Economics 206 Problem Set 1 **Winter 2007**
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This problem set, which is optional, covers the material in the first half of the course, roughly in the order in which topics are discussed in the lectures. The problems are meant to help you think about some of the issues we discuss in lectures, and also as practice for the first take-home mini-exam: at least two of its three questions will be selected from the questions below. Although once the mini-exam is posted (by 4 p.m. Wednesday, February 7), you must work on it individually, without consulting anyone but me, before it is posted you are encouraged to work together and/or consult me as much as you wish.

1. You are risk neutral, and care only about your income. With probability p, you will catch a disease that reduces your income from y, its level when you are healthy, to y - k, where k > 0. A vaccine is available, at cost c, that reduces the probability of your catching the disease from p to q < p.

(a) Suppose that you know the values of p, q, y, k, and c, so that the only thing about which you are uncertain is whether you will catch the disease. Write the condition that determines whether or not you should buy the vaccine.

(b) Now suppose that you know y, k, and c, but neither p nor q. Which is more relevant to your decision, the percentage amount by which the vaccine reduces the probability of catching the disease (what is usually reported in the press), or the absolute amount? Explain.

(c) How do your answers to (a) and (b) change if you are a risk-averse expected-utility maximizer?

2. (Attributed to Richard Zeckhauser) In Russian Roulette, a number of bullets are loaded into a revolver with six chambers; an individual then points the revolver at his head, pulls the trigger, and is killed if and (hopefully) only if the revolver goes off. Assume that the individual must play this game; that he is an expected-utility maximizer; and that each chamber is equally likely to be in firing position, so that if the number of bullets is b his probability of being killed is b/6. Suppose further that the maximum amount he is willing to pay to have one bullet removed from a gun initially containing only one bullet is \$x, and the maximum amount he is willing to pay to have one bullet removed from a gun initially containing 4 bullets is \$y, where x and y are both finite. Finally, suppose that he prefers more money to less and that he prefers life (even after paying \$x or \$y) to death. Let U_D denote his von Neumann-Morgenstern utility when dead, which is assumed to be independent of how much he paid (as suggested by empirical studies of the demand for money); and let U_{A0}, U_{Ax}, and U_{Ay} denote his von Neumann-Morgenstern utilities when alive after paying \$0, \$x, or \$y respectively.

(a) What restrictions are placed on U_D, U_{A0}, U_{Ax}, and U_{Ay} by the assumption that he prefers more money to less when alive?

(b) What restrictions are placed on U_D, U_{A0}, U_{Ax}, and U_{Ay} by the assumption that he prefers life (even after paying $x or $y) to death?

(c) Is it possible to tell from the information given above whether $x > y$ for an expected utility maximizer? Does it matter whether he is risk-averse? Explain.

3. Suppose that an individual must choose among distributions some of which are unbounded (the sets of outcomes to which they assign positive probability are unbounded), and that the von Neumann—Morgenstern utilities that rationalize his preferences over bounded probability distributions make the expected utility of some of these unbounded distributions infinite. Assume that his preferences are consistent with expected-utility maximization, in the sense that it determines his preferences in the usual way when the alternatives in question have finite expected utilities; and that he always prefers a distribution with infinite expected utility to one with finite expected utility.

(a) Show that his preferences over probability distributions cannot be continuous in the sense of the axiom used in the von Neumann—Morgenstern theorem. (Hint: suppose that the distribution P has infinite expected utility, but that $0-for-certain and $5-for-certain have finite expect utilities. Then consider the individual's preferences between the alternatives $5-for-certain {aP + (1-a)$0} as a approaches zero.).

4. Suppose that there are two states of the world, s_1 and s_2, and that an individual who knows the probabilities, p_1 and p_2 respectively, of the two states chooses among state-contingent consumption bundles as if to maximize the expectation of a state-independent, strictly increasing von Neumann-Morgenstern utility function.

(a) Suppose that the individual is risk-neutral, and that he is indifferent between (8, 2) and (4, 4). What must the value of p_1 be?

(b) Now suppose that the individual may be either risk-averse or risk-loving. What is the lowest possible value of p_1 for which the individual could weakly (or strictly) prefer the state-contingent consumption bundle (6, 2) to the bundle (2, 6)? Do his risk preferences affect your answer? Why or why not?

(c) Now suppose that the individual is risk-averse, and that he is indifferent between (6, 2) and (2, 6). Show (graphically or algebraically, whichever you prefer) that he must weekly prefer (4, 4) to either of these bundles.

5. Consider an expected utility-maximizing student, who cares only about his income. Cheating on his 206 exam adds a given amount to his income, whether or not he is caught at it. Suppose, however, that a student who is caught cheating is fined a given amount. It is observed that a 1% increase in the probability of being caught lowers the student's expected utility of cheating by more than a 1% increase in the amount of the fine.

(a) Is the student a risk-averter or a risk-lover? Explain.

6. An expected utility-maximizing person has von Neumann—Morgenstern utility function $u(\cdot)$, with $u'(\cdot) > 0$, and deterministic initial wealth w. He is just indifferent between losing $x > 0$ for certain, and losing $y > x$ with probability $p > 0$ and losing nothing with probability $1 - p$. (In other words, x is the most he will pay to be insured against a random loss of y with probability p.)

