1	2	3	4	5	6	Total
/10	/15	/15	/20	/20	/20	/100

Economics 172A Final Exam NAME________________________________

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Your grade from this exam is 55% of your course grade. The exam ends at 11:00, so you have three hours. You may not use books, notes, calculators or other electronic devices. There are six questions, weighted as indicated. Answer them all. If you cannot give a complete answer, try to explain what you understand about the answer. Write your name in the space above, now. Write your answers below the questions, on the back of the page, or on separate sheets. Explain your arguments and show your work. Good luck!

1. Graph the feasible region and use your graph to solve the following problem:

Minimize $Z = 3x_1 + 2x_2$,

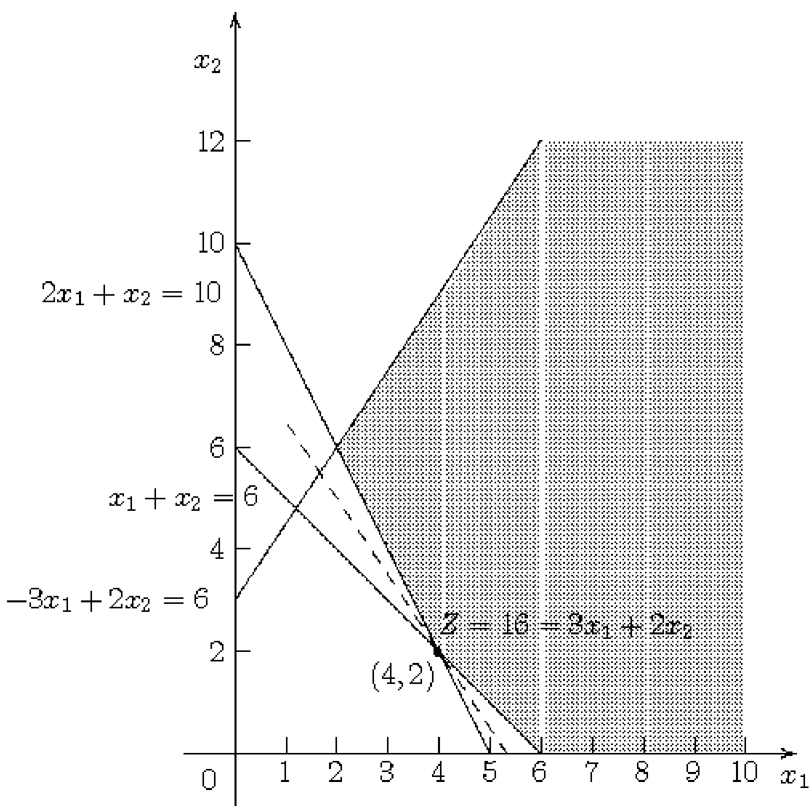
subject to $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1 \geq 0, x_2 \geq 0.$

The optimal solution is $(x_1, x_2) = (4, 2)$ with Z = 16.



2. Slim-Down Manufacturing makes a line of nutritionally complete weight-reduction beverages. One of their products is a strawberry shake which is designed to be a complete meal. The strawberry shake consists of several ingredients. Some information about each of these ingredients is given in the table below.

Ingredient	Calories from fat (per tbsp)	Total Calories (per tbsp)	Vitamin Content (mg/tbsp)	Thickeners (mg/tbsp)	Cost (¢/tbsp)
Strawberry flavoring	1	50	20	3	10
Cream	75	100	0	8	8
Vitamin supplement	0	0	50	1	25
Artificial sweetener	0	120	0	2	15
Thickening agent	30	80	2	25	6

The nutritional requirements are as follows. The beverage must total at least 380 calories. No more than 20% of the total calories must come from fat. There must be at least 50 milligrams (mg) of vitamin content. There must be at least two tablespoons (tbsp) of strawberry flavoring for each tbsp of artificial sweetener. Finally, there must be exactly 15 mg of thickeners in the beverage. Management would like to select the quantity of each ingredient for the beverage which would minimize cost while meeting the above requirements.

(a) Formulate a linear programming model for this problem, and put the constraints into standard "≥ constant" or "= constant" form. Please use the following notation for the decision variables: S = Tablespoons of strawberry flavoring, C = Tablespoons of cream, V = Tablespoons of vitamin supplement, A = Tablespoons of artificial sweetener, T = Tablespoons of thickening agent, and Z = Total cost.

