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Notes, Comments, and Letters to the Editor

Comparative Statics in Matching Markets

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This paper studies the comparative statics of adding agents to matching markets that generalize the marriage and college-admissions markets of D. Gale and L. Shapley (*Amer. Math. Monthly* **69**, 1962, 9–15). It is shown, for the wide class of matching markets studied by A. Roth (*Econometrica* **52**, 1984, 47–57), that adding an agent to one side of the market weakens the competitive positions of the other agents on that side and strengthens the competitive positions of the agents on the other side. *Journal of Economic Literature* Classification Number: 026. © 1991 Academic Press, Inc.

1. INTRODUCTION

The theory of matching markets spawned by Gale and Shapley's [6] analysis of the marriage market has been the subject of much recent research. Matching markets are markets with two *sides*, identifiable independent of market conditions, such that an agent on one side can benefit only by dealing with agents on the other side, and a feasible allocation is a reciprocal, all-or-nothing assignment, or *matching*, of agents on one side to agents on the other. In Gale and Shapley's marriage market, for example, the sides are a set of men and a set of women (whose members do not depend on market conditions!); each man must choose between marrying one of the women or remaining single, and vice versa; and a man marries a woman if and only if she marries him.

Gale and Shapley suggested that frictionless competition in a marriage market would yield a matching that is *stable*, in the sense that no man and woman would prefer each other to their mates. They demonstrated the

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existence of a stable matching by constructing an algorithm they called the "deferred-acceptance procedure," in which men begin by proposing to their favorite women and women tentatively accept their favorite proposals, rejecting any others they receive. Rejected men then propose to their next-favorite women, and the process continues as before until no more rejections are issued. Gale and Shapley showed that this algorithm converges, in a finite number of steps, to a stable matching. Moreover, when all agents have strict preferences over prospective mates this matching has the surprising property that all men weakly prefer it to any other stable matching. The analogous algorithm in which women propose converges to a (generally different) stable matching that all women weakly prefer to any other stable matching. Thus, competition allows a range of outcomes, which are coherently ordered in agents' preferences, and market institutions exert a systematic influence on which of these outcomes arises.

Gale and Shapley's analysis of the marriage market has since been extended, by allowing agents to decide how to organize their relationships and share the resulting surplus, to settings that more closely resemble the markets usually studied by economists.[1] Some extensions also relax the marriage-market assumption that matching must be one-to-one, allowing many-to-one matching, in which agents on one side of the market may form partnerships with more than one agent on the other side, as in Gale and Shapley's college-admissions market, and sometimes even allowing many-to-many matching. Under reasonable restrictions on agents' preferences, described below, much of Gale and Shapley's analysis remains valid for these more general models. The resulting theory of matching markets complements the traditional theory of market competition, and enjoys an advantage in realism for some important applications.

Perhaps the most important advantage of the matching approach is its robustness to heterogeneity. A traditional competitive equilibrium cannot exist in general unless the goods traded in each market are homogeneous, because all goods in the same market must sell at the same price. A traditional model of a labor market with the degree of heterogeneity normally encountered therefore has the structure of a multi-market general equilibrium model. But because the markets in such a model are very thin, the usual arguments in support of price-taking are strained. The theory of matching markets replaces this collection of thin markets with a single market game, in which the terms of partnerships are determined endogenously, along with the matching, via negotiations between prospective partners. Gale and Shapley's notion of stability, suitably generalized, formalizes the idea of competition, and thereby makes it possible to evaluate the robustness of traditional competitive analysis to heterogeneity.

[1] See Roth and Sotomayor [11] for an excellent overview and bibliography.

(Stable outcomes in matching markets can in fact be viewed as traditional competitive equilibria when prices are allowed to reflect the differences between matches; see, for example, Shapley and Shubik [13].)

The matching approach readily yields comparative statics results about the effects of increased competition. One would normally expect the advent of a new worker, for example, to weaken the competitive positions of other workers (even those for whom he is only an imperfect substitute) and to strengthen the competitive positions of firms. This paper uses a generalization of Gale and Shapley's [6] deferred-acceptance procedure to prove results that confirm this intuition. These results are nontrivial generalizations of those originally obtained for many-to-one matching by Kelso and Crawford [7] and are, to my knowledge, the strongest available for many-to-one (or many-to-many) matching. Their proofs also provide more economic intuition than the alternative arguments (discussed below) that are now available for one-to-one matching.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 states and proves the results. Section 4 discusses related work and open questions.

