VC
ECON 200C
SPRING 94

Notes on Risk (Including Proofs for Machina-Rothschild)

Theorem ("$F^*(\cdot)$ is riskier than $F(\cdot)$"):

Assume $F^*(\cdot)$ and $F(\cdot)$ are distributed on $[0, M]$.
Then the following four conditions are equivalent:

(A) $F^*(\cdot)$ and $F(\cdot)$ have the same mean and
$\int_0^M u(x)\, dF^*(x) \leq \int_0^M u(x)\, dF(x)$ for all concave (not necessarily increasing) vN-M utility functions $u(\cdot)$.

(B) $F(\cdot)$ and $F^*(\cdot)$ are the c.d.f.s of the random variables $\tilde{x}$ and $\tilde{x} + \tilde{\varepsilon}$, where $E(\tilde{\varepsilon} \mid x) \stackrel{x}{\equiv} 0$.

(C) $F^*(\cdot)$ may be obtained from $F(\cdot)$ by a finite sequence, or as the limit of an infinite sequence, of mean-preserving spreads.

(C′) $\int_0^x [F^*(t) - F(t)]\, dt \geq 0, \quad \forall x \in (0, M]$ ("spread")
and $= 0, \quad x = M$ ("mean-preserving")

Remarks: 1. Don't need $u(\cdot)$ increasing in (A) because the means of $F^*(\cdot)$ and $F(\cdot)$ are the same; a different version, discussed later, allows for changes in the mean.

2. In (B) it is not required that the realizations of the random variables be related as $\tilde{x}$ and $\tilde{x} + \tilde{\varepsilon}$, just that the distributions are — the realizations can be rearranged if necessary.

3. The condition $E(\tilde{\varepsilon} \mid x) \stackrel{x}{\equiv} 0$ is weaker than independence (which would require that the whole conditional distribution of $\tilde{\varepsilon} \mid x$ be independent of x) but stronger than uncorrelatedness (in that it holds for all x).

4. (C) is a geometric concept, defined here informally by pictures.

5. (C′), called the "integral condition", is clearly a partial ordering of distributions in that it's required to hold for all $x \in (0, M]$. It can be viewed as the transitive closure of the single-crossing relation that holds for a single mean-preserving spread (but not for a sequence of them; see geometric example).

6. (C') implies a transitive (though partial) ordering because if $\int_0^x [F^{**}(t) - F^*(t)]\,dt \geq 0$ and $\int_0^x [F^*(t) - F(t)]\,dt \geq 0$, $\forall x \in (0, m]$, clearly $\int_0^x [F^{**}(t) - F^*(t)]\,dt + \int_0^x [F^*(t) - F(t)]\,dt = \int_0^x [F^{**}(t) - F(t)]\,dt \geq 0$, $\forall x \in (0,m]$ as well.

7. that the "spread" part of (C') captures the idea of a sequence of (mean-preserving) spreads will be "proved" partly by appeal to geometric intuition; Rothschild + Stiglitz I give a real (and tedious) proof. That the "mean-preserving" part is right is proved below.

<u>Proof</u> (assuming $u(\cdot)$ is twice continuously differentiable and $F(\cdot)$ and $F^*(\cdot)$ have densities $f(\cdot)$ and $f^*(\cdot)$):

a. (A) $\Leftrightarrow$ (C') $\int_0^m u(x) f(x)\,dx \overset{\text{IBP}}{=} [\underset{\text{"u"}}{u(x)} \underset{\text{"v"}}{F(x)}]_0^m - \int_0^m \underset{\text{"v"}}{F(x)} \underbrace{u'(x)\,dx}_{\text{"du"}}$

$$= u(m) - \int_0^m F(x) u'(x)\,dx, \text{ since } F(0) = 0.$$

(corollary when $u(x) \equiv x$: $Ex = m - \int_0^m F(x)\,dx = \int_0^m [1 - F(x)]\,dx$, hence the "mean-preserving" part of (C').)