(a) Minimize $Z = 10S + 8C + 25V + 15A + 6T$

subject to $50S + 100C + 120A + 80T \geq 380$

$S+75C+30T \leq 0.2(50S+100C+120A+80T)$, **rewrite as** $9S - 55C + 0V + 24A - 14T \geq 0$

$20S + 50V + 2T \geq 50$

$S \geq 2A$, **rewrite as** $S - 2A \geq 0$

$3S + 8C + V + 2A + 25T = 15$

$S \geq 0, C \geq 0, V \geq 0, A \geq 0, T \geq 0.$

(b) Write the dual of your linear programming problem from part (a), using Y to denote the dual objective function value and F, G, H, I, and J to denote the dual variables associated with the five primal constraints in the order listed in the verbal statement (so that F is the dual variable associated with the primal constraint that says the beverage must total at least 380 calories, G is the dual variable associated with the primal constraint that says no more than 20% of the total calories must come from fat, and so on). Make sure you have the primal constraints in standard "≥ constant" or "= constant" form before you do this. There is no need to explain your answer here, as long as it is correct; but if you're unsure, explanations of why you did what you did it might help.

(b) The primal constraints (with dual prices in parentheses on the right) are:

$50S + 100C + 120A + 80T \geq 380$ (F)
$9S - 55C + 0V + 24A - 14T \geq 0$ (G)
$20S + 50V + 2T \geq 50$ (H)
$S - 2A \geq 0$ (I)
$3S + 8C + V + 2A + 25T = 15$ (J)
$S \geq 0, C \geq 0, V \geq 0, A \geq 0, T \geq 0$.

Using the standard recipe to write the dual:

Maximize Y =	**$380F + 0G + 50H + 0I + 15J$**
subject to	**$50F + 9G + 20H + 1I + 3J \leq 10$**
	$100F - 55G + 0H + 0I + 8J \leq 8$
	$0F + 0G + 50H - 2I + 1J \leq 25$
	$120F + 24G + 0H + 0I + 2J \leq 15$
	$80F - 14G + 2H + 0I + 25J \leq 6$
	$F \geq 0, G \geq 0, H \geq 0, I \geq 0$, J unrestricted.

(c) Suppose you have solved the primal, and you find that for the optimal values of S, V, and T, $20S + 50V + 2T > 50$. What must be true of the optimal value of H in the dual?

(c) $H^* = 0$.

(d) What is the interpretation of the optimal value of F in the dual for how the minimized value of the primal objective function when the data of the problem (objective function coefficients and/or constraint constants) change? What must be true about the optimal basis before and after a change in the data of the problem for this interpretation to be exact (rather than an approximation)?

(d) F^* = the derivative of minimized primal value with respect to the constraint constant whose initial value is 380. The optimal basis must be the same before and after the change (though the optimal solution might change).

3. Consider the problem choose x (a scalar) to solve minimize 2x subject to

$$x \le 5$$
$$x \ge b$$
$$x \ge 0$$

where b (also a scalar) ≥ 0.

(a) For what values of $b \ge 0$ does the problem have a nonempty feasible region? For what values of $b \ge 0$ does the problem have a solution?
(a) $b \le 5$. $b \le 5$.

(b) For all values of $b \ge 0$ for which the problem has a solution, graphically or by inspection, whichever you prefer, write x*(b), the optimal value of x, as a function of the parameter b. Your answer must tell what the optimal value of x is for any value of $b \ge 0$; that is, it must be a clearly specified function of b.
(b) Clearly x*(b) = b for all b such that $0 \le b \le 5$.

(c) Put the primal constraints into standard ("≥ constant") form and write the dual, using y_i to represent the dual variable that is the shadow price of the ith constraint in the primal.
(c) Rewrite the first primal constraint as $-x \ge -5$, so that all the constraints are in standard form. Using the standard recipe to write the dual:
Choose y_1, y_2 to solve maximize $-5y_1 + by_2$ subject to

$$-y_1 + y_2 \le 2$$
$$y_1 \ge 0,\ y_2 \ge 0.$$

(d) For what values of $b \ge 0$ does the dual have a nonempty feasible region?
(d) The dual has a nonempty feasible region for all values of $b \ge 0$: $y_1 = y_2 = 0$ is always feasible.