(a) Prove that for any given values of w and y, x is an increasing function of p.

(b) Prove that for any given values of w and p, x is an increasing function of y.

(c) Prove that if the person is risk-averse, then $x > py$.

(d) How does x vary with w when $u(w) \equiv a - be^{-cw}$ with $b, c > 0$, so that the person has constant absolute risk aversion? (Here, e is the base of natural logarithms.)

7. An individual has initial wealth W and holds a lottery ticket that will be worth zA with probability p and –zB with probability $1 - p$. Here, A, B, and z are positive, with $pA \leq (1-p)B$. Let X be the maximum amount the individual would pay someone to take this ticket off his hands. Prove that if the individual is risk-averse, then X is an increasing function of z.

8. Consider a risk-averse, expected-utility maximizing agent with von Neumann—Morgenstern utility function $u(\cdot)$ and initial wealth y.

(a) Show how to determine (by giving an expression that implicitly defines it) the minimum probability of winning, p, needed to get the agent to accept a binary bet in which the outcomes are winning or losing $z > 0$. (Assume he will accept if indifferent.).

(b) Use Taylor's Theorem to derive an approximate expression for p when z is small, and use your expression to show that p is then an increasing function of z.

(c) What aspect of the agent's risk preferences determines whether p is an increasing function of y for small z? Explain.

9. Formulate and prove the statement that a first-order stochastically dominating shift in the distribution of a random variable that is continuously distributed on a bounded interval always increases its mean. Briefly compare the strengths and weaknesses, for the purpose of describing individual decisions under uncertainty, of ordering probability distributions by first-order stochastic dominance and by their means.

10. Formulate and prove the statement that a mean-preserving spread in the distribution of a random variable that is continuously distributed on a bounded interval always increases its variance. Briefly compare the strengths and weaknesses, for the purpose of describing individual decisions under uncertainty, of ordering probability distributions by mean-preserving increases in risk and buy their variances.

11. Two securities are available today at the same price. The value of security A tomorrow is a uniform random variable, distributed between 0 and a. The value of security B tomorrow is also a uniform random variable, but is distributed between 0 and b. Prove directly (that is, without invoking any general theorems) that if $a > b$, then any individual with a strictly increasing von Neumann—Morgenstern utility function prefers A to B. What general theorem does this result follow from?

12. The random variable x is uniformly distributed on the interval $[a, b]$, where a and b are constants such that $0 < a < b < M$, so that the interval $[a, b]$ is always contained in the interval $[0, M]$.

(a) Show that a change in a and/or b induces a first-order stochastically dominating increase in the distribution of x if and only if it weakly increases both a and b. (A carefully drawn graph of the distribution function may help here.) Does such a change necessarily increase the mean of x? Does a change that increases the mean necessarily induce a first-order stochastically dominating increase? Explain.

(b) What kinds of change in a and/or b induce mean-preserving spreads to the distribution of x? Do such changes necessarily increase the variance of x? Does a change that increases the variance necessarily induce a mean-preserving spread? Explain.

13. The random variable x has a trinomial distribution, with $\Pr\{x = A\} = a$, $\Pr\{x = B\} = b$, and $\Pr\{x = C\} = 1 - a - b$, with $a, b, c > 0$ and $A < B < C$.

(a) What kinds of change in a, b, and c (with A, B, and C constant) induce a first-order stochastically dominating increase in the distribution of x? (A carefully drawn graph may help.) Does such a change necessarily increase the mean of x? Does a change that increases the mean necessarily induce a first-order stochastically dominating increase? Explain.

(b) What kinds of change in a, b, and c (with A, B, and C constant) induce a mean-preserving spread in the distribution of x? Does such a change necessarily increase the variance of x? Does a change that increases the variance necessarily induce a mean preserving spread? Explain.

(c) How would your answers to parts (a) and (b) change if $C < B < A$?

14. (Suggested by a futile argument with Gordon Tullock) A rich uncle gives you a gift certificate that entitles you to an insurance contract, of your own choosing, with an actuarial value (an expected return) of $1 million. The only uncertainty you face us about whether you will be involved in an accident that leaves you paralyzed; this will happen with probability 1/2. You may allocate the $1 million in actuarial value however you wish between the two states, paralyzed and not paralyzed. It has been observed that, when faced with this choice, some people choose to receive more money in the paralyzed state, some choose to receive less money in the paralyzed state, and some choose to receive equal amounts in both states.

(a) Which of these choices is/are consistent with expected-utility maximization with a strictly concave, differentiable, state-independent von Neumann—Morgenstern utility function? Explain.

(b) Which of these choices is/are consistent with expected-utility maximization with a strictly concave, differentiable state-dependent von Neumann—Morgenstern utility function? Explain.

15. There are two states of the world, 1 and 2, and a single consumption good; the state-contingent consumption vector $e \equiv (e_1, e_2)$ represents consumption of e_i units of the consumption good if state i occurs. The probability of state i is p_i. Suppose that an individual chooses among state-contingent consumption vectors to maximize the expectation of the state-independent von Neumann—Morgenstern utility function $u(\cdot)$.

(a) Write the equation of a typical indifference curve for the individual.

(b) Derive an expression for $MRS_{12}(e_1, e_2)$, the individual's marginal rate of substitution between consumption in states 1 and 2 at consumption vector (e_1, e_2).