Thus $\Delta \equiv \int_0^m u(x) f(x)\,dx - \int_0^m u(x) f^*(x)\,dx = \int_0^m \underset{\text{"u"}}{u'(x)} \underbrace{[F^*(x) - F(x)]\,dx}_{\text{"dv"}}$

This result will be used to prove the basic result characterizing first-order stochastic dominance preference, stated formally below. Integrating by parts again,

$$\Delta = [\underset{\text{"u"}}{u'(x)} \underbrace{\int_0^x [F^*(t) - F(t)]\,dt}_{\text{"v"}}]_0^m - \int_0^m \underbrace{[\int_0^x [F^*(t) - F(t)]\,dt]}_{\text{"v"}} \underbrace{u''(x)\,dx}_{\text{"du"}}$$

$$= u'(m) \int_0^m [F^*(t) - F(t)]\,dt - \int_0^m [\int_0^x [F^*(t) - F(t)]\,dt]\, u''(x)\,dx.$$

This result will be used to prove the characterization of second-order stochastic dominance preference, stated formally below.

By the mean-preservingness condition, the first term $= 0$, so

$$\Delta = -\int_0^m [\int_0^x [F^*(t) - F(t)]\,dt]\, u''(x)\,dx.$$

thus, if (C′) is satisfied and $u''(x) \leq 0$, $\forall x \in [0,m]$, $\Delta \geq 0$, which implies (A). Conversely, if (C′) is violated, can take $u(\cdot)$ concave but linear except in interval where $\int_0^x [F^*(t) - F(t)]dt < 0$ and get $\Delta < 0$, contradicting (A). Thus (A) $\Leftrightarrow$ (C′).

Note that we needed (A) for all distributions (with the same mean) and (C′) for all $x \in [0,m]$ to get the clean equivalence demonstrated.

b. (C) $\Rightarrow$ (C′) proved geometrically; (C′) $\Rightarrow$ (C) in RS I.

c. (B) $\Rightarrow$ (A) (assuming that $\tilde{X}$ and $\tilde{\epsilon}$ have a joint density): If $h(x,\epsilon)$ is the joint density,

$$\iint u(x+\epsilon)\,h(x,\epsilon)\,d\epsilon\,dx \leq \int\Big[\int [u(x)+\epsilon u'(x)]\,h(x,\epsilon)\,d\epsilon\Big]dx$$

(by $u''(x) \leq 0$ and Taylor's Theorem)

$$= \iint u(x)\,h(x,\epsilon)\,d\epsilon\,dx + \int u'(x)\int \epsilon\,h(x,\epsilon)\,d\epsilon\,dx$$

$$= \iint u(x)\,h(x,\epsilon)\,d\epsilon\,dx$$ by (B), which requires $\int \epsilon\,h(x,\epsilon)\,d\epsilon \overset{x}{\equiv} 0$. Note that this would just be Jensen's Inequality if x were deterministic.

RS I do (A) $\Rightarrow$ (B), indirectly.

This completes the proof, except for the omitted parts.

Sample application of (B) $\Rightarrow$ (A), "diversification pays": Can invest w_0 in any proportions in two i.i.d. securities w/ rates of return r and s. If x is amount invested in "r", then $x^* = \frac{w_0}{2}$ for all riskaverters.

Proof: final wealth $w = (1+r)x + (1+s)(w_0 - x)$

$$\equiv w_0 + \frac{w_0}{2}(r+s) + \left(x - \frac{w_0}{2}\right)(r-s).$$

(Both terms random, so Jensen's Inequality doesn't apply.)

Note $E(r-s \mid r+s) \overset{r+s}{\equiv} E(r \mid r+s) - E(s \mid r+s) \equiv 0$ because r,s i.i.d.

Thus if $x \neq \frac{w_0}{2}$, third term adds conditional mean zero noise.

<u>Theorem</u> ("$F(\cdot)$ First-Order Stochastically Dominates $F^*(\cdot)$"):

Assume $F(\cdot)$ and $F^*(\cdot)$ are distributed on $[0, m]$.