(e) For what values of $b \ge 0$ does the dual have a solution? For those values, graphically or by inspection, compute y_1*(b) and y_2*(b), the optimal values of y_1 and y_2, as functions of the parameter b. Your answer must tell what the optimal values of y_1 and y_2 are for any value of b.
(e) The dual has a solution for all values of $b \ge 0$ such that $b \le 5$: Clearly y_1*(b) = 0, y_2*(b) = 2 is then the unique solution. The dual does not have a solution for $b > 5$: Then setting $y_1 = y_2 \ge 0$ is always feasible, and letting y_1 and y_2 increase indefinitely yields unbounded objective function value.

(f) Use the Duality Theorem to show that your solutions to the primal in (b) and the dual in (c) are both optimal for all values of b for which the primal and dual have solutions.
(f) With x*(b) = b for all b such that $0 \le b \le 5$, the primal objective function value is 2b. With y_1*(b) = 0, y_2*(b) = 2 for all b such that $0 \le b \le 5$, the dual objective function value is 2b. Since both solutions are feasible for their respective problems, and they yield equal objective function values, both are optimal by the Duality Theorem.

(g) For all values of b for which the primal in (a) and the dual in (c) have solutions, verify directly that your solutions to the primal and dual satisfy Complementary Slackness, saying clearly what Complementary Slackness requires.

(g) Complementary Slackness requires that if a primal (dual) control variable is strictly positive, then the associated dual (primal) constraint must be binding; and that if a primal (dual) constraint is slack (non-binding), then the associated dual (primal) control variable must be zero. Checking: (i) $y_1^*(b) = 0$ for all b such that $0 \leq b \leq 5$, so the requirement for the first primal constraint is satisfied; (ii) $y_2^*(b) = 2 > 0$ for all b such that $0 \leq b \leq 5$, but $x^*(b) = b$ for all such b, so the requirement for the second primal constraint is satisfied; (iii) $x^*(b) = b > 0$ (unless b = 0) but $-y_1^*(b) + y_2^*(b) = 0 + 2 = 2$, so the requirement for the dual constraint is satisfied.

(h) For all values of b for which the primal in (a) and the dual in (c) have solutions, compute V(b), the maximized value of the primal objective function, as a function of b. Check that $V'(b) = y_2^*(b)$.

(h) For all b such that $0 \leq b \leq 5$, $V(b) = 2x^*(b) = 2b$ and $V'(b) = 2 = y_2^*(b)$.

4. Consider an assignment problem with four workers, A, B, C, and D, and three jobs, 1, 2, and 3:

	1	2	3
A	4	6	5
B	3	-1	7
C	9	2	8
D	7	6	9

(a) Put into standard form for the Hungarian Method, increasing or decreasing costs and creating dummy workers or dummy jobs as necessary. (For ease of grading, please do your costs increases or decreases before creating dummy workers or jobs.) Explain why your cost increases or decreases don't distort the optimal assignment of workers to jobs. Explain why your assignment of costs to dummy workers or jobs doesn't distort the optimal assignment of real workers to real jobs.

(a) There are two problems with the problem: one cost is < 0, and there are more workers than jobs. First add one to all costs, making them all nonnegative. Then create a dummy job with all costs 0, to finish putting the problem into standard form (the order doesn't really matter, but do the former first for ease of grading):

	1	**2**	**3**	**4**
A	**5**	**7**	**6**	**0**
B	**4**	**0**	**8**	**0**
C	**10**	**3**	**9**	**0**
D	**8**	**7**	**10**	**0**

(b) Start to solve the problem in standard form by the Hungarian Method, by doing row reduction first and then column reduction. Explain why row and column reduction don't distort the optimal assignment of workers to jobs. (You are *not* asked to keep track of the dual variables.)

(b) Row reduction doesn't change the problem at all in this case, because there are 0s in all entries in the fourth column. It wouldn't distort the optimal assignment anyway because all people must be assigned, and so it is like subtracting a constant from the cost of all assignments. Column reduction doesn't distort the optimal assignment for the same reason, and changes the problem to:

	1	**2**	**3**	**4**
A	**1**	**7**	**0**	**0**
B	**0**	**0**	**2**	**0**
C	**6**	**3**	**3**	**0**
D	**4**	**7**	**4**	**0**

(c) Continue solving the problem by identifying a maximal set of independent zeros (for ease of grading, please identify them by *s as I did in class) and a minimal cover with the same number of lines (please identify them by +s as I did in class). Identify the smallest uncovered entry and do a pivot step to obtain a new reduced cost matrix. Use the new matrix to find the optimal assignment, and calculate its cost in the original matrix.