(c) Suppose that $e^a \equiv (\underline{e}_1, \underline{e}_2)$, $e^b \equiv (\underline{e}_1, \bar{e}_2)$, $e^c \equiv (\bar{e}_1, \underline{e}_2)$, and $e^d \equiv (\bar{e}_1, \bar{e}_2)$, so that these four consumption vectors form a rectangle in (e_1, e_2)-space. Show that $MRS_{12}(e^a)/MRS_{12}(e^b) = MRS_{12}(e^c)/MRS_{12}(e^d)$.

(d) Does the result of part (c) remain valid when the utility function is state-dependent? Explain.

(e) Does the result of part (c) remain valid when the individual is not an expected-utility maximizer? Explain.

16. Suppose that there are three money outcomes, x_1, x_2, and x_3, with $x_1 < x_2 < x_3$, and that you can observe which values of p make a person prefer getting x_2 for certain to getting a random outcome $\{x_1$ with probability p, x_3 with probability $(1-p)\}$. Is this enough to determine a person's preferences over arbitrary probability distributions over x_1, x_2, and x_3:

(a) if he is an expected-utility maximizer? Explain.

(b) if he chooses among distributions to maximize some differentiable preference function, not necessarily consistent with expected-utility maximization? Explain.

17. An individual chooses among lotteries according to preferences that are complete, transitive, and continuous. He cares only about money, and all the lotteries he faces have the same three possible outcomes, \$1, \$2, and \$3; the probabilities of these outcomes are denoted p_1, p_2, and p_3, respectively. Labeling your diagrams carefully so that I can tell how they were constructed, draw (in either (p_1, p_3)-space or (p_2, p_3)-space, whichever you

prefer, but making clear which space you have chosen) an indifference map for an individual who:

(a) is risk-loving, likes money (always prefers first-order stochastically dominating shifts in the distribution of money outcomes), and satisfies the independence axiom

(b) is risk-averse, likes money, but does not satisfy the independence axiom

(c) is risk neutral, but does not like money. Is it possible for a risk-neutral individual whose preferences depend only on the probability distribution of money outcomes to violate the independence axiom? Explain why or why not.

18. There are three possible states of the world, with known, objective probabilities $q_1 = 2/6$, $q_2 = 3/6$, and $q_3 = 1/6$. There is one consumption good, which can be purchased contingent on which state occurs. Consumption in state i is denoted c_i, and the price of this contingent commodity is denoted p_i. (That is, a contract to deliver c_i units of the consumption good if and only if state i happens costs $p_i c_i$.) There are two persons, Mr. A and Ms. B, both risk-averse, with state-independent, differentiable von Neumann—Morgenstern utility functions denoted U(c) and V(c) respectively. Mr. A and Ms. B have the same income, denoted I.

(a) Derive and interpret the first-order conditions for an interior solution of Mr. A's problem of allocating his income optimally among consumption in the three states. (Do not assume that the p_i are proportional to the q_i.)

(b) Can Mr. A's second-order conditions fail to be satisfied under the stated assumptions?

(c) Do the stated assumptions rule out the possibility of a corner solution?

(d) When Mr. A faces the prices $p_{1A} = 8/25$, $p_{2A} = 10/25$, and $p_{3A} = 3/25$, he purchases $c_{1A} = 7$, $c_{2A} = 8$, and $c_{3A} = 9$; and when Ms. B faces the prices $p_{1B} = 8/25$, $p_{2B} = 6/25$, and $p_{3B} = 8/25$, she purchases $c_{1B} = 8$, $c_{2B} = 9$, and $c_{3B} = 7$. What can you conclude about their comparative levels of risk aversion?

(e) Now suppose that Mr. A faces the prices $p_{1A} = 9/25$, $p_{2A} = 10/25$, and $p_{3A} = 2/25$, and purchases $c_{1A} = 7$, $c_{2A} = 8$, and $c_{3A} = 10$. Why does this additional information demonstrate that Mr. A. cannot in fact be an expected-utility maximizer?

19. You are bidding for a \$100 bill against a single opponent. You must choose your bid before observing his bid, which is uniformly distributed on the interval from \$0 to \$s, where $0 < s \leq 100$. If your bid is higher than his bid, you win the \$100, but must pay the amount you bid (to a third party); if your bid is lower than his bid, you win nothing but pay nothing. (Ignore ties, which have zero probability because his bid is continuously distributed.) You have a twice continuously differentiable von Neumann—Morgenstern utility function u(y), were y is your final wealth, with $u'(y) > 0$ for all y.

(a) Formulate the problem that determines your expected-utility maximizing bid. Do not assume, in your formulation, that your bid necessarily lies in the interval from \$0 to \$s. (Hint: Start by writing your probability of winning as a function of your bid and s.)

(b) Making further assumptions only if necessary, show that it is never optimal to bid \$0 or less. Does it matter if you are risk-averse? Why or why not?

(c) Making further assumptions only if necessary, show that it is never optimal to bid more than \$s.

(d) Show that it is always optimal to bid exactly \$s when s is near 0, and that it is never optimal to bid exactly \$s when s is near 100.

