Economics 172A: Introduction to Operations Research **Winter 2008**
Problem set 2 **Due Thursday, March 13 at start of class (no late papers)**

Instructions

Unless otherwise noted, you are required to supply complete answers and explain how you got them. Simply stating a numerical answer is insufficient. For graphs, clearly label the graph and identify what's on it.

1. Consider the problem choose x (a scalar) to solve maximize 3x subject to

$$x \le 5$$
$$x \le b$$
$$x \ge 0$$

where b (also a scalar) > 0.

(a) Graphically or by educated guess, whichever you prefer, compute x*(b), the optimal value of x, as a function of the parameter b. Your answer must tell what the optimal value of x is for any value of b > 0; that is, it must be a clearly specified function of b.
(Hint: Given that b > 0, could there ever be a solution with x* = 0? Which constraint would you expect to determine the solution when b < 5? When b > 5?)

(a) Clearly x*(b) > 0 for all b > 0. For b > (<) 5, the second (first) constraint is redundant. One of these constraints must always be binding, so x*(b) = b if b ≤ 5, and x*(b) = 5 if b > (or ≥) 5.

(b) Write the dual, using y_i to represent the dual control variable that is the shadow price of the ith constraint in the primal.

(b) Choose y_1, y_2 to solve minimize $5y_1 + by_2$ subject to

$$y_1 + y_2 \ge 3$$
$$y_1 \ge 0$$
$$y_2 \ge 0.$$

(c) Graphically or by educated guess, whichever you prefer, compute $y_1^*(b)$ and $y_2^*(b)$, the optimal values of y_1 and y_2, as functions of the parameter b. Your answer must tell what the optimal values of y_1 and y_2 are for any value of b. (When b = 5, identify all the possible optimal values of $y_1^*(b)$ and $y_2^*(b)$.)

(c) If b < (or ≤) 5, $y_1^*(b) = 0$ and $y_2^*(b) = 3$. If b > (or ≥) 5, $y_1^*(b) = 3$ and $y_2^*(b) = 0$. If b = 5, any y_1 and y_2 with $y_1 + y_2 = 3$ and $y_1 \ge 0$ and $y_2 \ge 0$ is optimal, including but not limited to the values specified above.

(d) Use the Duality Theorem to show that your solutions to the primal in (a) and the dual in (c) are both optimal for all values of b.

(d) The primal objective function value is 3b if $b \leq 5$ and 15 if $b > 5$. The dual objective function value is 3b if $b \leq 5$ and 15 if $b > 5$. Since the solutions are feasible for the primal and the dual, and the objective function values are equal in each case, the Duality Theorem shows that they are both optimal in each case.

(e) Verify directly that your solutions to the primal in (a) and the dual in (c) satisfy Complementary Slackness, saying clearly what Complementary Slackness requires.

(e) Complementary Slackness requires that if a primal (dual) control variable is strictly positive, then the associated dual (primal) constraint must be binding; and that if a primal (dual) constraint is slack (non-binding), then the associated dual (primal) control variable must be zero. Checking: (i) $y_1^*(b) + y_2^*(b) = 3$ for all b, so CS in the dual constraint is satisfied; (ii) if $b < 5$, $x^*(b) < 5$ but $y_1^*(b) = 0$, and $x^*(b) = b$, so CS in the primal constraints are satisfied; and (iii) if $b > 5$, $x^*(b) = 5$, and $x^*(b) < b$ but $y_2^*(b) = 0$, so CS in the primal constraints are again satisfied.

(f) Compute V(b), the maximized value of the primal objective function, as a function of b. Graph V(b), and check that its slope = $y_2^*(b)$ for almost all values of b. What happens to the optimal basis that makes V(b) have a kink at b = 5? What is the relationship between the slopes to the left and right of the kink and the possible optimal values of $y_2^*(b)$ when b = 5?

