5. Interpretation of Nash Equilibria.

> "Ordinary people are not familiar with the Nash concept. There is no reason to expect them to play such equilibrium strategies."

(1) Do you agree?

(2) When should the Nash concept be applied?

(3) Is there any alternative?

(4) Are there any good reasons for predicting that players will play a mixed-strategy Nash equilibrium in a strategic-form game?

6. Private Provision of Public Goods. Consider the following *public goods* game:

	Contribute	Don't
Contribute	$1-c_1$, $1-c_2$	$1-c_1$, 1
Don't	1, $1-c_2$	0, 0

The benefits (1 to each player if at least one player contributes) are commonly known, but the costs of contributing c_i is private information to player i. A *strategy* for player i in this game specifies an action ("Contribute" or "Don't") for each possible value of c_i.

(1) Suppose it is common knowledge that $c_1 = \frac{1}{4}$, but Player 1 does not know c_2. Player 1 believes $c_2 = \frac{1}{4}$ with probability $\frac{1}{2}$ and $c_2 = 2$ with probability $\frac{1}{2}$.

  (a) If $c_2 = 2$, does Player 2 have a dominant strategy? If so, what is it? Does Player 2 have a dominant strategy if $c_2 = \frac{1}{4}$?

  (b) Let z denote the probability that Player 2 contributes if the cost is $c_2 = \frac{1}{4}$. What is Player 1's expected payoff from "Don't", given his beliefs, in terms of z? Does Player 1 have a dominant strategy?

  (c) What is the Bayesian-Nash equilibrium?

(2) Suppose that Player i believes $\Pr[c_j = \frac{1}{4}] = \Pr[c_j = 2] = \frac{1}{2}$ for $i = 1, 2$ and $j = 1, 2$, $j \neq i$. What is the Bayesian-Nash equilibrium of the game?

(3) Suppose that both players believe costs are drawn independently from a uniform distribution on the interval $[0, 2]$. What is the Bayesian-Nash Equilibrium now?[4]

[4]Hint: Show that an equilibrium strategy has the trigger form of contributing whenever $c_i \leq c_i^*$, and note the uniform distribution of costs.

7. An Entry Game. Consider the following two-period game with no discounting.

- In period 1, four firms simultaneously and independently decide whether or not to pay 1 to enter an industry.
- In period 2, all firms that chose to enter now simultaneously and independently choose production levels, with fixed cost $F = 5$ and zero marginal cost $c = 0$. (That is, a firm in the industry can either produce no output and incur no costs in period 2, or can produce any positive output and incur a total cost of 5 in period 2.) All production is sold at price $10 - Q$ where Q is total industry output.

Consider the five possible post-period-1 outcomes: n firms enter, where $n = 0, 1, 2, 3, 4$.

(1) Consider the Nash equilibria of the five different possible period-2 subgames corresponding to $n = 0, 1, 2, 3, 4$ entrants. (You should have a good understanding of these from the earlier question.) Which period-1 outcomes are consistent with all firms choosing pure strategies in the Nash equilibrium of the whole game that are subgame perfect (i.e., consistent with backwards induction logic)? Explain.

(2) One of the outcomes you found in part (1) (you should have found more than one) is inconsistent with forwards induction logic. Which is it? Explain.

(3) In addition to the subgame-perfect outcomes, there is one outcome consistent with all firms choosing pure strategies in a Nash equilibrium of the whole game that is imperfect. Which is it? Explain.

(4) Of the remaining period-1 outcomes, which are consistent with Nash equilibrium behaviour (perhaps including mixed strategies).

8. A Bayesian Trading Game. Suppose that a buyer has a valuation for a good v_b, uniformly distributed on $[0, 1]$. The seller has a valuation v_s independently and identically distributed on $[0, 1]$. They each observe their own valuation, but not that of the other player. Simultaneously they each announce a price, p_b and p_s respectively. If $p_b \geq p_s$ a sale takes place at a price halfway between their announcements, $p = (p_b + p_s)/2$, and the buyer receives the good, yielding payoffs of

$$u_b(p; v_b) = v_b - p \quad \text{and} \quad u_s(p; v_s) = p - v_s.$$

Otherwise, there is no sale and both players receive 0.[5]

(1) Suppose that the seller uses a linear strategy of the form $p_s(v_s) = \alpha + \beta v_s$. Show that the buyer's expected payoff may be written

$$\frac{p_b - \alpha}{\beta}\left[v_b - \frac{1}{2}\left\{p_b + \frac{\alpha + p_b}{2}\right\}\right].$$

(2) Hence calculate the buyer's best reply to the seller's linear strategy, and show that it is also linear.

(3) Now suppose the buyer uses a linear strategy of the form $p_b(v_b) = \gamma + \delta v_b$. By calculating the seller's expected payoff, find the best reply of the seller to this strategy, and show that it is also linear.

(4) Calculate values of α, β, γ, and δ such that these strategies constitute a linear Bayesian-Nash equilibrium of the trading game.

(5) For what values of v_b and v_s is trade mutually advantageous? For what values of v_b and v_s does trade take place? Comment.

(6) Now suppose the buyer and seller use the following strategies:

$$p_b = \tfrac{1}{2} \quad \text{if} \quad v_b \geq \tfrac{1}{2} \quad \text{and} \quad p_b = 0 \quad \text{otherwise,}$$
$$p_s = \tfrac{1}{2} \quad \text{if} \quad v_s \leq \tfrac{1}{2} \quad \text{and} \quad p_s = 1 \quad \text{otherwise.}$$

Argue that these strategies constitute a Bayesian-Nash equilibrium of the trading game. Without doing any further calculations, are there any other Bayesian-Nash equilibria of this game? Are any of these efficient?