2. The Model

This section describes the model, which is taken from Roth [9]. His model was based on that of Kelso and Crawford [7], which was in turn based on those of Gale and Shapley [6], Shapley and Shubik [13], and Crawford and Knoer [2]. Roth's paper and the others just referenced provide further motivation for the assumptions.

For concreteness, the model is described as a labor market. The agents consist of a finite set W of workers and a finite set F of firms. For each firm-worker pair, the finite set $X(i, j)$ represents the feasible job descriptions at which worker i could be employed by firm j. Jobs offered by different firms or held by different workers are considered distinct, so that $X(i, j)$ is disjoint from $X(k, l)$ unless $(i, j) = (k, l)$. Also associated with each agent k is an alternative u_k, which represents remaining unmatched. For each firm j, $X(j)$ denotes the union of the sets $X(i, j)$ over all workers i; and for each worker i, $X(i)$ denotes the union of the sets $X(i, j)$ over all firms j. For each agent k, $\bar{X}(k)$ denotes $X(k) \cup \{u_k\}$, the set of job descriptions possible for agent k including the possibility of remaining unmatched.

Firms have preferences over sets of employees and workers have preferences over sets of employers. (More precisely, firms and workers have preferences over sets of employee- and employer-specific job descriptions.) For each firm j, let $Y(j)$ be the set of all nonempty subsets of $X(j)$ that contain at most one element from any $X(i, j)$, and for each worker i, let $Y(i)$

be the set of all nonempty subsets of $X(i)$ that contain at most one element of any $X(i, j)$. Each agent k has a complete and transitive binary preference relation defined on $\bar{Y}(k) \equiv Y(k) \cup \{u_k\}$. (Note that this characterization of preferences implicitly assumes that workers are indifferent to which other workers their employers hire and that firms are indifferent to which other jobs their employees hold.) If a and b are elements of $\bar{Y}(k)$, then $a\, P_k\, b$ means that agent k strictly prefers a to b, and $a\, R_k\, b$ means that agent k weakly prefers a to b. It is assumed throughout that each agent k has preferences over the set $\bar{Y}(k)$ that are *strict* in the sense that $a\, R_k\, b\, R_k\, \{u_k\}$ implies that either $a = b$ or $a\, P_k\, b$; the consequences of relaxing this assumption are discussed below. For any agent k and any subset S of $\bar{X}(k)$, $C_k(S)$ is k's most preferred element of $\bar{Y}(k)$ that is a subset of S.

Whether matching is one-to-one, many-to-one, or many-to-many is formally a property of agents' preferences. For instance, if each agent k prefers $\{u_k\}$ to any element of $Y(k)$ that contains more than one job description, then matching is one-to-one.

An *outcome* in the market just described specifies which workers each firm employs, which firms employ each worker, the job description for each firm–worker pair, and which workers and firms remain unmatched. If E is the subset of employed workers at a given outcome and $\hat{E}$ is the subset of firms with at least one employee, then the assignment of workers to firms is described by a correspondence f from $E \cup \hat{E}$ to itself, where $f(j)$ denotes the set of workers firm j employs and $f(i)$ denotes the set of firms who employ worker i. The correspondence f takes E into $\hat{E}$ and $\hat{E}$ into E in such a way that $i \in f(j)$ if and only if $j \in f(i)$. A market outcome is given by a vector x^f, with one component for each agent, such that for each k in $E \cup \hat{E}$, x^f_k is the set of feasible job descriptions held by agent k, and for each k not in E or $\hat{E}$, $x^f_k = u_k$, indicating that agent k is unmatched. x^f_i includes a job description x_{ij} if and only if $j \in f(i)$, and vice versa, in which case x^f_j also includes x_{ij}. There is at most one element of $X(i, j)$ in any x^f_i or x^f_j.

The analysis requires two further assumptions, introduced by Kelso and Crawford [7] and adapted to the present model by Roth [9], that restrict how agents' preferences over job descriptions depend on which other job descriptions are also in their "portfolios." Both are vacuously satisfied for one-to-one matching.

Pareto-Separability. For each $i \in W$ and $j \in F$, the set $X(i, j)$ contains a Pareto-efficient subset $\hat{X}(i, j) \equiv \{\hat{x}^1_{ij}, ..., \hat{x}^p_{ij}\}$. If x^f_i and y^f_i differ only in the job description of worker i at job j, and $\hat{x}^q_{ij} \in x^f_i$ and $\hat{x}^r_{ij} \in y^f_i$, then firm j prefers y^f_j to x^f_j if and only if $q > r$.