20. Your fairy godmother gives you a signed blank check, telling you to fill in whatever amount you wish before trying to cash it. If the amount you fill in, \$x, does not exceed the amount in her account, \$y, you get that amount; otherwise the check "bounces" and you get nothing. You do not know y when you choose x, but you do know that y is distributed on the interval [0, 100] with differentiable distribution function $F(Y)$ ($\equiv \Pr\{y \leq Y\}$) and density $f(Y)$ ($\equiv F'(Y)$), where $F(0) = 0$, $F(100) = 1$, and $f(Y) > 0$ for all Y in [0, 100]. Your initial wealth is w, and you choose x to maximize the expectation of a strictly increasing, differentiable von Neumann—Morgenstern utility function, $u(\cdot)$.

(a) Formulate the problem that determines your optimal choice of x subject to the constraint that $x \geq 0$. Do not assume that x must be ≤ 100. (Hint: what is your utility if $x > y$? If $x \leq y$? What is the probability that $x \leq y$? Use the answers to these questions to derive a simple expression for your expected utility.)

(b) Show that it is never optimal to set $x = 0$.

(c) Write the first- and second-order conditions that characterize the optimal choice of x when it is strictly positive.

(d) Show that it is never optimal to set $x \geq 100$.

21. Formulate and prove the statement that an expected utility-maximizing risk averter who prefers more money to less and has a deterministic initial wealth will never take a bet that does not have a strictly positive expected return.

22. A monopolistic firm faces a demand function of the form $D(p,a) \equiv d(p)a$, where p is price and a is a parameter distributed on [0, M] with density f(a). The firm chooses price and quantity to maximize expected profit (it is risk-neutral), but it must choose these variables before the demand parameter a is observed. Suppose that output can be produced at constant marginal cost, c; that unsold output has no value; and that realized demand in excess of the quantity produced cannot be met. Prove that a mean-preserving spread in the distribution of a lowers *maximized* expected profit. (Hint: Write expected profit, for each p and q, as the sum of expected profit given that demand is greater than q,

and given that demand is less than q. Then integrate by parts to simplify this expression for the firm's objective function, so that the integral condition can be used to characterize the effect of increasing risk on the firm's problem. Explain each step of your argument.)

23. A risk-averse, state-independent expected-utility maximizer must divide his wealth between investment in two risky assets, whose returns are independently and identically distributed. (These two assets are the only choices; there is no safe asset.) Prove that in the optimal portfolio, he diversifies by investing half his wealth in each asset.

24. Consider an expected-utility maximizer who prefers more money to less and has a deterministic initial wealth, who is offered a bet with a strictly positive expected return, with the option to scale it up or down proportionally as much as he wishes. Making whatever assumptions you find necessary, formulate the problem that determines his optimal choice of scale, and prove that he will always take the bet at some strictly positive scale. Use your argument to decide whether this conclusion depends on whether he is risk-averse or on whether his von Neumann-Morgenstern utility function is differentiable. Explain.

25. An individual with a state-independent, differentiable von Neumann—Morgenstern utility function, $u(\cdot)$, with $u'(\cdot) > 0$, and deterministic initial wealth, W, is offered a bet that will add a random amount, X, to his initial wealth, where $EX > 0$.

(a) Suppose it is known that he is either (globally) risk-averse or (globally) risk-loving. What can you infer about his risk preferences if he declines the bet?

(b) Use your answer to (a) to show that if he declines the bet, he must also decline all scaled bets of the form aX with $a > 1$, but might accept some bets of this form with $a < 1$.

26. A risk-averse, state-independent expected utility-maximizing investor must decide how to divide his portfolio between two assets. (These are the only two possible investments, and he cannot borrow or lend.) There are two states of the world, such that \$1 invested in asset 1 yields \$(1+r) > \$1 in state 1 and \$1 in state 2, and \$1 invested in asset 2 yields \$(1+s) > \$1 in state 2 and \$1 in state 1.

(a) Letting x_i denote the investor's final wealth if state i occurs, graph his opportunity set, given initial wealth I, in (x_1, x_2)-space, (putting x_1 on the horizontal axis). Label your graph clearly to show how I, r, and s, determine the opportunity set.

(b) Compute the portfolio that makes the investor's final wealth independent of the state. Is it always optimal for a risk-averse investor to choose this portfolio? Explain.

(c) Suppose that the investor has constant relative risk aversion. Draw the path in (x_1, x_2)-space that shows how the state-contingent final wealths generated by his optimal portfolios vary with I. What theorem justifies your answer, and why? (Please explain carefully.) Can you tell whether the investor's coefficient of absolute risk aversion is increasing or decreasing and wealth in this case?

27. A risk-averse, state-independent expected utility-maximizing investor who likes money must decide how much of his initial wealth to invest in a risky asset, investing whatever remains in a safe asset. (These are the only two possible investments, and he cannot borrow or lend.) There are two states of the world, with known probabilities, such that \$1 invested in the safe asset yields \$(1+s) no matter which state occurs, and \$1 invested in the risky asset yields \$(1+r) if state 1 occurs and \$1 if state 2 occurs.

(a) Suppose that $0 < s < r$. Letting x_i denote the investor's final wealth if state i occurs, graph his opportunity set, given initial wealth I, in (x_1, x_2)-space, (putting x_1 on the horizontal axis). Label your graph clearly to show how I, r, and s, determine the set.

