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ECON 200 C
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Arrow-Pratt Characterization of Comparative Risk Aversion

Theorem: The following four conditions on a pair of (increasing, twice differentiable) VN-M utility functions, $u_a(\cdot)$ and $u_b(\cdot)$, are equivalent:

(a1) $u_a(\cdot)$ is a concave transformation of $u_b(\cdot)$, i.e. $u_a(x) \equiv \rho(u_b(x))$ for some (necessarily increasing) concave function $\rho(\cdot)$.

(a2) The Arrow-Pratt coefficients of absolute risk aversion satisfy the inequality

$$-\frac{u_a''(x)}{u_a'(x)} \geq -\frac{u_b''(x)}{u_b'(x)} \quad \text{for all } x.$$

(a3) If c_a and c_b are such that $u_a(c_a) = E_F\, u_a(x)$ and $u_b(c_b) = E_F\, u_b(x)$ for some distribution $F(\cdot)$, then $c_a \leq c_b$.

(a4) Suppose that $u_a(\cdot)$ and $u_b(\cdot)$ are concave. Then, if r is known and >0, x is uncertain with $E_F x > r$ and $\text{Prob}(x < r) > 0$, and α_a and α_b respectively solve $\max\limits_{0 \leq \alpha \leq I} E_F\, u_a((I-\alpha)r + \alpha x)$ and $\max\limits_{0 \leq \alpha \leq I} E_F\, u_b((I-\alpha)r + \alpha x)$, $\alpha_a \leq \alpha_b$.

Remarks: 1. It is the equivalence of the mathematical conditions (a1) and (a2) to the behavioral conditions (a3) and (a4) (and their respective equivalences to each other) that makes (a1) and (a2) interesting.

2. Risk aversion is plainly related to concavity, hence to $u_a''(\cdot)$ and $u_b''(\cdot)$. But these, unlike the Arrow-Pratt coefficients $-\frac{u_a''(\cdot)}{u_a'(\cdot)}$ and $-\frac{u_b''(\cdot)}{u_b'(\cdot)}$, are not invariant to increasing linear transformations, and therefore cannot be linked as closely to the behavioral conditions (a3) and (a4) as the theorem's conclusion requires.

3. The conclusions of (a3) and (a4), about certainty equivalents and absolute levels of investment in safe vs. risky assets, are both intuitively equivalent to "$u_a(\cdot)$ is more risk-averse than $u_b(\cdot)$." But it is somewhat surprising that they are equivalent to each other (and it is *not* true for many plausible generalizations, e.g. if initial wealth is uncertain).

4. The "extra" conditions in (a4) ensure that the investor who invests in the safe asset can't lose all his wealth, and that the risky asset has higher expected return than the safe asset but some chance of yielding less. Otherwise, either the risky asset or the safe asset would be unambiguously better, the problems would have corner solutions (even with concave $u_a(\cdot)$ and $u_b(\cdot)$), and these solutions could not be as sensitive to the forms of $u_a(\cdot)$ and $u_b(\cdot)$ as the theorem's conclusion requires. (a4) is concerned with absolute amounts invested, which is why the coefficients that appear in (a2) are called "absolute."

5. (a1) and (a2) clearly define *partial* orderings of the set of (increasing, concave) vN-M utility functions, which is all you should expect, given their equivalence to the strong behavioral conditions (a3) and (a4), whose conclusions hold independent of $F(\cdot)$.

6. Weak inequality statements give strict inequality and equality statements in usual way.

The Arrow-Pratt theorem can be interpreted in several different ways, with $u_a(\cdot)$ and $u_b(\cdot)$ viewed as the vN-M utility functions of different individuals, or of the same individual at different levels of initial wealth, and with risks expressed in absolute or proportional terms. The first two of these interpretations yield especially useful corollaries, stated here before proving the theorem:

Corollary ("Decreasing Absolute Risk Aversion"):

The following four conditions on an (increasing, twice differentiable) vN-M utility function, denoted $u(\cdot)$, are equivalent:

(b1) $u(Ir + x)$, viewed as a function of x, is a concave transformation of $u(I^*r + x)$ whenever $I^*r > Ir$.

(b2) $$-\frac{u''(Ir+x)}{u'(Ir+x)} \geq -\frac{u''(I^*r+x)}{u'(I^*r+x)} \text{ for all } x,$$
whenever $I^*r > Ir$.