Pareto-separability ensures that there is a well-defined Pareto-efficient subset of the set of job descriptions feasible for i and j, which is independent of which other job descriptions are in i's and j's portfolios. For an arbitrary Pareto-efficient job description $\hat{x}^s_{ij}$, s will be called the *generalized salary* associated with the job description, because it parametrizes the set of Pareto-efficient job descriptions in such a way that the worker prefers higher values of s and the firm prefers lower values of s.

Substitutability. Let $x^f_k = C_k(x^f_k \cup y^g_k)$. Then, for any $x_{ij} \in x^f_k$, $x_{ij} \in C_k(y^g_k \cup \{x_{ij}\})$.

Substitutability would be violated for a firm's preferences, for instance, if the firm would choose to hire a particular worker at a given job description as part of a given portfolio of job descriptions, but not if some of the rest of the portfolio were unavailable. Thinking of job descriptions in terms of their generalized salaries makes it clear that this notion is analogous to the concept of substitutability in consumer and producer theory.

An outcome x^f is *stable* if it is individually rational and if there exist no outcome y^g, set of workers $S \subset W$, and set of firms $\hat{S} \subset F$, with $S = g(\hat{S})$ and $\hat{S} = g(S)$, for which $y^g_k \subset C_k(x^f_k \cup y^g_k)\ P_k x^f_k$ for all agents $k \in S \cup \hat{S}$. Thus, if x^f is a stable outcome, there is no coalition of firms and workers that can propose among themselves a set of job descriptions, different from those assigned to them at x^f, that every agent in the coalition would include in his most preferred feasible set of job descriptions drawn from his new proposals together with those already assigned him. (The set of stable outcomes is thus closely related to the core of the market game; Roth and Sotomayor [11, Chap. 6] discuss this relationship in detail.) Note that a stable outcome x^f must have the property that $x^f_k = C_k(x^f_k)$ for each agent k, since otherwise it would be unstable via agent k and the empty set S (or $\hat{S}$) of agents on the other side of the market. An outcome x^f is *pairwise stable* if $x^f_k = C_k(x^f_k)$ for all agents k and x^f satisfies the definition of stability when the sets S and $\hat{S}$ are restricted to be singletons. Note that stability trivially implies pairwise stability, but not *vice versa.*

Kelso and Crawford [7, Theorem 1] showed, under the assumptions maintained here and in Roth [9], that there always exists a stable outcome when matching is many-to-one. Blair [1] then showed that those assumptions ensure the existence of a pairwise stable outcome when matching is many-to-many, but that a stable outcome may not exist in that case, even with substitutable preferences.[2]

Blair's nonexistence result poses an interesting challenge for research in cooperative game theory, but it need not stand in the way of using the

[2] The example used to make the latter point, which is reproduced in Roth and Sotomayor [11, Example 6.9], also shows, of course, that a pairwise stable outcome need not be stable when matching is many-to-many. Note that Blair's [1] and Roth and Sotomayor's [11] "stable" is equivalent to my "pairwise stable," and that Roth and Sotomayor's "group stable" is equivalent to my "stable."

theory of matching markets to describe the effects of market competition. In a labor market, for instance, it seems most realistic to allow firms to make agreements with workers or groups of workers, but not with other firms; and to allow workers to make agreements with firms (or perhaps groups of firms when matching is many-to-many), but not with other workers. Thus, under the restrictions on agents' preferences that define many-to-many matching, it seems best for my purposes to disallow only coalitions that include more than one member from each side.

Because such coalitions can sometimes improve upon outcomes that smaller coalitions cannot, the resulting notion is weaker than stability. For many-to-many matching, the proof of Roth and Sotomayor's [11, Proposition 6.4] shows that it is in fact equivalent to pairwise stability. For one-to-one and many-to-one matching, pairwise stability and stability are equivalent (in the former case trivially; in the latter by Roth and Sotomayor's [11, Proposition 6.4]), and both clearly correspond to the traditional idea of competition. Thus, in each case, the effects of competition can be characterized by requiring only pairwise stability. My results (and those in Roth [9, 10]) hold for stability (or pairwise stability) when matching is many-to-one, but only for pairwise stability when matching is many-to-many. This qualification is expressed below by using "(pairwise) stable" as a shorthand for "pairwise stable, and stable if matching is many-to-one."