(f) Form (d), V(b) = 3b if $b \leq 5$ and 15 if $b > 5$. (Easy graph, not shown.) $y_2^*(b) = 3$ when $b < 5$, and 0 when $b > 5$, in each case $y_2^*(b) = V'(b)$. V(b) has a kink at b = 5 because the optimal basis changes there. The slopes to the left and right of the kink are 3 and 0, respectively. Thus the possible optimal values of $y_2^*(b)$, which range from 0 to 3, are exactly the slopes between those to the left and right of the kink.

2. Consider the problem:

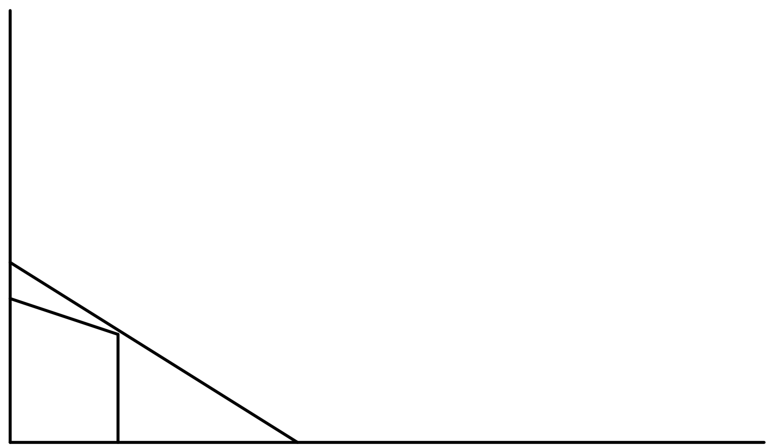
Choose x_1 and x_2 to solve maximize $x_1 + 6x_2$ s.t. $x_1 \leq 3$

$x_1 + 10x_2 \leq 20$

$x_1 \geq 0,\ x_2 \geq 0$

(a) Solve the problem graphically (here and below, with x_2 on the vertical axis) when there are no integer restrictions.

(a) $x_1^* = 3$, $x_2^* = 1.7$ (from $x_1 = 3$ and $x_1 + 10x_2 = 20$).



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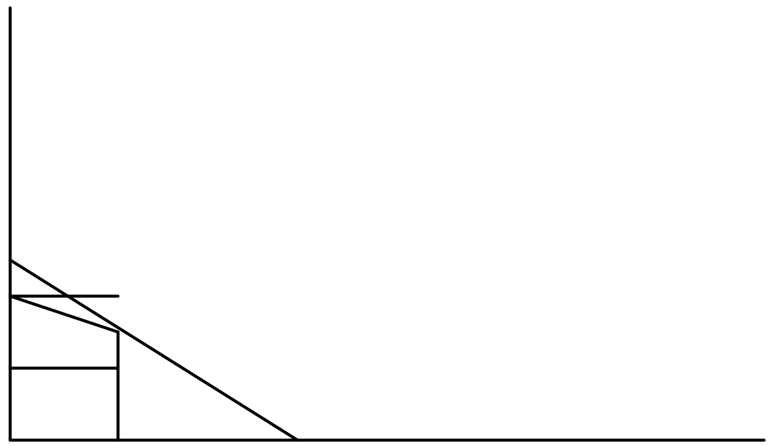
(b) Now solve the problem graphically when x_1 (but not x_2) must be an integer.

(b) Same solution as (a), because the solution with no integer restrictions has $x_1^* = 3$.

(c) Now solve the problem graphically when x_2 (but not x_1) must be an integer.

(c) There's a new solution, because the solution with no integer restrictions has $x_2^* = 1.7$. (Because there is only one variable with an integer restriction, it can be shown that the optimal x_2 respecting the integer restrictions is adjacent to 1.7, either 1 or 2.) Graphically, the feasible region is like a zebra with horizontal stripes at integer values of x_2. It's clear by inspection that $x_1 = 0$, $x_2 = 2$ is optimal.



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(d) Now use the branch and bound method to solve the problem when both x_1 and x_2 must be integers.

(d) Set v_ = −∞. First solve the relaxed problem without integer restrictions, as in (a). Then branch on the first variable (as the x_i are numbered) that is restricted to be an integer, and isn't an integer in the solution to the relaxed problem: $x_2 = 1.7$. Branch by creating two subproblems from the original problem, one with added constraint (i) $x_2 \leq 1$ and one with added constraint (ii) $x_2 \geq 2$.

