1	2	3	4	5	6	Total
/10	/15	/15	/20	/20	/20	/100

Economics 172A Final Exam **NAME________________________________**
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Your grade from this exam is 55% of your course grade. The exam ends at 11:00, so you have three hours. You may not use books, notes, calculators or other electronic devices. There are six questions, weighted as indicated. Answer them all. If you cannot give a complete answer, try to explain what you understand about the answer. Write your name in the space above, now. Write your answers below the questions, on the back of the page, or on separate sheets. Explain your arguments and show your work. Good luck!

1. Graph the feasible region and use your graph to solve the following problem:

Minimize $Z = 3x_1 + 2x_2$,
subject to
$2x_1 + x_2 \geq 10$
$-3x_1 + 2x_2 \leq 6$
$x_1 + x_2 \geq 6$
$x_1 \geq 0, x_2 \geq 0.$

2. Slim-Down Manufacturing makes a line of nutritionally complete weight-reduction beverages. One of their products is a strawberry shake which is designed to be a complete meal. The strawberry shake consists of several ingredients. Some information about each of these ingredients is given in the table below.

Ingredient	Calories from fat (per tbsp)	Total Calories (per tbsp)	Vitamin Content (mg/tbsp)	Thickeners (mg/tbsp)	Cost (¢/tbsp)
Strawberry flavoring	1	50	20	3	10
Cream	75	100	0	8	8
Vitamin supplement	0	0	50	1	25
Artificial sweetener	0	120	0	2	15
Thickening agent	30	80	2	25	6

The nutritional requirements are as follows. The beverage must total at least 380 calories. No more than 20% of the total calories must come from fat. There must be at least 50 milligrams (mg) of vitamin content. There must be at least two tablespoons (tbsp) of strawberry flavoring for each tbsp of artificial sweetener. Finally, there must be exactly 15 mg of thickeners in the beverage. Management would like to select the quantity of each ingredient for the beverage which would minimize cost while meeting the above requirements.

(a) Formulate a linear programming model for this problem, and put the constraints into standard "≥ constant" or "= constant" form. Please use the following notation for the decision variables: S = Tablespoons of strawberry flavoring, C = Tablespoons of cream, V = Tablespoons of vitamin supplement, A = Tablespoons of artificial sweetener, T = Tablespoons of thickening agent, and Z = Total cost.

(b) Write the dual of your linear programming problem from part (a), using Y to denote the dual objective function value and F, G, H, I, and J to denote the dual variables associated with the five primal constraints in the order listed in the verbal statement (so that F is the dual variable associated with the primal constraint that says the beverage must total at least 380 calories, G is the dual variable associated with the primal constraint that says no more than 20% of the total calories must come from fat, and so on). Make sure you have the primal constraints in standard "≥ constant" or "= constant" form before you do this. There is no need to explain your answer here, as long as it is correct; but if you're unsure, explanations of why you did what you did it might help.

(c) Suppose you have solved the primal, and you find that for the optimal values of S, V, and T, $20S + 50V + 2T > 50$. What must be true of the optimal value of H in the dual?

(d) What is the interpretation of the optimal value of F in the dual for how the minimized value of the primal objective function when the data of the problem (objective function coefficients and/or constraint constants) change? What must be true about the optimal basis before and after a change in the data of the problem for this interpretation to be exact (rather than an approximation)?

3. Consider the problem choose x (a scalar) to solve minimize 2x subject to

$$x \le 5$$
$$x \ge b$$
$$x \ge 0$$

where b (also a scalar) ≥ 0.

(a) For what values of $b \ge 0$ does the problem have a nonempty feasible region? For what values of $b \ge 0$ does the problem have a solution?

(b) For all values of $b \ge 0$ for which the problem has a solution, graphically or by inspection, whichever you prefer, write $x^*(b)$, the optimal value of x, as a function of the parameter b. Your answer must tell what the optimal value of x is for any value of $b \ge 0$; that is, it must be a clearly specified function of b.

(c) Put the primal constraints into standard ("$\ge$ constant") form and write the dual, using y_i to represent the dual variable that is the shadow price of the ith constraint in the primal.