3. Comparative Statics

This section states and proves the comparative statics results. The proofs make use of the following algorithm from Roth [9], which is a version of Gale and Shapley's [6] deferred-acceptance procedure, descended from the versions of Crawford and Knoer [2] and Kelso and Crawford [7].

Algorithm. *Step* 1. (a) Each firm j proposes its most preferred set $x_j(1)$ in $\bar{Y}(j)$. (b) Each worker i accepts his choice set from u_i and the set $x_i(1) \equiv \{x_{ij} \mid x_{ij} \in x_j(1) \text{ for some } j\}$ of alternatives proposed to him, and rejects the rest.

⋮

Step k. (a) Each firm j proposes its most preferred set $x_j(k)$ in $\bar{Y}(j)$ with the property that no $x_{ij} \in x_j(k)$ has been rejected at an earlier step. (b) Each worker i accepts his choice set from the set of alternatives not yet rejected by step $k-1$ together with those proposed at step k (his *portfolio* in step k), and rejects the rest.

The algorithm terminates at the first step, t, in which no rejections are issued, and results in the outcome x^f that assigns to each worker i his current choice set $x_i^f = C_i(x_i(t) \cup \{u_i\})$ and to each firm j its latest proposal $x_j^f = x_j(t)$.

Roth [9, Theorem 2] showed, for the model just described, that this algorithm converges to a (pairwise) stable outcome that is *firm-optimal* in the sense that each firm weakly prefers it to any other (pairwise) stable outcome. Because Roth treated firms and workers symmetrically, his result implies that the analogous algorithm with firms' and workers' roles reversed converges to a (pairwise) stable outcome that is *worker-optimal* in an analogous sense. (By contrast, Kelso and Crawford [7, Theorem 4] treated firms and workers asymmetrically by restricting workers to at most one job, and showed only that their version of the algorithm converges to a firm-optimal stable outcome.)

In what follows, the matching market with sets of firms and workers F and W and preferences P_k for agent k is denoted $(F, W; P)$, where P is the vector whose kth component is P_k. The expression $c\ R_F\ d$ means that each firm in F weakly prefers outcome c to outcome d; and $c\ R_W\ d$ means that each worker in W weakly prefers c to d.

THEOREM 1. *Suppose that W is contained in $\hat{W}$, that μ_F is the firm-optimal outcome for the market $(F, W; P)$, and that $\hat{\mu}_F$ is the firm-optimal outcome for $(F, \hat{W}; \hat{P})$, where $\hat{P}$ agrees with P on F and W. Then $\mu_F\ R_W\ \hat{\mu}_F$ under P and $\hat{\mu}_F\ R_F\ \mu_F$ under $\hat{P}$.*

Remarks. Theorem 1 says that adding one or more workers to the market makes the firm-optimal (pairwise) stable outcome weakly worse for all workers already in the market and weakly better for all firms. Only weak preferences can be established because, even though agents' preferences are strict, μ_F and $\hat{\mu}_F$ may be identical in how they treat some of the agents. By virtue of the symmetry of the model, the theorem also shows that adding firms makes the worker-optimal (pairwise) stable outcome weakly worse for all firms already in the market, and weakly better for all workers.

Proof. I shall use the term *offer* to refer to a proposed job description; recall that these are specific to firm-worker pairs. The strategy of the proof is to show that any worker in W who ever rejects an offer when the algorithm is applied to the market $(F, \hat{W}; \hat{P})$ also rejects that offer (at some point) in the market $(F, W; P)$. It follows that, because firms begin with their most preferred sets of offers and work "down," they all weakly prefer the outcome generated by the algorithm in $(F, \hat{W}; \hat{P})$ to that generated in $(F, W; P)$. Further, because workers lose offers only by rejecting them, and reject offers only in favor of better offers or combinations of offers, all workers in W weakly prefer the outcome generated in $(F, W; P)$ to that generated in $(F, \hat{W}; \hat{P})$. The desired conclusion then follows immediately from Roth [9, Theorem 2].

The proof proceeds by induction on the hypothesis: If a worker $w \in W$ rejects an offer x_{wh} by step k of the algorithm in the market $(F, \hat{W}; \hat{P})$, then he also rejects x_{wh} by step k in the market $(F, W; P)$.[3]

"$k = 1$". By the rules of the algorithm, worker $w \in W$ rejects offer x_{wh} in step 1 in $(F, \hat{W}; \hat{P})$ only if his portfolio of offers also includes an offer or combination of offers he prefers to any combination that includes x_{wh}. Because firms' preferences are substitutable and W is contained in $\hat{W}$, worker w must receive (at least) the same offers in step 1 in $(F, W; P)$ as in $(F, \hat{W}; \hat{P})$. Thus, because w's preferences are substitutable, he also rejects x_{wh} in $(F, W; P)$.