Next, attempt to fathom the two branch subproblems, starting with the first, by solving their relaxed versions.

Solving branch subproblem (i) yields $x_1 = 3$, $x_2 = 1$; this satisfies the integer restrictions, so this branch is fathomed. $x_1 = 3$, $x_2 = 1$ becomes the incumbent solution, with objective function value 9, a lower bound on optimal value. Set v_ = 9.

Branch subproblem (ii) has only one feasible point: $x_1 = 0$, $x_2 = 2$, which satisfies the integer restrictions and is therefore optimal in this branch, so this branch is fathomed, and $x_1 = 0$, $x_2 = 2$, which has objective function value 12 > 9, becomes the new incumbent solution. Set v_ = 12. (You can also see that this solution is better than the previous incumbent solution by mentally shifting the objective function contour downward in the graph for part (c).)

Because all branches are now fathomed, the latest incumbent solution is optimal.

3. Reconsider the job assignment problem from Problem Set 1: You must assign three people, A, B, and C, to fill five jobs, 1, 2, 3, 4, and 5. Each person *must* be given either one or two jobs, but you are otherwise free to make the assignment in any way you like. The costs are given in the following table; if a person is assigned to two jobs, the total cost of that part of the assignment is computed by adding the costs for the two jobs. c_{ij} is the cost of having worker i assigned to job j. In the table, I have already cloned each worker (A becoming a and A, and so on) as needed to do the problem by linear programming.

	1	2	3	4	5
a	4	9	3	5	3
A	4	9	3	5	3
b	3	6	2	6	1
B	3	6	2	6	1
c	1	7	7	3	4
C	1	7	7	3	4

(a) What else do you need to do to this problem, if anything, to formulate it as an optimal assignment problem suitable for the Hungarian Method? Do it, and explain why your change yields a problem whose solution will yield the solution to the original problem.

(a) You need to create a dummy job to make the numbers of people and jobs equal. As long as the cost of that job is the same for all workers, your solution to the problem will be a solution to the original problem, where it doesn't matter, other things equal, who is not assigned.

(b) Solve the reformulated problem by the Hungarian Method, explaining your steps.

(b) First create the new cost matrix with the dummy job:

	1	2	3	4	5	6
a	4	9	3	5	3	0
A	4	9	3	5	3	0
b	3	6	2	6	1	0
B	3	6	2	6	1	0
c	1	7	7	3	4	0
C	1	7	7	3	4	0

Because there's a 0 in every row, row reduction doesn't change the matrix at all. Column reduction gives you the completed reduced cost matrix:

	1	2	3	4	5	6
a	3	3	1	2	2	0*
A	3	3	1	2	2	0
b	2	0*	0	3	0	0
B	2	0	0*	3	0	0
c	0*	1	5	0	3	0
C	0	1	5	0*	3	0

Next, find a maximal set of k independent zeros and a minimal cover of k lines, for example the zeros with *s in the matrix, which are all covered by a single vertical line in column 6 and four horizontal lines in columns b, B, c, and C. The fact that these five zeros can all be covered by five lines means that there cannot be more than five independent zeros in the matrix.

Next, subtract the smallest cost entry that is not covered by your k lines from all entries not covered by a line, and add that same number to all entries covered by two lines. The smallest uncovered entry is one of the 1s in column 3 (doesn't matter which one you choose). The result is the new cost matrix:

	1	2	3	4	5	6
a	2	2	0	1	1	0*
A	2	2	0*	1	1	0
b	2	0*	0	3	0	1
B	2	0	0	3	0*	1
c	0*	1	5	0	3	1
C	0	1	5	0*	3	1

Again, find a maximal set of k independent zeros and a minimal cover of k lines, for example the new set of zeros with *s in the matrix, which are all covered by either six horizontal or six vertical lines. That there are six independent zeros (the number of people and jobs) implies by Complementary Slackness that the assignment identified by the *s is optimal. Its total cost, from the original matrix, is 1 + 6 + 3 + 3 + 1 + 0 = 14. There are multiple solutions, but the *s here identify the same optimal assignment that Solver should have given you in your answer to Problem Set 1 #3(c).