Then the following four conditions are equivalent:

(E1) $F(\cdot)$ may be obtained from $F^*(\cdot)$ by a finite sequence, or as the limit of an infinite sequence, of rightward shifts of probability mass.

(E2) $F(x) \leq F^*(x)$, $\forall x \in [0, m]$.

(E3) $F^*(\cdot)$ and $F(\cdot)$ are the c.d.f.s of the random variables $\tilde{x}$ and $\tilde{x} + \tilde{\epsilon}$, where $\tilde{\epsilon} \geq 0$.

(E4) $\int_0^m u(x)\, dF(x) \geq \int_0^m u(x)\, dF^*(x)$ for all nondecreasing functions $u(\cdot)$.

Remarks: 1. (E1) is defined here geometrically.

2. If (E2) seems upside down, it's because the c.d.f. is defined as the probability of being $\leq x$.

3. In (E3), it's not required that the r.v.s be related that way, just the distributions, but if the r.v.s are it's usually the easiest way to check the condition.

4. Note that, given $Ex = \int_0^m [1 - F(x)]\, dx$, (E2) implies that the mean of $F(\cdot)$ is greater than that of $F^*(\cdot)$, but is much stronger (give picture); and, unlike the mean, (E2) implies a partial ordering.

5. Note that concavity or convexity of $u(\cdot)$ is irrelevant.

<u>Proof:</u> (assuming $u(\cdot)$ is once continuously differentiable and $F(\cdot)$ and $F^*(\cdot)$ have densities $f(\cdot)$ and $f^*(\cdot)$):

a. (E1) $\Leftrightarrow$ (E2) "proved" geometrically.

b. (E2) $\Leftrightarrow$ (E3) clear from definition of c.d.f., fact that $(\tilde{x} + \tilde{\epsilon} \leq x$ and $\epsilon \geq 0) \Rightarrow \tilde{x} \leq x$.

C. (E2) $\Leftrightarrow$ (E4) follows from
$\Delta \equiv \int_0^m u'(x)\,[F^*(x) - F(x)]\,dx$, proved in (A') $\Leftrightarrow$ (C') above, this immediately gives (E2) $\Rightarrow$ (E4). If not (E2), can take $u(\cdot)$ nondecreasing but constant except in interval where $F(x) > F^*(x)$, making $\Delta < 0$ and violating (E4).

Theorem ("$F(\cdot)$ Second-order Stochastically dominates $F^*(\cdot)$"):
Same as "$F^*(\cdot)$ is riskier than $F(\cdot)$" theorem but:
In (A), drop "same mean" condition and require $u(\cdot)$ to be nondecreasing.
In (B), replace $E(\tilde{\epsilon}\,|\,x) \overset{x}{=} 0$ by $E(\tilde{\epsilon}\,|\,x) \le 0, \forall x$.
In (C), $F(\cdot)$ differs from $F^*(\cdot)$ by a finite sequence, or as the limit of an infinite sequence, of FOSD shifts and/or mean-preserving reductions in risk.
In (C'), drop the $\int_0^m [F^*(t) - F(t)]\,dt = 0$ condition.

The proof is essentially the same. Return, e.g., the the second expression for Δ. Now $\int_0^x [F^*(t) - F(t)]\,dt \ge 0, \forall x \in [0, m]$, by the new "spread" condition, and $F^*(t) - F(t) \ge 0$, $\forall t$, by the FOSD condition. Thus, given $u'(m) \ge 0$, $\Delta \ge 0$ as before, so ~~the new~~ (E4) holds. Can prove converse as before.

<u>Summary</u>: $F(\cdot)$ and $F^*(\cdot)$ are distributions on $[0,m]$.

F "FOSD"s F^* iff $F(x) \le F^*(x)$, $\forall x \in [0,m]$.

F "SOSD"s F^* iff $\int_0^x F(t)\,dt \le \int_0^x F^*(t)\,dt$, $\forall x \in [0,m]$.