"k" implies "$k+1$". To complete the proof, it is necessary to show that, given the induction hypothesis, if worker $w \in W$ rejects offer x_{wh} by step $k+1$ in $(F, \hat{W}; \hat{P})$, he also rejects x_{wh} by step $k+1$ in $(F, W; P)$. Worker w can reject offer x_{wh} by step $k+1$ in $(F, \hat{W}; \hat{P})$ only if by then he has received both x_{wh} and offers that include a combination he prefers to any combination that includes x_{wh}. For w to have received these offers by step $k+1$ in $(F, \hat{W}; \hat{P})$, at least one offer in each proposal preferred by the firms who made them must have been rejected by step k. By the induction hypothesis, those offers must also have been rejected by step k in $(F, W; P)$. It then follows from the substitutability of firms' preferences that any offer worker w has received by step $k+1$ in $(F, \hat{W}; \hat{P})$, he has also received by step $k+1$ in $(F, W; P)$. The substitutability of w's preferences then implies (via Roth [9, Proposition 3], which shows that a worker with substitutable preferences never wishes to recall rejected offers) that w rejects x_{wh} by step $k+1$ in $(F, W; P)$ as well. ∎

THEOREM 2. *Suppose that F is contained in $\bar{F}$, that μ_F is the firm-optimal outcome for the market $(F, W; P)$, and that $\bar{\mu}_F$ is the firm-optimal outcome for $(\bar{F}, W; \bar{P})$, where $\bar{P}$ agrees with P on F and W. Then $\bar{\mu}_F \, R_W \, \mu_F$ under $\bar{P}$ and $\mu_F \, R_F \, \bar{\mu}_F$ under P.*

Remarks. Theorem 2 completes the chiasm, showing that adding firms makes the firm-optimal (pairwise) stable outcome weakly better for all workers and weakly worse for all firms already in the market. The theorem also shows, by symmetry, that adding workers makes the worker-optimal

[3] This formulation applies unmodified to cases where the algorithm halts before step k in one market but not in the other.

(pairwise) stable outcome weakly better for all firms and weakly worse for all workers already in the market.

Proof. The proof is similar but not identical to the proof of Theorem 1. It proceeds by induction on the hypothesis: If a worker $w \in W$ rejects an offer x_{wh} by step k of the algorithm in the market $(F, W; P)$, then he also rejects x_{wh} by step k in the market $(\bar{F}, W; \bar{P})$.

"$k = 1$". By the rules of the algorithm, worker $w \in W$ rejects offer x_{wh} in step 1 in $(F, W; P)$ only if his portfolio of offers also includes an offer or combination of offers he prefers to any combination that includes x_{wh}. Because F is contained in $\bar{F}$, worker w must receive (at least) the same offers in step 1 in $(\bar{F}, W; \bar{P})$ as in $(F, W; P)$. Because w's preferences are substitutable, he therefore also rejects x_{wh} in $(\bar{F}, W; \bar{P})$.

"k" implies "$k+1$". To complete the proof, it is necessary to show that, given the induction hypothesis, if worker $w \in W$ rejects offer x_{wh} by step $k+1$ in $(F, W; P)$, he also rejects x_{wh} by step $k+1$ in $(\bar{F}, W; \bar{P})$. Worker w can reject offer x_{wh} by step $k+1$ in $(F, W; P)$ only if by then he has received both x_{wh} and offers that include a combination he prefers to any combination that includes x_{wh}. For w to have received these offers by step $k+1$ in $(F, W; P)$, at least one offer in each proposal preferred by the firms who made them must have been rejected by step k. By the induction hypothesis, those offers must also have been rejected by step k in $(\bar{F}, W; \bar{P})$. It then follows from the substitutability of firms' preferences that any offer worker w has received by step $k+1$ in $(F, W; P)$, he has also received by step $k+1$ in $(\bar{F}, W; \bar{P})$. The substitutability of w's preferences then implies (via Roth [9, Proposition 3]) that w rejects x_{wh} by step $k+1$ in $(\bar{F}, W; \bar{P})$ as well. ▌