(c) * identify a maximal set of three independent 0s, covered by the three row or column lines marked by +s. The smallest uncovered entry is 3, and the pivot step yields the second reduced cost matrix, in which a maximal set of four independent zeroes are marked by *s. These are optimal, with total cost (translating back into the original problem) 3 + 2 + 5 + 0 = 10.

	1	**2**	**3**	**4+**
A+	**1**	**7**	**0***	**0**
B+	**0**	**0***	**2**	**0**
C	**6**	**3**	**3**	**0***
D	**4**	**7**	**4**	**0**

	1	**2**	**3**	**4**
A	**1**	**7**	**0***	**3**
B	**0***	**0**	**2**	**3**
C	**3**	**0***	**0**	**0**
D	**1**	**4**	**1**	**0***

(d) Solve the original assignment problem (with four workers, three jobs, and a negative cost) by the branch and bound method, without putting it into standard form. Branch by solving the relaxed version of the problem in which you can fill each job with whomever you wish, without regard to duplication. Then branch on how to fill job 1, job 2, etc.

(d) Return to the original problem:

	1	2	3
A	**4**	**6**	**5**
B	**3**	**-1**	**7**
C	**9**	**2**	**8**
D	**7**	**6**	**9**

Solving the relaxed problem yields B1, B2, A3, infeasible. The entire problem is a "remaining problem." Branch on how to fill job 1: A1, B1, C1, or D1.
A1: Solving the relaxed problem yields A1, B2, A3, infeasible, cost 8.
B1: Solving the relaxed problem yields B1, B2, A3, infeasible, cost 7.
C1: Solving the relaxed problem yields C1, B2, A3, feasible, fathomed, incumbent solution with cost 13.
D1: Solving the relaxed problem yields D1, B2, A3, feasible, fathomed, new incumbent solution with cost 11.
C1 is fathomed, but A1 and B1 are not fathomed. Branch them on how to fill job 2:
A1, B2: Solving the relaxed problem yields A1, B2, A3, infeasible, cost 8, not fathomed.
A1, C2: Solving the relaxed problem yields A1, C2, A3, infeasible, cost 11 ≥ 11, fathomed.
A1, D2: Solving the relaxed problem yields A1, D2, A3, infeasible, cost 15 > 11, fathomed.
B1, A2: Solving the relaxed problem yields B1, A2, A3, infeasible, cost 14 > 11, fathomed.
B1, C2: Solving the relaxed problem yields B1, C2, A3, feasible, cost 10 < 11, fathomed, new incumbent solution.
B1, D2: Solving the relaxed problem yields B1, D2, A3, feasible, cost 14 > 10, fathomed.
Now B1 and C1 are fathomed, but A1, B2 is not yet fathomed. Branch A1, B2 on how to fill job 3:
A1, B2, C3: feasible, cost 11 > 10, fathomed.
A1, B2, D3: feasible, cost 12 > 10, fathomed.
Now everything is fathomed, so the latest incumbent solution, B1, C2, A3 with cost 10, is optimal. This is the same solution obtained above by the Hungarian Method.

5. Consider the two-person zero-sum game, with only the Row player's payoffs shown:

	L	**C**	**R**
T	**0**	**-1**	**2**
B	**5**	**4**	**-3**

(a) Restricting attention to the pure (unrandomized) strategies, T and B for Row and L, C, and R for Column, find the Row player's security-level-maximizing pure strategy and his associated security level. Find the Column player's security-level-maximizing pure strategy and her associated security level. Taking into account that Column's payoffs are minus Row's payoffs, are these security levels consistent (that is, could both players realize them simultaneously playing the game)?

(a) T for Row, with security level -1 in Row's payoffs. R for Column, with security level 2 in Row's payoffs, -2 in Column's. They are not consistent.

(b) Now consider mixed (randomized) strategies. Letting x_1 and x_2 denote the probabilities with which Row plays his strategies T and B, respectively, and letting v denote his resulting security level, write the linear programming problem that determines Row's security-level maximizing mixed strategy.