(b) Draw the path in (x_1, x_2)-space that shows how the state-contingent final wealths generated by his optimal portfolio vary with I, when $0 < s < r$ and his Arrow-Pratt measure of absolute risk aversion is constant, independent of his wealth.

(c) Answer part (b) again, but when his relative risk aversion is constant.

(d) How do your answers to (b) and (c) change when $0 < r , s$?

28. A risk-averse, state-independent expected-utility maximizer must choose the proportions in which to invest his initial wealth, w, in two risky assets, A and B. There are two states the world, 1 and 2, with known probabilities $p_1 = 1/3$ and $p_2 = 2/3$. Every dollar invested in asset A yields \$(1+r) in state 1 and \$1 in state 2, and every dollar invested in asset B yields \$1 in state 1 and \$(1+r/2) in state 2.

(a) Show (graphically or analytically, whichever you prefer) that the optimal portfolio combines the two assets in the same proportions for all w. Identify those proportions.

(b) Does the fact that the optimal proportions are independent of w in this case imply that the person has constant relative risk aversion, given the Arrow-Pratt characterization? Explain.

29. According to Paul Samuelson, the mathematician Stanislaw Ulam defined a coward as someone who will not bet even when you offer him two-to-one odds and let him choose his side. (A gamble with two-to-one odds is one in which the individual wins \$2x if an event A occurs and loses \$x if A does not occur. Letting the individual choose his side means letting him choose between winning \$2x if A occurs and losing \$x if A does not occur, or winning \$2x if A does not occur and \$x if A occurs.)

(a) Show by example (graphical, if you prefer) that it is possible for an expected-utility maximizer who likes money to be a coward according to Ulam's definition.

(b) Show (graphically, if you prefer) that an expected-utility maximizer who likes money, and whose von Neumann—Morgenstern utility function is differentiable, cannot be a Ulam-coward for all values of $x > 0$.

30. A bookie (or bookmaker) is someone who offers bets on sporting events. A point spread of S points in favor team A means if someone who bets on team A wins either if team A wins or if team A loses by no more than S points. Bookie Bob Martin stated (in the 19 January 1980 edition of the L.A. Times (part III, pages 1, 9), in response to criticism from L.A. Rams coach Ray Malavasi) that he tried to set the point spread for games on which he offered bets not to represent his true subjective beliefs, but to try to insure that equal amounts of money would be bet on the two teams. Martin offered the rationale that if equal amounts were but on both teams, that his return (based on a 10% fee on the value of each bet on either team) would be safely independent of the outcome of the game. Assume that there is only one game, and that the total amount bet on it is independent of the point spread, so that the spread affects only which of the two teams bettors bet on. Also assume that Martin can only accept bets, not place them; and that he thinks he has some information about the teams not shared by all bettors, so that, given his beliefs, the point spread makes both teams appear equally attractive to Martin differs from the point spread that equalizes the expected amounts of money bet on the teams.

(a) Forumalate a model of the "market" for bets on the two teams. Let A and B represent the two teams; M the total amount bettors bet on them; and S and T, respectively, the point spread (in favor of team A) Martin offers bettors and the point spread that makes both teams appear equally attractive to him. Making reasonable assumptions, as necessary, about the behavior of bettors and the probabilities of winning the various possible bets, show how Martin's expected utility depends on the point spread he sets.

(b) Are Martin's actions and the rationale he gave consistent with expected-utility maximization? Does your answer depend on whether Martin's von Neumann—Morgenstern utility function is concave? On whether it is differentiable? Explain.

31. Suppose that there are two states of the world, s_1 and s_2, and that an individual who knows the probabilities, p_1 and p_2 respectively, of the two states chooses among state-contingent consumption bundles to maximize the expectation of a state-independent von Neumann—Morgenstern utility function $u(x) \equiv \ln x$ (where $\ln x$ is the natural logarithm of x, so that $u'(x) \equiv 1/x$). Suppose that the individual's income is \$7 if state 1 occurs and \$2 if state 2 occurs, and that an insurance company offers him a contract whereby for each \$1 he pays (independent of the state, before observing it), he receives \$2 if state 2 occurs and nothing if state 1 occurs.

(a) Write the problem that determines the individual's demand for insurance, at this price, as a function of his estimate of the probability, p_2, that state 2 will occur.

(b) For what value of p_2 is expected profit 0 for the insurance company at this price?

(c) Using the first-order condition, show that if p_2 is higher than this value, the individual will buy so much insurance that he is actually better off if state 2 occurs than if state 1 occurs. Is this consistent with the insurance company staying in business? Explain.

32. There are two states of the world, 1 and 2, and a single consumption good; your state-contingent endowment of the consumption good is w_i units if state i occurs. The probability of state i is known to be p_i. You are a risk-averse expected-utility maximizer with state-independent von Neumann—Morgenstern utility function $u(\cdot)$. You have an opportunity to make an insurance contract, specifying your consumption contingent on the state of world, with a risk-neutral insurer. Characterize, as far as you can, the Pareto-efficient contract that gives the insurer zero expected profits.

33. There are two states of the world, 1 and 2, and a single consumption good. The probability of state i is known to be p_i. A risk-averse expected-utility maximizing consumer with state-independent von Neumann—Morgenstern utility function $u(\cdot)$ has state-contingent endowment of the consumption good of w_i units if state i occurs. He has an opportunity to make an insurance contract, specifying his consumption contingent on the state of the world, with a risk-neutral insurer.