F^* is an "MPS" of F iff

$$\int_0^x F(t)\,dt \le \int_0^x F^*(t)\,dt, \ \forall x \in [0,m]$$

and $$\int_0^m F(t)\,dt = \int_0^m F^*(t)\,dt.$$

Thus (F "FOSD"s F^*) $\Rightarrow$ (F "SOSD"s F^*)

but not $\Leftarrow$.

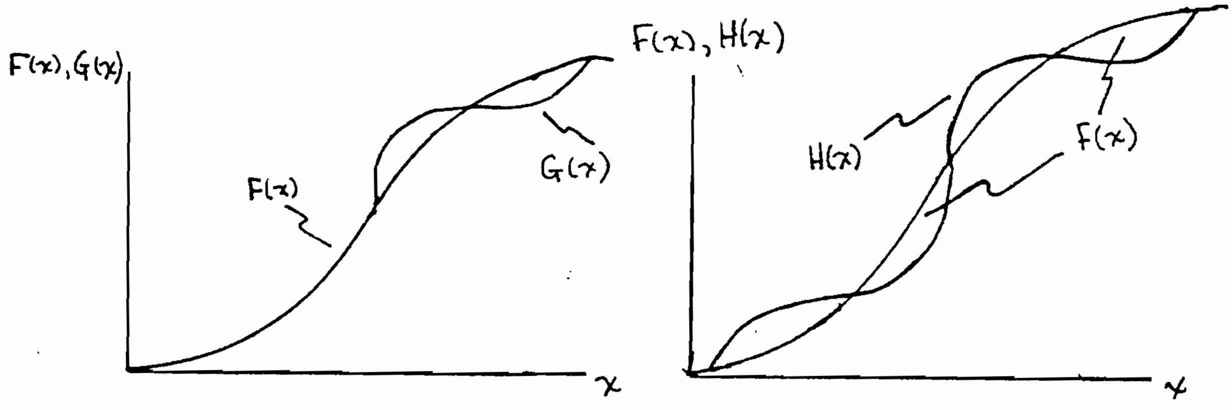
(F^* is an "MPS" of F) $\Rightarrow$ (F "SOSD"s F^*)

but not $\Leftarrow$ (because F may have higher mean)

Note that (F "FOSD"s F^*) $\not\Rightarrow$ (F^* is an "MPS" of F)

because F must have a higher mean (unless F and F^* are the same).

$F(x) \leqslant 0$ for all $x \geqslant k$ (k need not be unique).



A single MPS has the single-crossing property.

But The single-crossing property is not transitive, and more-risky-than should be a transitive relation, intuitively! Thus, the single-crossing property is not the aspect of mean-preserving spreads that is most relevant for our purposes. Note, however, that because G and F have the same means, the single-crossing property guarantees that condition the integral ~~(*)~~ is satisfied as well. Condition The integral ~~(*)~~ from any given upper limit does generate a transitive relation among distributions, because $\int (H\text{-}F) \equiv \int (H\text{-}G) + \int (G\text{-}F)$, because real #'s are linearly ordered. (Stieltjes has partial ordering, because uniform) It can be shown that if the integral cond ~~(*)~~ holds, and holds with equality for $x = 1$, then $G(x)$ differs from $F(x)$ by a sequence of mean-preserving spreads. ~~Social-choice jocks will recognize the ordering induced by (*) as the transitive closure of the ordering induced by the single-crossing property. (*) implies that you can get from F to G by a (possibly infinite) sequence of mean-preserving spreads.~~

Now we want to show that (*) is equivalent to the requirement that all risk-averters prefer F to G. Let $D \equiv \int_0^1 u(x)\, f(x)\, dx - \int_0^1 u(x)\, g(x)\, dx$. We want to show that (*) implies $D > 0$. Using integration by parts,

$$\int_0^1 u(x)\, f(x)\, dx = [U(x)F(x)]_0^1 - \int_0^1 u'(x)\, F(x)\, dx = u(1) - \int_0^1 u'(x)\, F(x)\, dx.$$