(b3) If c and c^* are such that $u(Ir+c) = E_F u(Ir+x)$ and $u(I^*r + c^*) = E_F u(I^*r + x)$ for some distribution $F(\cdot)$, then $c \leq c^*$ whenever $I^*r > Ir$.

(b4) Suppose that $u(\cdot)$ is concave. Then, if r is known and >0, $\tilde{x}$ is uncertain with $E_F x > r$ and Prob. $(x < r) > 0$, and $\alpha(I)$ solves
$$\max_{0 \leq \alpha \leq I} E_F u(Ir + \alpha(x - r)), \quad \alpha(I) \leq \alpha(I^*)$$
whenever $I^*r > Ir$.

Remarks: 1. Strange $u(Ir+x)$ formulation is to simplify statement of (b4) and clarify its relation to (a4); note that $Ir + \alpha(x-r) \equiv (I-\alpha)r + \alpha x$, and that for $r > 0$ (as is natural), $I^*r > Ir$ iff $I^* > I$.
2. Given the definition of "absolutely more risk averse than" implicit in the Arrow-Pratt Theorem, (b1) - (b4) can be viewed as equivalent translations of "absolute risk aversion is decreasing in initial wealth." The Corollary can be run "backwards" to give an analogous characterization of increasing absolute risk aversion.

Proof (given Arrow-Pratt Theorem): Let $u(Ir+x) \overset{x}{\equiv} u_a(x)$, $u(I^*r+x) \equiv u_b(x)$, $\alpha(I) = \alpha_a$, $\alpha(I^*) = \alpha_b$ and apply the Theorem.

Corollary ("Decreasing Relative Risk Aversion"):

The following four conditions on an (increasing, twice differentiable) vN-M utility function, denoted $u(\cdot)$, are equivalent:

(C1) $u(I\epsilon)$, viewed as a function of ϵ, is a concave transformation of $u(I^*\epsilon)$ whenever $I < I^*$.

(C2)
$$\frac{-I\epsilon\, u''(I\epsilon)}{u'(I\epsilon)} \geq \frac{-I^*\epsilon\, u''(I^*\epsilon)}{u'(I^*\epsilon)} \text{ for}$$
all ϵ, whenever $I < I^*$. (These are the Arrow-Pratt coefficients of relative risk aversion for wealth levels I and I^*, so-called because of the equivalence of (C2) and (C4) below.)

(C3) If e and e^* are such that $u(Ie) = E_F\, u(I\epsilon)$ and $u(I^*e^*) = E_F\, u(I^*\epsilon)$ for some distribution $F(\cdot)$, then $e \leq e^*$ whenever $I \leq I^*$.

(C4) Suppose that $u(\cdot)$ is concave. Then, if γ is known and > 0, ξ is uncertain with $E\xi > \gamma$ and $\mathrm{Prob}(\xi < \gamma) > 0$, and $\beta(I)$ solves $\max_{0 \leq \beta} E\,u((1-\beta)I\gamma + \beta I \xi)$, then $\beta(I) \leq \beta(I^*)$ whenever $I < I^*$.

Proof (given Arrow-Pratt Theorem): Let $u(I\epsilon) \overset{\epsilon}{\equiv} u_a(\epsilon)$, $u(I^*\epsilon) \overset{\epsilon}{\equiv} u_b(\epsilon)$, note that β in (C4) plays a role analogous to that of α in (a4) (because $(1-\beta)I\gamma + \beta I \xi \equiv I(\gamma + \beta(\xi - \gamma))$, which is linear in β with mean increasing in β, just as $(I-\alpha)r + \alpha x = Ir + \alpha(x - r)$ is in α), and apply the Theorem again. Note, in particular, that if $u_a(\epsilon) \overset{\epsilon}{\equiv} u(I\epsilon)$, then
$$-\frac{u_a''(\epsilon)}{u_a'(\epsilon)} \equiv -\frac{I^2 u''(I\epsilon)}{I u'(I\epsilon)} \equiv \frac{-I u''(I\epsilon)}{u'(I\epsilon)},$$
hence (C2) (cancelling the ϵ's from both sides, with $\epsilon > 0$).