(a) Characterize the Pareto-efficient contract that gives the insurer zero expected profit.

(b) Characterize the contract that maximizes the insurer's expected profit, subject to the constraint that the consumer weakly prefers it to going without insurance.

(c) Do your characterizations remain valid when the consumer's von Neumann—Morgenstern utility function is state-dependent? Explain.

34. (In memory of Jack Hirshleifer) Suppose that there are two expected utility-maximizing consumers, X and Y, and two commodities, A and B. A and B are perfect substitutes in consumption; think of them as corn grown in different locations. If v is the total amount of A and B consumed by a consumer, his von Neumann—Morgenstern utility is $\ln(1+v)$. X's initial endowment is one unit of A (and no B) and Y's initial endowment is one unit of B (and no A). There are two, equally likely states of the world; think of them as flood locations. In state A all of the B is destroyed but none of the A; and in state B all of the A is destroyed but none of the B.

(a) Suppose there are markets for two contingent commodities: corn in state A and corn in state B. (In principle contracts for these commodities could be fulfilled by delivering either A or B, because consumers cannot distinguish them, but feasibility requires fulfillment with A corn in state A and B corn in state B.) Compute the competitive equilibrium (relative) prices of A and B and each consumer's trades and expected utility.

(b) Suppose that the market structure is the same as in part (a), but that it is publicly announced which state will occur before X and Y have an opportunity to trade the contingent commodities. Taking into account the prior probability of each possible state announcement and what will happen in competitive markets after the announcement is made, compute each consumer's ex ante (before the state is announced) expected utility. Is the social value of better information always positive? Why or why not?

35. Let there be N jobs and N workers. Suppose that total income is fixed, and that the income to be associated with each job is to be decided by majority vote. All workers are risk-averse and care only about their incomes, and each has subjective probability (1/N) of being assigned to each job. What can you say about the distribution of income? Does it matter whether workers all have the same risk preferences?

36. Time is discrete and a person may live up to three periods. (First he is an Assistant Person, then—with luck—an Associate Person and finally a Full Person.) With known probability $p_t > 0$ he dies at the end of period t; thus if q_t is the probability that he is alive during period t, $q_1 = 1$, $q_2 = 1 - p_1$, and $q_3 = 1 - p_1 - p_2$. There is a single perfectly divisible consumption good, which can be purchased at a price of 1 per unit each period. The person's income per period is y when alive and 0 when dead; and he may borrower lend at 0 interest as much as he wants, except that he cannot risk dying in debt. He does not discount future consumption just because it is in the future, but he gets utility from consumption only when he is alive. He chooses a consumption plan to maximize lifetime expected utility. His preferences over consumption streams are stationary and additively separable across periods; and consumption c_t in period t yields additional utility $u_t(c_t)$ when alive, where $u_t(\cdot)$ is differentiable with $u_t''(\cdot) < 0$, and 0 when dead.

(a) Write the person's lifetime expected utility as a function of the c_t and the q_t. Then write the constraints on his consumption plan, given that he cannot risk dying in debt.

(b) Suppose that his optimal consumption plan satisfies the constraints for periods 1 and 2 strictly, so that only the constraint for period 3 is binding. Write the first-order conditions that determine his optimal plan, assuming that the optimal c_t are strictly positive.

(c) Use the first-order conditions from (b) to analyze the qualitative response of the optimal consumption plan to an increase in y. (Here and below, you are not required to compute the comparative statics derivatives, just to determine the signs of the responses. It may be easier to do this by reasoning directly from the first-order conditions.)

(d) What kinds of changes in the q_t constitute mean-preserving spreads in the distribution of the death date? Use the first-order conditions from (b) to analyze the qualitative response of the optimal consumption plan to such a mean-lifetime-preserving spread.

(e) Now suppose that the person can purchase life insurance at its actuarial value. It can be shown (although you are not asked to show it) that this allows him to replace the constraints on consumption in (a) by the single constraint that expected lifetime consumption equals expected lifetime income; assume that this constraint is binding. Answer the questions in (b), (c), and (d) again for this case.

37. Greedy Oil Company has purchased desert land in Arizona. The firm's head geologist estimates that the probability is 0.2 that there is oil beneath the land. It would cost \$100,000 to drill in search of oil. If the firm drills and oil is found, it will earn \$400,000, or \$300,000 over and above drilling costs. If oil is not found, the \$100,000 will have been totally wasted. Assume Greedy is risk-neutral and that the land is worthless without oil.

(a). If the only options are to drill or not to drill, what should Greedy do?

(b) What is the expected value of perfect information in this situation?

(c) Now supposed that at a cost of \$25,000, Greedy has already drilled down 50 feet in search of stone formation A. It is known that when there is oil further down, the probability of finding A at this step is 0.6, while if there is no oil further down, the probability is only 0.2. Greedy's test drilling found A. Should it drill for oil at this point are not? (Assume that the \$25,000 counts against the cost of any further drilling, which is therefore \$75,000 equals \$100,000 -\$25,000 at this point.).

(d) If you were Greedy's president, would you have authorized the test drilling described in (c)? Justify your answer in terms of the expected value of sample information.