(c) Now suppose, as in Problem Set 1 #3(d), that job 5 has been eliminated, but the rest of the problem is unchanged. In this case you should have found that the problem cannot be formulated as a linear programming problem, because the way we did this in the first part (and any other way anyone has ever thought of) might assign the two dummy jobs to the same person, which is not really feasible (in fact this happens if you try to do it this way). Can you, nonetheless, do this version of the problem by the branch and bound method? If so, do it and illustrate at least the first couple of steps, explaining what you are doing.

(c) Yes. Start with the original matrix, without job 5 but still cloning workers:

	1	2	3	4
a	**4**	**9**	**3**	**5**
A	**4**	**9**	**3**	**5**
b	**3**	**6**	**2**	**6**
B	**3**	**6**	**2**	**6**
c	**1**	**7**	**7**	**3**
C	**1**	**7**	**7**	**3**

Set $\bar{v} = \infty$. First, solve the relaxed version of the problem in which you can fill each job with whomever you wish, without regard to duplication. (You could also do this by assigning each worker however you wish, but doing it the way I suggest is easier and more natural.) This yields the assignment (for example) c1, b2, B3, C4. It's not feasible because A is not assigned—recall that each person *must* be given either one or two jobs—so it yields no bound on attainable cost and nothing is yet fathomed. Next, branch on how to fill job 1. (Again, this is not the only way to do it, but the most promising.) There are only three branch subproblems, using the equivalence of a1 and A1 etc.: a1, b1, c1.

In branch a1, solving the relaxed version of the problem in which you imagine you can fill each job other than job 1 with whomever you wish other than person a—crossing out job 1 and person a but otherwise not enforcing feasibility in the rest of the problem—yields a1, b2, B3, c4, which is feasible, so this branch is fathomed, and a1, b2, B3, c4 becomes the incumbent solution, with total cost 4 + 6 + 2 + 3 = 15. Set $\bar{v}$ = 15.

	1	2	3	4
a	**4**	**9**	**3**	**5**
A	**4**	**9**	**3**	**5**
b	**3**	**6**	**2**	**6**
B	**3**	**6**	**2**	**6**
c	**1**	**7**	**7**	**3**
C	**1**	**7**	**7**	**3**

In branch b1, solving the relaxed version of the problem yields b1, B2, B3, c4, which is infeasible and yields total cost 3 + 6 + 2 + 3 = 14 < $\bar{v}$ = 15, so this branch is not yet fathomed.

In branch c1, solving the relaxed version of the problem yields c1, B2, B3, c4, which is infeasible and yields total cost 1 + 6 + 2 + 3 = 12 < $\bar{v}$ = 15, so this branch is not yet fathomed.

Branch a1 is fathomed, so branch further on the next (or last-checked, whichever you prefer) unfathomed branch, say b1.

	1	2	3	4
a	4	9	3	5
A	4	9	3	5
b	3	6	2	6
B	3	6	2	6
c	1	7	7	3
C	1	7	7	3

Branching within b1 on how to fill job 2 yields three subbranches: b1, a2; b1, B2; and b1, c2.

In branch b1, a2, solving the relaxed version of the problem in which you imagine you can fill each job other than 1 and 2 with whomever you wish other than b and a—crossing out jobs 1 and 2 and persons b and a but otherwise not enforcing feasibility—yields b1, a2, B3, c4, which is feasible, so this branch is fathomed, and with total cost 3 + 9 + 2 + 3 = 17 > $\bar{v}$ = 15 the optimal solution cannot be in this branch.

In branch b1, B2, solving the relaxed version of the problem yields b1, B2, a3, c4, which is feasible and yields total cost 3 + 6 + 3 + 3 = $\bar{v}$ = 15, so this branch is fathomed. The objective function value is that same as for the incumbent solution a1, b2, B3, c4, so there is no need to replace it or to update $\bar{v}$.