Remarks: 1. Note that (C4) deals with proportions invested in safe and risky assets, whereas (a4) and (b4) deal with absolute amounts.
2. Can run "backwards" to characterize increasing relative risk aversion.

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Proof of Arrow-Pratt Theorem:

We need a complete set of implication arrows linking (a1)-(a4). Given transitivity, this can be built up out of 1. (a1) $\Leftrightarrow$ (a2), 2. (a1) $\Leftrightarrow$ (a3), and 3. (a2) $\Leftrightarrow$ (a4). I shall prove 1, 2, and 3′ (b2) $\Leftrightarrow$ (b4), which is equivalent to 3 (a2) $\Leftrightarrow$ (a4), less tedious, and just as illuminating.

1. (a1) $\Leftrightarrow$ (a2). $u_a(x) \equiv \rho(u_b(x)) \Longrightarrow$

$$u_a'(x) \equiv \rho'(u_b(x))\, u_b'(x),$$

$$u_a''(x) \equiv \rho'(\cdot)\, u_b''(x) + \rho''(\cdot)\,(u_b'(x))^2. \text{ Thus}$$

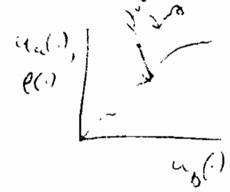
$$\frac{-u_a''(x)}{u_a'(x)} \equiv \frac{-\rho'(\cdot)\, u_b''(x) - \rho''(\cdot)\,(u_b'(x))^2}{\rho'(\cdot)\, u_b'(x)}$$

$$\equiv -\frac{u_b''(x)}{u_b'(x)} - \frac{\rho''(\cdot)}{\rho'(\cdot)}\, u_b'(x) \geq -\frac{u_b''(x)}{u_b'(x)}, \text{ because}$$

$\rho(\cdot)$ is increasing and concave.

This proves (a1) $\Rightarrow$ (a2). To prove (a1) $\Leftarrow$ (a2), ~~note that~~ since $u_a(\cdot)$ and $u_b(\cdot)$ are both increasing functions of one variable, they can always be related, as in (a1), by an increasing transformation $\rho(\cdot)$. (a2) then shows that this $\rho(\cdot)$ must be concave.

Since both differentiable, if $u_a'(\cdot), u_b'(\cdot) > 0$ everywhere, then $\rho(\cdot)$ differentiable



2. (a1) $\Leftrightarrow$ (a3) depends on an important lemma known as Jensen's inequality, which is true "in general," but proven here assuming twice differentiability:

<u>Lemma:</u> If $f(y)$ is a concave function of one variable, then $E f(y) \leq f(Ey)$.

<u>Proof:</u> By Taylor's Theorem,

$$f(y) = f(Ey) + (y - Ey) f'(Ey) + \frac{(y-Ey)^2}{2} f''(z)$$

for some z between y and Ey. Since $f(\cdot)$ is concave, $f''(y) \le 0$ everywhere, hence at $y = z$. Taking expectations then yields

$$Ef(y) = Ef(Ey) + E[(y-Ey)f'(Ey)] + E\left[\frac{(y-Ey)^2}{2} f''(z)\right]$$

$$= f(Ey) + E\left[\frac{(y-Ey)^2}{2} f''(z)\right] \le f(Ey), \text{ as desired.}$$

Now, by Jensen's Inequality and (a1), for any $F(\cdot)$,

$$u_a(c_a) = E_F u_a(x) = E_F \rho(u_b(x)) \le \rho(E_F u_b(x))$$

$= \rho(u_b(c_b)) = u_a(c_b)$; since $u_a(\cdot)$ is increasing, this implies that $c_a \le c_b$ as desired.

This proves that (a1) $\Rightarrow$ (a3). To prove that (a1) $\Leftarrow$ (a3), note that $c_a \le c_b$ for any $F(\cdot)$, and that $u_a(\cdot)$ and $u_b(\cdot)$ can always be related by an increasing transformation $\rho(\cdot)$ as in (a1). If this transformation were not concave in some interval, picking an $F(\cdot)$ concentrated in that interval and using Jensen's Inequality on $-\rho(\cdot)$ would yield a contradiction to (a1) $\Rightarrow$ (a3). (Here, we need a slightly strengthened version of the Inequality.)