38. You are an expected net benefit-maximizing planner with a zero discount rate. You have a total of one unit of perfectly divisible land, which can be developed in either or both of two periods. Let D_1 denote the amount of land developed in period 1 and D_2 the total amount developed in period 2 (including land already developed in period 1, if it remains developed). The net benefit per unit of land developed in period 1 is known with certainty to be α when D_1 is chosen, where $-1 \leq \alpha \leq 1$. The net benefit per unit of land developed in period 2, β, is known when D_2 is chosen, but uncertain when D_1 is chosen, with continuous probability distribution function, $F(\beta)$, where $F(-1) = 0$ and $F(1) = 1$.

(a) Formulate the problem that determines your optimal choices of D_1 and D_2 when development in period 1 is reversible (so that land developed in period 1 need not remain developed in period 2). Be careful to include any constraints on your choices. Show how the realized values of α and β and the distribution $F(\cdot)$ determine the optimal D_1 and D_2.

(b) Analyze the effects on the optimal policy and the maximized expected net benefits of a first-order stochastically dominating increase in the distribution of β, and then of a mean-preserving spread in the distribution of β. (Use the integral condition with care!)

(c) Answer the questions in part (a) when development in period 1 is irreversible (so that land developed in period 1 must remain developed in period 2), and use your answer to show that irreversibility makes the criterion for development in period 1 more stringent.

(d) Answer the questions in part (b) when development in period 1 is irreversible.

39. You are an expected net benefit-maximizing planner with a zero discount rate. You have a total of one unit of perfectly divisible land, which can be developed in either or

both of two periods. The net benefit per unit of land developed in period 1 is known with certainty to be b_1. The net benefit per unit of land developed in period 2 is uncertain, and takes the value b_2' with probability p and b_2'' with probability 1-p. Assume that $b_2' < 0$ and $b_2'' > 0$, and (to rule out ties), that neither b_1 nor $pb_2' + (1-p)b_2''$ equals 0. Let D_1 denote the amount of land developed in period 1 and let D_2' and D_2'' respectively denote the total amount developed in period 2 (including land already developed in period 1, if it remains developed), when the benefit is b_2' and b_2''. Assume first that the uncertainty about the benefit in period 2 is resolved before the decision about development in period 2 must be made.

(a) Formulate the problem that determines your optimal choices of D_1 and D_2 when development in period 1 is reversible (so that land developed in period 1 need not remain developed in period 2). Be careful to include any constraints on your choices. Show how the realized values of α and β and the distribution $F(\cdot)$ determine the optimal D_1 and D_2.

(b) Analyze the effects on the optimal policy and the maximized expected net benefits of a first-order stochastically dominating increase in the distribution of β, and then of a mean-preserving spread in the distribution of β. (Use the integral condition with care!)

(c) Answer the questions in part (a) when development in period 1 is irreversible (so that land developed in period 1 must remain developed in period 2), and use your answer to show that irreversibility makes the criterion for development in period 1 more stringent.

(d) Answer the questions in part (b) when development in period 1 is irreversible.

40. Suppose I'm trying to sell my car. I am risk-neutral and do not discount the future. I know that potential offers are independently and uniformly distributed between zero and M. It costs me $c < M/2$ to generate an offer. What is the lowest price I should accept for the car? Does it matter if I have perfect recall? Explain.

41. TV sets at different stores differ only in price and you wish to buy exactly one. You are risk-neutral, and you do not discount future costs and benefits. You therefore seek to minimize the expected total cost of acquiring your TV set, including search costs; searching costs a constant $c > 0$ for each store you visit. Recall is possible, in the sense that you can buy your TV set at some store other than the last one you visit, if you wish, without paying the cost of visiting that store again. The prices at stores are independently, identically distributed random variables with a common, known distribution function, $F(p)$. You can search sequentially (instead of deciding in advance how many stores to visit), and there is no limit on the number of different stores you can visit.

(a) Show that the optimal search policy involves setting a reservation price and searching until you find a price at least as good. Write the condition that characterizes the optimal reservation price. Does your answer depend on whether recall is possible? Explain. Does your answer depend on whether the distribution is known? Explain.
(b) Suppose that the distribution of prices at each store shifts downward to a new distribution, $G(\cdot)$, in such a way that $G(\cdot)$ is first-order stochastically dominated by $F(\cdot)$.

(Be careful not to confuse up and down translating this condition into mathematics.) Using your condition from (a), explain how and why this changes your optimal search policy. What, if anything, can you say about how this change affects the probability distribution of the number of searches implied by the optimal policy?

(c). Now suppose that the distribution of prices at each store shifts to a new distribution, $H(\cdot)$, that differs from $F(\cdot)$ by a mean-preserving increase in risk. Using your condition from part (a), explain how and why this changes your optimal search policy. What, if anything, can you say about how this change affects the probability distribution of the number of searches implied by the optimal policy?

(d) Suppose that R^* is your optimal reservation price and $p > R^*$ is the best price you have found so far. Show that your minimized total additional expected cost (including both search costs and the price you pay) equals $c + R^*$. (Hint: Use the fact that if you could buy a TV set at your reservation price by searching one more time, you would be just indifferent between taking your reservation price and continuing to search.) How do the distribution changes in (b) and (c) affect your minimized additional expected cost?