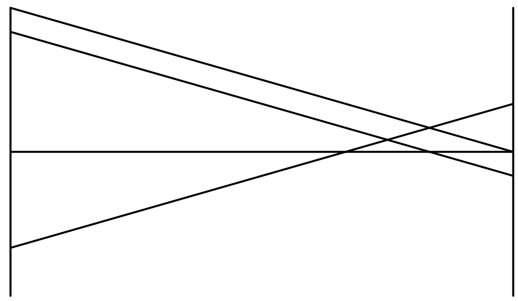
(b) Choose x_1, x_2 to maximize v subject to

$$v - 0x_1 - 5x_2 \leq 0$$
$$v + 1x_1 - 4x_2 \leq 0$$
$$v - 2x_1 + 3x_2 \leq 0$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0.$$

The constraints ensure that Row's security level is at least v because any mixed strategy Column uses will yield Row expected payoffs that are a weighted average of the expected payoffs of his pure strategies, and these are constrained to be at least v.

(c) Letting $x_2 = 1 – x_1$ and simplifying the constraints, solve the problem in (b) graphically, and identify the optimal values of x_1, x_2, and v.

(c) The constraints become:
$v \leq 5x_2 = 5 – 5x_1$, $v \leq –x_1 + 4x_2 = 4 – 5x_1$, and $v \leq 2x_1 – 3x_2 = 5x_1 – 3$. With x_1 on the horizontal axis and v on the vertical axis, these graph as:



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Clearly the $v \leq 5x_2 = 5 – 5x_1$ is redundant C (corresponding to the fact that C is dominated for Column), and v is maximized at the intersection of the two other constraints, where $v = 4 – 5x_1 = 5x_1 – 3$, so $x_1^* = 7/10$, $x_2^* = 3/10$, and $v^* = ½$.

(d) Does Column have any dominated strategies? If so, how do they show up in your graph from (c)?

(d) Yes, L is dominated by C for Column, and it shows up in that the associated constraint in Row's security-level-maximizing problem is redundant.

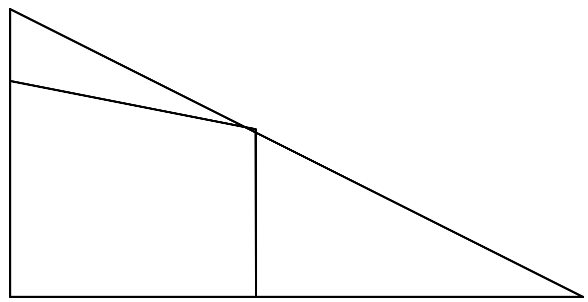
6. Consider the problem:

Choose x_1 and x_2 to solve maximize $3x_1 + 15x_2$ subject to

$$x_1 + 10x_2 \leq 20$$
$$x_1 \leq 2$$
$$x_1 \geq 0,\ x_2 \geq 0$$

(a) Solve the problem graphically (with x_2 on the vertical axis).

(a) $x_1^* = 2$, $x_2^* = 1.8$ (from $x_1 = 2$ and $x_1 + 10x_2 = 20$).



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(b) Now solve the problem graphically when x_1 (but not x_2) must be an integer.

(b) Same solution as (a), because the solution with no integer restrictions has $x_1^* = 2$.

(c) Now solve the problem graphically when x_2 (but not x_1) must be an integer.

(c) There's a new solution, because the solution with no integer restrictions has $x_2^* = 1.8$. Graphically, the feasible region is like a zebra with horizontal stripes at integer values of x_2; by inspection $x_2 = 2$, $x_1 = 0$ is optimal.

(d) Now use the branch and bound method to solve the problem when both x_1 and x_2 must be integers.

(d) First solve the relaxed problem without integer restrictions, as in (a). Then branch on the first variable (as the x_i are numbered) that is restricted to be an integer, and isn't an integer in the solution to the relaxed problem: $x_2 = 1.8$. Branch by creating two subproblems from the original problem, one with added constraint (i) $x_2 \leq 1$ and one with added constraint (ii) $x_2 \geq 2$.

Next, attempt to fathom the two branch subproblems, starting with the first, by solving their relaxed versions.

Solving branch subproblem (i) yields x_1 = 2, x_2 = 1; this satisfies the integer restrictions, so this branch is fathomed, and x_1 = 2, x_2 = 1 becomes the incumbent solution, with objective function value 21, a lower bound on the optimal value. Branch subproblem (ii) has only one feasible point: x_1 = 0, x_2 = 2, which satisfies the integer restrictions and is therefore optimal in this branch, so this branch is fathomed, and x_1 = 0, x_2 = 2, which has objective function value 30 > 21, becomes the new incumbent solution. Because all branches are now fathomed, the latest incumbent solution is optimal.