The conclusions of Theorems 1 and 2, as stated, depend on the assumption that agents have strict preferences; without this assumption, firm- and worker-optimal (pairwise) stable outcomes may not exist (see Roth and Sotomayor [11, Example 2.15]). Theorems 1 and 2 can be extended to allow non-strict preferences, following Roth and Sotomayor [11, Chap. 2], by imagining that there is a tie-breaking rule that describes how agents choose when confronted with alternatives between which they are indifferent. Such a rule effectively converts non-strict preferences into strict preferences; it follows from Roth [9, Theorem 2] that when agents obey it, the algorithm with firms proposing converges to an outcome that is firm-optimal for those strict preferences. This outcome might not be firm-optimal for the original preferences, because an outcome can be (pairwise) stable for them but not for the strict preferences. But the conclusions of Theorems 1 and 2 continue to hold when the firm-optimal (pairwise) stable outcomes are taken to be those associated with the strict preferences in their respective markets: Otherwise, the theorems would fail if those strict preferences were the true preferences.

It is likely that Theorems 1 and 2 (and Roth's [9] results as well) can also be extended to *continuous* matching markets, in which agents can compensate each other for matching with a perfectly divisible, desirable good, along the lines suggested by the proofs of Crawford and Knoer [2, Theorem 2], Kelso and Crawford [7, Theorem 2], and Demange, Gale, and Sotomayor [5, Theorems 3 and 4]. If the conclusions of Theorems 1 and 2 did not hold for the continuous analogs of the discrete matching markets studied here, they would presumably also fail to hold for sufficiently "fine" discrete markets, contradicting the theorems. Because the difference between discrete and continuous markets is of little practical importance, I do not pursue this line of investigation here.

4. Related Work and Open Questions

Kelso and Crawford [7, Theorem 5] were the first to prove comparative statics results about the effects of adding agents to matching markets. Their model allowed many-to-one matching, and generalized Gale and Shapley's [6] marriage and college-admissions markets by letting agents compensate each other for matching and allowing substitutable preferences over sets of potential partners.[4] Kelso and Crawford showed that (when it is the firms that are allowed multiple partners) adding one or more firms to the market makes the firm-optimal stable outcome weakly better for all workers, and adding one or more workers makes the firm-optimal stable outcome weakly better for all firms. This paper's results generalize Kelso and Crawford's by determining the welfare effects for many-to-one matching for both sides of the market, and for the worker-optimal stable outcome as well as the firm-optimal stable outcome; and by extending these results, for the firm- and worker-optimal pairwise stable outcomes, to many-to-many matching.

The first hints that comparative statics results of this kind might be available came much earlier. Shapley [12] showed that in a linear optimal-assignment problem, the marginal product of an agent on one side of the

[4] Allowing compensation for matching corresponds to allowing variable job descriptions in the present model; although the latter formulation appears more general, the two turn out to be essentially equivalent. As noted above, substitutability is vacuously satisfied when agents are allowed at most one partner. It is significantly more general than the separability (Roth and Sotomayor's [11, Chap. 5] "responsiveness") assumed in Gale and Shapley's [6] college-admissions analysis; see Kelso and Crawford [7, Sect. 6].

associated matching market (defined as the maximized value with the agent minus the maximized value without him) weakly decreases when another agent is added to that side and weakly increases when an agent is added to the other side. Shapley and Shubik [13] then showed that solving the dual of a linear optimal-assignment problem yields a stable outcome in the associated matching market. Demange [3] and Leonard [8] (see also Roth and Sotomayor [11, Lemma 8.15]) then showed, under the same assumptions, that in a stable outcome that is optimal for the agents on one side of the market those agents are, in effect, paid their marginal products as Shapley [12] defined them. Combining these results yields comparative statics results for continuous matching markets like those established here, under the strong assumptions that matching is one-to-one and the perfectly divisible good with which agents can compensate each other for matching enters their preferences linearly.

It is likely that this line of argument can be extended to allow many-to-one matching and even many-to-many matching, as long as agents' preferences remain separable across prospective partners and linear in money. With nonlinear, substitutable preferences, the market no longer solves an optimization problem in general, and the analysis becomes correspondingly more difficult. Demange and Gale [4, Corollary 3] obtained results like those established here for one-to-one matching in continuous markets with nonlinear preferences. The argument sketched at the end of Section 3 suggests that it may be possible to generalize their results, in their framework, to allow many-to-one and even many-to-many matching with substitutable preferences.

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