[Note: Problems 42 and 43 are instructive and rewarding, but I think they are too computationally difficult to ask on the mini-exam (and I won't do it).]

42. (in memory of Bennett Crain) You are sailing a match race (a race against a single opponent), and wish to maximize your probability of winning. The race has two "legs," numbered 1 and 2; and your net gain in time ahead in leg i is a (possibly negative) random variable x_i, so that (ignoring ties, which will occur with probability zero) you win if and only if $x_1 + x_2 > 0$. x_1 and x_2 are independently and uniformly (but not identically; see below) distributed, with zero means. Your opponent has no influence on these distributions, so this is an individual decision problem, not a game. You have no influence on their means, but you control their variances, within limits. Specifically, x_i has c.d.f. (cumulative distribution function) $F(x, a_i)$, where $F(x, a)$ is the c.d.f. of a random variable x uniformly distributed on $[-a, a]$, so that $F(x, a) = 0$ when $x < -a$; $F(x, a) = (x+a)/2a$ when $-a \leq x \leq a$; and $F(x, a) = 1$ when $x > a$. (It is not hard to show that increasing a induces a mean-preserving spread in the distribution of x, and increases its variance; but you are not asked to show this.) At the start of the first leg you may choose any a_1 such that $\underline{a} \leq a_1 \leq \bar{a}$, where $\underline{a}$ and $\bar{a}$ are known, exogenous parameters satisfying $0 < \underline{a} < \bar{a}$. At the start of the second leg, after observing x_1, you may choose any a_2 (different from a_1 if you wish) such that $\underline{a} \leq a_2 \leq \bar{a}$.

(a) Formulate and solve the problem that determines your optimal choice of a_2, as a function of x_1. Call the answer $a_2^*(x_1)$. Derive an expression for $V(x_1)$, the maximized probability of winning obtained by setting $a_2 = a_2^*(x_1)$ when x_1 is realized. (Hints: (i) Common sense suggests that in some circumstances is optimal to be as conservative as possible in the second leg, and another is it is optimal to "go for broke." Express this intuition mathematically. (ii) The difference between < and ≤ is not crucial here, because the probability of the continuously distributed x_i taking any particular value is zero.)

(b) Formulate and solve the problem that determines your optimal choice of a_1. Call the answer a_1^*. Derive an expression for V^*, the maximized prior probability of winning obtained by setting $a_1 = a_1^*$ and then setting $a_2 = a_2^*(x_1)$. (Hints: (i) The prior probability of winning equals the mathematical expectation, before x_1 is revealed, of $V(x_1)$ from part (a). (ii) You will find (after some tedious calculations) that a_1^* is determined by $\underline{a}$ and $\bar{a}$ in a strikingly simple way, and that V^* varies just as you would expect, on reflection, as $\underline{a}/\bar{a}$ approaches 0 or 1.)

43. (In honor of the first European to discover San Francisco Bay, either Juan Rodriguez Cabrillo—who was probably the first but didn't bother to write it down—or Sir Francis Drake, who had better a public relations department behind him; see Samuel Eliot Morison, *The European Discovery of America*, Oxford 1971) You are sailing in the ocean west of the mouth of a bay on the coast. You are far enough offshore that the distance to the coast, which is assumed to be a straight line running north and south, featureless except for the bay mouth, is approximately independent of which straight-line course you steer to reach it. You wish to minimize the expected distance you need to sail to find the mouth of the bay, assumed for simplicity to be a point. The mouth cannot be seen from any significant distance offshore; you must be "right on top of it" to find it (which is approximately true for San Francisco Bay). Thus, because the distance to the coast is approximately constant, minimize the expected distance sailed amounts to minimizing the expected distance sailed after you reach the coast. Initially, you know which course to steer to reach the mouth, or any other desired point on the coast, in the absence of errors. Your course made good, however, includes a random error y, so that if you steer for a point x miles north of the mouth (with points south represented by negative values of x), you actually reach the coast $x + y$ (possibly negative) miles north of the mouth. Assume for simplicity that the distribution of y is independent of x, and that it is continuously distributed on [-10, 10] with known c.d.f. $F(y)$, and density $f(y)$ that is symmetric about 0 and single-peaked, so that $f(0) > 1/20$. In the absence of errors it would clearly be optimal to set $x = 0$. However, with errors you will generally be uncertain when you reach the coast whether to turn north or south to reach the mouth. Setting $x \neq 0$ might help by making one direction more likely than the other. Your navigator notes that because the error density $f(y)$ is symmetric about 0, as is the coast, there is no loss of generality in setting $x \geq 0$, so that (unless by extraordinary luck $x + y = 0$) it is optimal to turn south when you first reach the coast. He also notes that your crew will insist that you sail far enough in the direction you choose to be certain it is wrong (give the bounded error distribution) before turning back.

(a) If you do not find the mouth of the bay to the south of the point where you first reach the coast, how far south will your crew make you sail before turning north to retracte your path and continue the search?

(b) Using your answer to (a), write the expected distance sailed to find the mouth after you reach the coast, as a function of x and the other data of the problem.

(c) Characterize the expected distance-minimizing choice of x as far as you can. Can it ever be optimal to set $x = 0$? To set $x = 10$? Why or why not?