(d) For what values of $b \ge 0$ does the dual have a nonempty feasible region?

(e) For what values of $b \ge 0$ does the dual have a solution? For those values, graphically or by inspection, compute $y_1^*(b)$ and $y_2^*(b)$, the optimal values of y_1 and y_2, as functions of the parameter b. Your answer must tell what the optimal values of y_1 and y_2 are for any value of b.

(f) Use the Duality Theorem to show that your solutions to the primal in (b) and the dual in (c) are both optimal for all values of b for which the primal and dual have solutions.

(g) For all values of b for which the primal in (a) and the dual in (c) have solutions, verify directly that your solutions to the primal and dual satisfy Complementary Slackness, saying clearly what Complementary Slackness requires.

(h) For all values of b for which the primal in (a) and the dual in (c) have solutions, compute V(b), the maximized value of the primal objective function, as a function of b. Check that $V'(b) = y_2^*(b)$.

4. Consider an assignment problem with four workers, A, B, C, and D, and three jobs, 1, 2, and 3:

	1	2	3
A	4	6	5
B	3	-1	7
C	9	2	8
D	7	6	9

(a) Put into standard form for the Hungarian Method, increasing or decreasing costs and creating dummy workers or dummy jobs as necessary. (For ease of grading, please do your costs increases or decreases before creating dummy workers or jobs.) Explain why your cost increases or decreases don't distort the optimal assignment of workers to jobs. Explain why your assignment of costs to dummy workers or jobs doesn't distort the optimal assignment of real workers to real jobs.

(b) Start to solve the problem in standard form by the Hungarian Method, by doing row reduction first and then column reduction. Explain why row and column reduction don't distort the optimal assignment of workers to jobs. (You are *not* asked to keep track of the dual variables.)

(c) Continue solving the problem by identifying a maximal set of independent zeros (for ease of grading, please identify them by *s as I did in class) and a minimal cover with the same number of lines (please identify them by +s as I did in class). Identify the smallest uncovered entry and do a pivot step to obtain a new reduced cost matrix. Use the new matrix to find the optimal assignment, and calculate its cost in the original matrix.

(d) Solve the original assignment problem (with four workers, three jobs, and a negative cost) by the branch and bound method, without putting it into standard form. Branch by solving the relaxed version of the problem in which you can fill each job with whomever you wish, without regard to duplication. Then branch on how to fill job 1, job 2, etc.

5. Consider the two-person zero-sum game, with only the Row player's payoffs shown:

	L	**C**	**R**
T	**0**	**-1**	**2**
B	**5**	**4**	**-3**

(a) Restricting attention to the pure (unrandomized) strategies, T and B for Row and L, C, and R for Column, find the Row player's security-level-maximizing pure strategy and his associated security level. Find the Column player's security-level-maximizing pure strategy and her associated security level. Taking into account that Column's payoffs are minus Row's payoffs, are these security levels consistent (that is, could both players realize them simultaneously playing the game)?

(b) Now consider mixed (randomized) strategies. Letting x_1 and x_2 denote the probabilities with which Row plays his strategies T and B, respectively, and letting v denote his resulting security level, write the linear programming problem that determines Row's security-level maximizing mixed strategy.

(c) Letting $x_2 = 1 - x_1$ and simplifying the constraints, solve the problem in (b) graphically, and identify the optimal values of x_1, x_2, and v.

(d) Does Column have any dominated strategies? If so, how do they show up in your graph from (c)?

6. Consider the problem:

Choose x_1 and x_2 to solve maximize $3x_1 + 15x_2$ subject to

$$x_1 + 10x_2 \le 20$$
$$x_1 \le 2$$
$$x_1 \ge 0,\ x_2 \ge 0$$

(a) Solve the problem graphically (with x_2 on the vertical axis).

(b) Now solve the problem graphically when x_1 (but not x_2) must be an integer.

(c) Now solve the problem graphically when x_2 (but not x_1) must be an integer.

(d) Now use the branch and bound method to solve the problem when both x_1 and x_2 must be integers.