3. (b2) $\Leftrightarrow$ (b4). Note first that $D_\alpha E_F u(Ir + \alpha(x-r))$

$$\equiv E_F[u'(Ir+\alpha(x-r))(x-r)]\Big|_{\alpha=0} = u'(Ir)E_F(x-r) > 0,$$

an instance of the "favorable-bet theorem" (often called "fair-bet theorem", but should then be "more-than-fair bet theorem").

In general, the first-order condition for the problem in (b4) can be written as

$$E_F[x u'(Ir+\alpha(x-r))] - r E_F u'(Ir+\alpha(x-r)) = 0.$$

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The second-order (sufficient) condition is

$$E_F\left[(x-r)^2 u''(Ir+\alpha(x-r))\right] < 0,$$

satisfied everywhere for any risk-averter. Totally differentiating the first-order condition with respect to I (as justified by the Implicit Function Theorem) yields, denoting the solution of $\max\limits_{0\le\alpha\le I} E_F\, u(Ir+\alpha(x-r))$ $\alpha(I)$,

$$\frac{d\alpha(I)}{dI} \equiv \frac{-r\,E_F\left[(x-r)\,u''(Ir+\alpha(x-r))\right]}{E_F\left[(x-r)^2 u''(Ir+\alpha(x-r))\right]}.$$

Given that $r>0$ and the second-order condition is satisfied, $\frac{d\alpha(I)}{dI}$ has the same sign as $E_F\left[(x-r)\,u''(Ir+\alpha(x-r))\right]$. Let $R_a(y) \triangleq \frac{-u''(y)}{u'(y)}$.

Then $E_F\left[(x-r)\,u''(Ir+\alpha(x-r))\right]$

$$\equiv E_F\left[(x-r)\left(-R_a(Ir+\alpha(x-r))\right)u'(Ir+\alpha(x-r))\right] + R_a(Ir)\,E_F\left[(x-r)\,u'(Ir+\alpha(x-r))\right]$$

(because the latter term equals zero by the first-order condition)

$$= -E_F\left[(x-r)\left[R_a(Ir+\alpha(x-r))-R_a(Ir)\right]u'(Ir+\alpha(x-r))\right].$$

If $R_a'(y)<0$, given that $\alpha\geq 0$, this expectation (with the minus sign that precedes it) is positive, so $\frac{d\alpha(I)}{dI}>0$. If $R_a'(y)>0$, $\frac{d\alpha(I)}{dI}<0$. Note ~~strictly~~ necessary & sufficient if $R_a' \neq 0$ (by assumption)

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Thus, the "Engel curves" in the asset-demand-as-consumer-choice analogy slope upward (so that risky investment is "normal") iff the individual's Arrow-Pratt measure of absolute risk aversion is decreasing in wealth. Partly for this reason, this is a widely accepted hypothesis about individual preferences under uncertainty.

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Some Particular Utility Functions

Following are some commonly-employed, one-argument, von Neumann-Morgenstern utility functions, along with their risk-aversion properties:

Quadratic: $u(x) = a + bx + cx^2$

$u' = b + 2cx$.

$u'' = 2c$; so c must be negative for risk-aversion, implying u' negative for $x > -b/2c$. Thus, b must be positive for u' positive for small positive x.

$R_a = (-2c)/(b+2cx)$; increasing.

$R_r = (-2cx)/(b+2cx)$; increasing.

Bernoulli: $u(x) = a + b \log(x)$

$u' = b/x$; so b should be positive.

$u'' = -b/x^2$.

$R_a = 1/x$; decreasing

$R_r = 1$; constant.

Constant Elasticity: $u(x) = a + b(1-c)x^{(1-c)}$ $(c \neq 1)$

$u' = b(1-c)^2x^{-c}$; so b should be positive.

$u'' = -bc(1-c)^2x^{-c-1}$. Thus, c should be positive for risk aversion.

$R_a = c/x$; decreasing.

$R_r = c$; constant.

Exponential: $u(x) = a - be^{-cx}$ (note that this function is bounded)

$u' = bce^{-cx}$; so (bc) should be positive.

$u'' = -bc^2e^{-cx}$; so both b and c should be positive for risk aversion.

$R_a = c$; constant.

$R_r = cx$; increasing.

Note that the Bernoulli and constant elasticity functions are the only ones with constant relative risk-aversion, while the exponential is the only one with constant absolute risk-aversion.