In branch b1, c2, solving the relaxed version of the problem yields b1, c2, B3, C4, which is infeasible and yields total cost 3 + 7 + 2 + 3 = $\bar{v}$ = 15, so this branch is fathomed.

[Corrected: Branch b1 is now completely fathomed, but branch c1 remains unfathomed. Branching within c1 on how to fill job 2 yields three subbranches: c1, a2; c1, B2; and c1, C2.

	1	2	3	4
a	4	9	3	5
A	4	9	3	5
b	3	6	2	6
B	3	6	2	6
c	1	7	7	3
C	1	7	7	3

In branch c1, a2, solving the relaxed version of the problem in which you imagine you can fill each job other than 1 and 2 with whomever you wish other than b and a—crossing out jobs 1 and 2 and persons c and a but otherwise not enforcing feasibility—yields b1, a2, B3, c4, which is feasible, so this branch is fathomed, and with total cost 3 + 9 + 2 + 3 = 17 > $\bar{v}$ = 15 the optimal solution cannot be in this branch.

In branch c1, B2, solving the relaxed version of the problem yields c1, B2, b3, C4, which is infeasible and yields (hypothetical) total cost 1 + 6 + 2 + 3 = 12 < $\bar{v}$ = 15, so this branch is not yet fathomed.

In branch c1, C2, solving the relaxed version of the problem yields c1, C2, b3, a4, which is feasible and yields total cost 1 + 7 + 2 + 5 = 15 = $\bar{v}$ = 15, so this branch is fathomed, and no need to update $\bar{v}$ or the incumbent solution.

The only remaining unfathomed problem is branch c1, B2. Branching within c1, B2 on how to fill job 3 yields three subbranches: c1, B2, a3; c1, B2, b3; and c1, B2, c3.

	1	2	3	4
a	4	9	3	5
A	4	9	3	5
b	3	6	2	6
B	3	6	2	6
c	1	7	7	3
C	1	7	7	3

In branch c1, B2, a3, solving the relaxed problem yields c1, B2, a3, C4, which is feasible and yields total cost 1 + 6 + 3 + 3 = 13 < $\bar{v}$ = 15, so this branch is fathomed, and c1, B2, a3, C4 with total cost 13 becomes the new incumbent solution. Reset $\bar{v}$ = 13.

In branch c1, B2, b3, solving the relaxed problem yields c1, B2, b3, C4, which is infeasible and yields (hypothetical) total cost 1 + 6 + 2 + 3 = 12 < $\bar{v}$ = 13, so this branch is not yet fathomed.

In branch c1, B2, c3, solving the relaxed problem yields c1, B2, c3, a4, which is feasible and yields total cost 1 + 6 + 7 + 5 = 19 > $\bar{v}$, so this branch is fathomed.

The only remaining unfathomed branch is c1, B2, b3. Further subdividing yields two subbranches, c1, B2, b3, a4 and c1, B2, b3, c4. c1, B2, b3, a4 is feasible and yields 1 + 6 + 2 + 5 = 14 > 13, so this branch is fathomed. c1, B2, b3, c4 is infeasible, so this branch is fathomed because it contains no feasible assignments.

Because all branches have now been fathomed, the latest (and only) incumbent solution c1, B2, a3, C4 with total cost 13 is optimal.]

4. Consider Matching Pennies with the payoff to R (and from C) for matching on Heads raised from 1 to 2 (where the Column player's payoffs are minus the Row player's):

	H	T
H	2	-1
T	-1	1

(a) Write the linear programming problem that determines the Row player's maximin (security level maximizing) mixed strategy, letting v be the Row player's security level, p_1 = Pr{Row plays H}, and p_2 = Pr{Row plays T}. Explain why your problem's constraints ensure that the Row player's security level is at least v, no matter what pure or mixed strategy the Column player uses.