[5] It would be equivalent to assume $u_s = p$ in the case of a sale and $u_s = v_s$ when there is no sale. Why?

9. Purification and Forward Induction. Chris and David attend a dinner at Morris College. They need to decide how to divide the remaining dessert wine and chocolates available. Both are particularly partial to the dessert wine (which is known to be very expensive), whereas the chocolates are known to be less good. They have one opportunity to choose either wine or chocolate before the end of dinner. If they both select the same product, neither get any satisfaction from the tiny amount they receive. Hence their preferences could be summarised in the following game:

	Wine	Choc
Choc	$3 + \varepsilon_d$ 1	0 0
Wine	0 0	1 $3 + \varepsilon_c$

where Chris is the row player (gaining the payoffs in the bottom left corners of the cells) and David is the column player (with payoffs in the top right). Chris and David have slightly different preferences over the wine types which differ from dinner to dinner: $\varepsilon_c \geq 0$ and $\varepsilon_d \geq 0$ are variables which reflect this.

(1) Suppose ε_c and ε_d are known. Calculate the Nash equilibria of this game.

(2) Suppose ε_c and ε_d are private information to Chris and David respectively, and that it is commonly known that ε_c and ε_d are independently drawn from an identical uniform distribution on $[0, \bar{\varepsilon}]$. Find a Bayesian Nash equilibrium in which neither player simply ignores their private information.

(3) What happens to this equilibrium as $\bar{\varepsilon} \to 0$?

(4) Does purification provide a justification for mixed strategy Nash equilibria?

(5) Suppose that $\varepsilon_c = \varepsilon_d = 0$. Before the game is played, David has an opportunity to bellow loudly for the wine to be passed to him. If he does so, he will receive the wine, and get a payoff of 2 whilst Chris gets a payoff of 0 (the lower payoffs are due to the shame attached to such boorish behaviour). Draw this new game in extensive form. What are the Nash equilibria? Use a forward induction argument to generate a prediction of play.

(6) Suppose again that $\varepsilon_c = \varepsilon_d = 0$. Before the 2×2 subgame is played, David has the opportunity to be rude to another guest. Doing so is simply bad; it lowers everyone's payoffs in every situation by 1. Once again, use a forward induction argument to generate a prediction of play.

10. Monetary Policy Game. Consider the following two-period game between an economy's monetary authority and its labour force. The timing of the game is:

- Nature picks the monetary authority's type. The monetary authority is *weak* with probability 0.4 and *strong* with probability 0.6.
- The monetary authority picks first period inflation to be either HIGH or LOW.
- The labour force forms *high* or *low* expectations of second period inflation.
- The authority picks second period inflation: HIGH if weak; LOW if strong.

Strong monetary authorities prefer low inflation; weak monetary authorities are less prepared to adopt policies that they believe may compromise national income in the short-run, and prefer a situation that results in higher inflation. The authority receives a payoff of 100 if it chooses its preferred first period inflation level, zero otherwise. In addition, the authority receives a bonus payoff if labour force expectations of second period inflation are low. These benefit the strong authority by keeping down inflationary pressures, gaining it a bonus payoff of 200; while the weak authority benefits from surprise inflation in the second period, gaining it a bonus of 50. (The (present value) of the bonus to the weak authority is lower since it cares less about the future.) The labour force simply wants to get its expectations of second period inflation correct. It receives a payoff of zero if it is correct, -100 otherwise.

(1) Draw this game in extensive form.

(2) Show that there is no equilibrium in which a weak monetary authority chooses LOW first period inflation.

(3) Is there a separating (perfect Bayesian) equilibrium in which a strong monetary authority chooses LOW first period inflation, and a weak monetary authority chooses HIGH first period inflation?

(4) What out-of-equilibrium beliefs would the labour force have to hold to support a pooling (perfect Bayesian) equilibrium in which both types of monetary authority chose HIGH first period inflation?

(5) Are these beliefs compatible with the intuitive criterion? If not, then why not?

11. Two Stage Game. Consider the following simultaneous-move stage game:

	L	C	R
T	1 3	0 0	0 5
M	1 2	2 1	1 3
B	2 1	1 0	4 4

This stage game is played twice, with the outcome from the first stage observed before the second stage begins. There is no discounting.

Can the payoff (4,4) be achieved in the first stage in a pure-strategy subgame-perfect Nash equilibrium? If so, describe a strategy profile that does so and prove that it is a subgame perfect Nash equilibrium. If not, prove why not.

12. Repeated Game. Consider an infinitely repeated game where the stage game is:

	L	R
U	9 9	10 1
D	1 10	7 7

Players discount the future using the common discount factor δ.

(1) What outcomes in the stage-game are consistent with Nash equilibrium play?

(2) Let v_R and v_C be the repeated game payoffs to Row and Column respectively. Draw the set of feasible payoffs from the repeated game, explaining any normalisation you use.

(3) Are all the payoffs in the feasible set obtainable from mixed-strategy combinations in the stage-game? (That is, for every point in the feasible set, can you find a p such that $0 \leq p \leq 1$ and a q such that $0 \leq q \leq 1$ that will give those expected payoffs from a single play?)

(4) What are the players' minmax values? Show the individually rational feasible set.

(5) Find a Nash equilibrium in which the players obtain the $(9, 9)$ payoff each period forever. What restrictions on δ are necessary?