her/his strategy first and Column gets to observe her/his choice (including the realization of randomization) before choosing her/his own strategy.

(c) The game is modified so that Row chooses her/his strategy first but Column does NOT get to observe her/his choice before choosing her/his own strategy.

(d) The game is modified so that Row chooses her/his strategy first and Column gets to observe her/his choice (including the realization of randomization) before choosing her/his own strategy, but then Row gets to observe Column's choice and (costlessly) revise her/his own choice if s/he wishes, and this decision ends the game (so Row cannot revise her/his choice). In this part, you are not asked to completely describe players' strategies or the entire set of equilibria, just describe the subgame-perfect equilibrium or equilibria. Hint: Does Row's initial choice have any effect on the subgames that follow?

8. (a) Give a clear and concise definition of the term "strategy".

(b) Explain why a strategy must specify what the player does off as well as on the equilibrium path.

(c) Give a clear and concise definition of the term "information set".

(d) Explain why a player must make the same decision at each node in one of his information sets.

For each of the following statements, say whether it is true or false. If it is true, explain why (or if you prefer, give a proof). If it is false, give an example of a game for which it is false.

(e) A subgame-perfect equilibrium in a sequential-move game must be a Nash equilibrium.

(f) A Nash equilibrium in a sequential-move game must be a subgame-perfect equilibrium.

9. Imagine a market setting with three firms. Firms 2 and 3 are already operating as monopolists in two different industries (they are not competitors). Firm 1 must decide whether to enter Firm 2's industry and compete with Firm 2, or enter Firm 3's industry and thus compete with Firm 3. Production in Firm 2's industry occurs at zero cost, while the cost of production in Firm 3's industry is 2 per unit. Demand in Firm 2's industry is given by $p = 9 - Q$, while demand in Firm 3's industry is given by $p' = 14 - Q'$, where p and Q denote price and total quantity in Firm 2's industry and p' and Q' denote price and total quantity in Firm 3's industry.

The firms interact as follows. First, Firm 1 chooses between E^2 and E^3, where E^2 means "enter Firm 2's industry" and E^3 means "enter Firm 3's industry." This choice is observed by Firms 2 and 3. Then, if Firm 1 chose E^2, Firms 1 and 2 compete as Cournot duopolists, where they select quantities q_1 and q_2. In this case, Firm 3 automatically gets the monopoly profit of 36 in its own industry. On the other hand, if Firm 1 chose E^3, then Firms 1 and 3 compete as Cournot duopolists, where they select quantities q_1' and q_3'. In this case, Firm 2 automatically gets the monopoly profit of 20¼ in its own industry.

(a) Calculate the subgame-perfect Nash equilibrium of this game and report the subgame-perfect equilibrium quantities. In the equilibrium, does Firm 1 enter Firm 2's industry or Firm 3's industry?

(b) Is there a Nash equilibrium (not necessarily subgame-perfect) in which Firm 1 selects E^2? If so, describe it. If not, briefly explain why.

10. This question concerns repeated Prisoner's Dilemma games with one-stage payoff matrix:

		Column: Cooperate	Column: Defect
	Cooperate	3 , 3	0 , 5
Row	Defect	5 , 0	1 , 1

The "Tit-for-tat" strategy defined by: Cooperate on the first play and from then on, in each period, play the other player's most recently observed pure strategy.

(a) Is (Tit For Tat, Tit For Tat) a subgame-perfect equilibrium in a finitely repeated Prisoner's Dilemma (with or without discounting)? Is it a Nash equilibrium? (You are not required to prove your answers here, just explain them briefly.)

Now suppose that you are playing an infinitely repeated Prisoner's Dilemma, with discount factor 0.99 and payoffs as above. The periods are numbered 1,2,3,….

(b) Is (Tit For Tat, Tit For Tat) a subgame-perfect equilibrium in an infinitely repeated Prisoner's Dilemma with discounting? Is it a Nash equilibrium? (You are not required to prove your answers here, just explain them briefly.)

(c) What is your best response to your partner's strategy, "Cooperate in even-numbered periods no matter what happened before, and Defect in odd-numbered periods no matter what happened before"? (That is, what strategy maximizes your expected discounted payoff, given the stated strategy for your partner? Be sure to specify your best-response strategy completely.)

(d) What is your best response to your partner's strategy, "Start out Cooperating, and Cooperate in any period in which the other player (that is, you) have not just Defected *twice* in a row" (this strategy is called Tit For Two Tats)? Does your answer depend on the discount factor? Explain.

(e) Is (Tit For Two Tats, Tit For Two Tats) a subgame-perfect equilibrium in the infinitely repeated Prisoner's Dilemma with discount factor 0.99? Is it a Nash equilibrium?