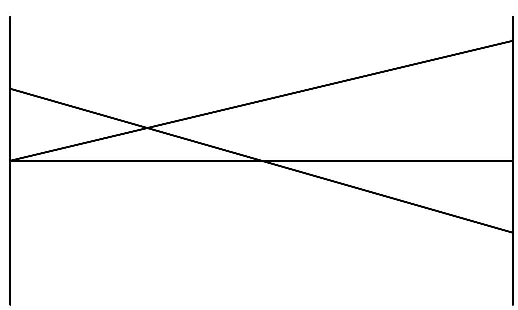
(a) Choose p_1, p_2 to maximize v subject to

$$v - 2p_1 + 1p_2 \le 0$$
$$v + 1p_1 - 1p_2 \le 0$$
$$p_1 + p_2 = 1$$
$$p_1, p_2 \ge 0.$$

The constraints ensure that Row's security level is at least v because any mixed strategy Column uses will yield Row expected payoffs that are a weighted average of the expected payoffs of his pure strategies, and these are constrained to be at least v.

(b) Solve the problem in (a) graphically, and identify the optimal values of p_1 and p_2.

(b) With $p_2 = 1 – p_1$, the constraints reduce to $v \le 2p_1 – p_2 = 3p_1 – 1$ and $v \le –p_1 + p_2 = 1 - 2p_1$. With p_1 on the horizontal axis, and v on the vertical axis, these graph as:



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Clearly v is maximized at the intersection of the two constraints, where $v = 3p_1 – 1 = 1 - 2p_1$, so $p_1^* = 2/5$ and $v^* = 1/5$.

(c) Use the analogous method (without showing details, unless you want to) to determine the Column player's optimal choice of q_1 = Pr{Column plays H}, and q_2 = Pr{Column plays T} and the resulting security level.

(c) The analogous graph shows that the optimal q_1 and q_2 equalize $2q_1 - q_2 = 3q_1 - 1$ and $-1q_1 + q_2 = 1 - 2q_1$, so $q_1^* = 2/5$ and $q_2^* = 3/5$.

(d) Comparing your solutions in (b) and (c) with the optimal mixed strategies in the standard, symmetric version of matching pennies (like this one, but with the payoff 2 changed to a 1), is Row's response to the increased payoff from matching on H intuitive? Is Column's response to Row's increased payoff from matching on Heads (and so Column's decreased payoff) intuitive? Why can't Row take advantage of the increased payoff by putting more rather then less probability on H? Why does he get a higher expected payoff, even though he puts less probability on H?

(d) Row's increased payoff from matching on H makes H a better strategy for him, other things equal, so Row's response is counterintuitive, while Column's is intuitive. If Row tried too hard for the higher payoff, and Column anticipated this, then Column could completely neutralize the benefits by setting $q_1 = 0$. Only by setting p_1 predictably and sufficiently less than ½ can Row give Column an incentive to set $q_1 > 0$, and only then can Row benefit from the higher payoff, obtaining a value 1/5 > his zero value in the symmetric version of the game.

(e) Now write the payoff matrix when R [I wrongly said C on the first version of this problem set] must choose between Heads and Tails first, and C can observe R's choice of pure strategy before making his own choice. Clearly identify players' pure strategies and explain your notation.

(e) R still has the same two pure strategies, H and T. C now has four pure strategies, (H if H, H if T) call it HH; (H if H, T if T) call it HT; TH; and TT. Deriving the new payoff matrix from the rules of the game:

	HH	HT	TH	TT
H	**2**	**2**	**-1**	**-1**
T	**-1**	**1**	**-1**	**1**

(f) As you did in (b), and using the same notation (v, p_1, and p_2), find R's security-level-maximizing strategy or strategies and his maximized security level, either graphically or by reasoning about the payoff matrix, or both. Find C's security-level-maximizing strategy or strategies and his maximized security level.

(f) You could draw a graph as for (b), with four lines, three of which would be redundant. Equivalently and more easily, you could just note that TH is a weakly dominant strategy for C, yielding at least as high a payoff as any other pure strategy and never a lower payoff. Thus whatever R does, it cannot raise his security level above -1, and any strategy, pure or mixed, is optimal for him. Similarly, C's security-level-maximizing strategy is TH and his maximized security level is 1.