COMPARATIVE STATICS OF INCREASING RISK (DIAMOND-STIGLITZ)

Consider an individual who chooses a control variable α to solve

$$\max_\alpha \int u(x,\alpha)\,dF(x;r) \equiv \int u(x,\alpha)\,f(x;r)\,dx,$$

where $F(\cdot)$ has a density shifted by the parameter r.
For example, x might be the return on a risky asset (investment in training for a job) and α the amount invested in it (level of training effort) (with the rest invested in a safe asset, and $u(\cdot)$ a kind of indirect utility), with r indexing the riskiness of the asset's return.

Assume the support of x is $[0,m]$ as before, so that $F(0;r) \overset{r}{\equiv} 0$ and $F(m;r) \overset{r}{\equiv} 1$. The first-order condition is $\int u_\alpha(x,\alpha)\,dF(x;r) \equiv \int u_\alpha(x,\alpha) f(x;r)\,dx = 0$.

Assume $u_{\alpha\alpha}(x,\alpha) < 0$, so the second-order condition $\int u_{\alpha\alpha}(x,\alpha) f(x;r)\,dx < 0$ is always satisfied, strictly.

By the Implicit Function Theorem, letting $\alpha(r)$ be the optimal α,

$$\frac{d\alpha(r)}{dr} = \frac{-\int u_\alpha(x,\alpha) f_r(x;r)\,dx}{\int u_{\alpha\alpha}(x,\alpha) f(x;r)\,dx}, \text{ so that}$$

[recall $F_x(x;r) \overset{x,r}{\equiv} f(x;r)$ so $F_{xr}(x;r) \overset{x,r}{\equiv} f_r(x;r)$]

$$\operatorname{sign}\frac{d\alpha(r)}{dr} = \operatorname{sign}\int \underbrace{u_\alpha(x,\alpha)}_{\text{"}u\text{"}}\,\underbrace{f_r(x;r)\,dx}_{\text{"}dv\text{"}}.$$

[recall $F_{rx}(x;r) \equiv f_r(x;r)$]

$$= \operatorname{sign}\left[\left\{\underbrace{u_\alpha(x,\alpha)}_{\text{"}u\text{"}}\,\underbrace{F_r(x;r)}_{\text{"}v\text{"}}\right\}_0^m - \int_0^m \underbrace{F_r(x;r)}_{\text{"}v\text{"}}\,\underbrace{u_{\alpha x}(x,\alpha)\,dx}_{\text{"}du\text{"}}\right]$$

$$= -\operatorname{sign}\int_0^m F_r(x;r)\,u_{\alpha x}(x,\alpha)\,dx \quad \text{since } F_r(m;r) = F_r(0;r) = 0.$$

Now a small increase in r induces an FOSD increase in $F(\cdot)$ iff $F_r(x;r) \le 0$, $\forall x, r$ (from the "finite" change condition (E2) and the Mean Value Theorem), thus $\frac{d\alpha(r)}{dr} \ge 0$ for all such changes iff $u_{\alpha x}(x,\alpha) \ge 0$, $\forall x, \alpha$, i.e. iff deterministic increases in x always shift the "marginal product" of α upward.

There's a "dual" result, which says if the problem is viewed as choosing r with α a parameter (required investment), the sign of $\frac{dr(\alpha)}{d\alpha}$ is the same as that of $\frac{d\alpha(r)}{dr}$.

Integrating by parts again we obtain

$$-\int_0^m \underbrace{u_{\alpha x}(x,\alpha)}_{\text{"}u\text{"}}\,\underbrace{F_r(x;r)\,dx}_{\text{"}dv\text{"}} = -\Big[\underbrace{u_{\alpha x}(x,\alpha)}_{\text{"}u\text{"}}\underbrace{\int_0^x F_r(s;r)ds}_{\text{"}v\text{"}}\Big]_0^m$$

$$+\int_0^m \underbrace{\Big[\int_0^x F_r(s;r)ds\Big]}_{\text{"}v\text{"}}\,\underbrace{u_{\alpha xx}(x,\alpha)\,dx}_{\text{"}du\text{"}}$$

$= \int_0^m \Big[\int_0^x F_r(s;r)ds\Big] u_{\alpha xx}(x,\alpha)\,dx$ because $\int_0^m F_r(s;r)ds = 0$ by the mean-preserving part of (C′). Now $\int_0^x F_r(s;r)ds \geq 0$, $\forall x$, by the spread part of (C′), so $\frac{d\alpha(r)}{dr} \geq 0$ for increases in r that induce mean-preserving spreads in $F(\cdot)$ for all $u(\cdot)$ iff $u_{\alpha xx}(\cdot) \geq 0$, $\forall \alpha, x$. (Same for $\frac{dr(\alpha)}{d\alpha}$ by "duality.") (Note use of the Mean Value theorem again to translate finite-change condition (C′) into calculus condition. Note that we used same expression to sign effects of changes in r that do different things: FOSD and MPS; this is weird but OK, because the formula is an identity and we never use both interpretations at the same time.)

Intuition: If $u_{\alpha xx}(\cdot) \geq 0$, $\forall x, \alpha$, raising α raises $u_{xx}(\cdot)$, making the person less risk-averse (yes, less!). Therefore the optimal response to a change in r that increases the riskiness of $F(\cdot)$, the distribution of x, is to move the control variable, α, in the direction that makes you less risk-averse. (Note that most of our intuition about the effects of changes in $F(\cdot)$ are first-order, as captured in the first result (on FOSD) above; the mean-preserving spread eliminates these effects by holding the mean of $F(\cdot)$ constant, and is entirely second-order, hence more subtle.

The two effects can be combined using the formula above, as in the second-order stochastic dominance characterization. Diamond-Stiglitz extend this to "Hicksian" compensated changes, holding Eu constant.