Economics 172A: Introduction to Operations Research **Winter 2008**
Problem set 2 **Due Thursday, March 13 at start of class (no late papers)**

Instructions

Unless otherwise noted, you are required to supply complete answers and explain how you got them. Simply stating a numerical answer is insufficient. For graphs, clearly label the graph and identify what's on it.

1. Consider the problem choose x (a scalar) to solve maximize 3x subject to

$$x \le 5$$
$$x \le b$$
$$x \ge 0$$

where b (also a scalar) > 0.

(a) Graphically or by educated guess, whichever you prefer, compute x*(b), the optimal value of x, as a function of the parameter b. Your answer must tell what the optimal value of x is for any value of b > 0; that is, it must be a clearly specified function of b.
(Hint: Given that b > 0, could there ever be a solution with x* = 0? Which constraint would you expect to determine the solution when b < 5? When b > 5?)
(b) Write the dual, using y_i to represent the dual control variable that is the shadow price of the ith constraint in the primal.
(c) Graphically or by educated guess, whichever you prefer, compute y_1*(b) and y_2*(b), the optimal values of y_1 and y_2, as functions of the parameter b. Your answer must tell what the optimal values of y_1 and y_2 are for any value of b. (When b = 5, identify all the possible optimal values of y_1*(b) and y_2*(b).)
(d) Use the Duality Theorem to show that your solutions to the primal in (a) and the dual in (c) are both optimal for all values of b.
(e) Verify directly that your solutions to the primal in (a) and the dual in (c) satisfy Complementary Slackness, saying clearly what Complementary Slackness requires.
(f) Compute V(b), the maximized value of the primal objective function, as a function of b. Graph V(b), and check that its slope = y_2*(b) for almost all values of b. What happens to the optimal basis that makes V(b) have a kink at b = 5? What is the relationship between the slopes to the left and right of the kink and the possible optimal values of y_2*(b) when b = 5?

2. Consider the problem:

Choose x_1 and x_2 to solve maximize $x_1 + 6x_2$ s.t. $x_1 \leq 3$

$x_1 + 10x_2 \leq 20$

$x_1 \geq 0, x_2 \geq 0$

(a) Solve the problem graphically (here and below, with x_2 on the vertical axis) when there are no integer restrictions.

(b) Now solve the problem graphically when x_1 (but not x_2) must be an integer.

(c) Now solve the problem graphically when x_2 (but not x_1) must be an integer.

(d) Now use the branch and bound method to solve the problem when both x_1 and x_2 must be integers.

3. Reconsider the job assignment problem from Problem Set 1: You must assign three people, A, B, and C, to fill five jobs, 1, 2, 3, 4, and 5. Each person *must* be given either one or two jobs, but you are otherwise free to make the assignment in any way you like. The costs are given in the following table; if a person is assigned to two jobs, the total cost of that part of the assignment is computed by adding the costs for the two jobs. c_{ij} is the cost of having worker i assigned to job j. In the table, I have already cloned each worker (A becoming a and A, and so on) as needed to do the problem by linear programming.

	1	2	3	4	5
a	4	9	3	5	3
A	4	9	3	5	3
b	3	6	2	6	1
B	3	6	2	6	1
c	1	7	7	3	4
C	1	7	7	3	4

(a) What else do you need to do to this problem, if anything, to formulate it as an optimal assignment problem suitable for the Hungarian Method? Do it, and explain why your change yields a problem whose solution will yield the solution to the original problem.

(b) Solve the reformulated problem by the Hungarian Method, explaining your steps.

(c) Now suppose, as in Problem Set 1 #3(d), that job 5 has been eliminated, but the rest of the problem is unchanged. In this case you should have found that the problem cannot be formulated as a linear programming problem, because the way we did this in the first part (and any other way anyone has ever thought of) might assign the two dummy jobs to the same person, which is not really feasible (in fact this happens if you try to do it this way). Can you, nonetheless, do this version of the problem by the branch and bound method? If so, do it and illustrate at least the first couple of steps, explaining what you are doing.

4. Consider Matching Pennies with the payoff to R (and from C) for matching on Heads raised from 1 to 2 (where the Column player's payoffs are minus the Row player's):

	H	T
H	2	-1
T	-1	1

(a) Write the linear programming problem that determines the Row player's maximin (security level maximizing) mixed strategy, letting v be the Row player's security level, p_1 = Pr{Row plays H}, and p_2 = Pr{Row plays T}. Explain why your problem's constraints ensure that the Row player's security level is at least v, no matter what pure or mixed strategy the Column player uses.

(b) Solve the problem in (a) graphically, and identify the optimal values of p_1 and p_2.

(c) Use the analogous method (without showing details, unless you want to) to determine the Column player's optimal choice of q_1 = Pr{Column plays H}, and q_2 = Pr{Column plays T} and the resulting security level.

(d) Comparing your solutions in (b) and (c) with the optimal mixed strategies in the standard, symmetric version of matching pennies (like this one, but with the payoff 2 changed to a 1), is Row's response to the increased payoff from matching on H intuitive? Is Column's response to Row's increased payoff from matching on Heads (and so Column's decreased payoff) intuitive? Why can't Row take advantage of the increased payoff by putting more rather then less probability on H? Why does he get a higher expected payoff, even though he puts less probability on H?

(e) Now write the payoff matrix when R [I wrongly said C on the first version of this problem set] must choose between Heads and Tails first, and C can observe R's choice of pure strategy before making his own choice. Clearly identify players' pure strategies and explain your notation.

(f) As you did in (b), and using the same notation (v, p_1, and p_2), find R's security-level-maximizing strategy or strategies and his maximized security level, either graphically or by reasoning about the payoff matrix, or both. Find C's security-level-maximizing strategy or strategies and his maximized security level.