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FIGURE 5

A Mean Preserving Spread of a Density Function

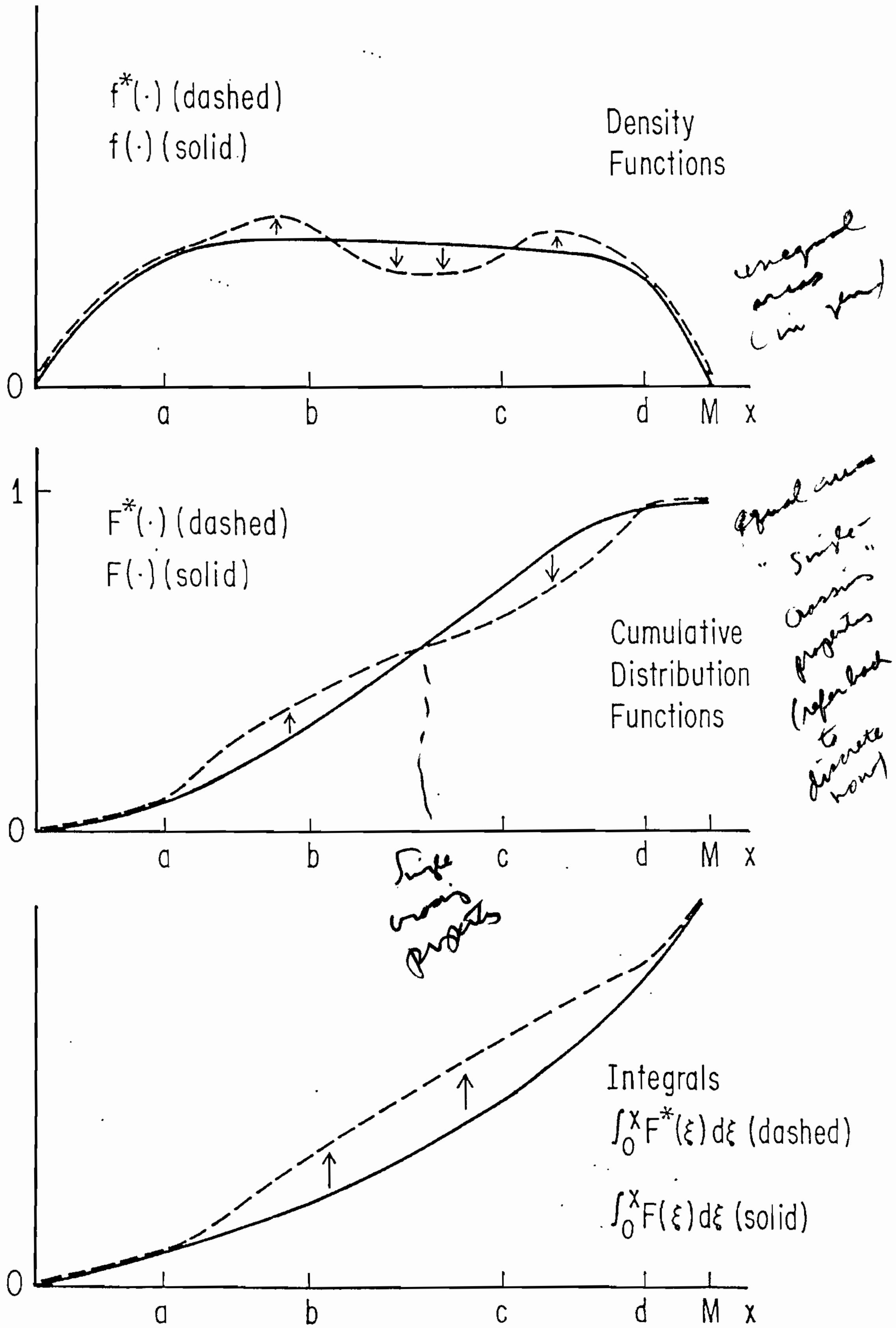

FIGURE 4

A Mean Preserving Spread of a Discrete Distribution