Economics 172A: Introduction to Operations Research **Winter 2008**
Problem set 1 Answers Due Thursday, January 31 at start of class (no late papers)

Instructions

Unless otherwise noted on homework assignments and on examinations, you are required to supply complete answers and explain how you got them. Simply stating a numerical answer is insufficient.

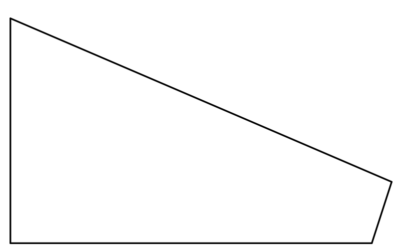
For this assignment, attach printouts of Excel spreadsheets when requested and indicate where to find the answers for each question the spreadsheet covers. This assignment asks you to solve many linear programming problems, but most are variations on the same basic problem. Set up one template for the Excel computations and then make changes to get the answers for variations of the problem. You need not include a separate printout for every simplex computation as long as you provide a clear description of how you got the answers. You are responsible for using the notes on Excel on the website to figure out how to get Excel answers yourself (I won't lecture on it).

For this assignment there is no need to provide answer reports and sensitivity reports, but please do indicate which cells on your spreadsheet have the solution. For graphs, clearly label the graph and show where the objective function is and how you identified a solution. If I ask you to solve a problem, please give both the solution (the optimal x) and the value (the objective function value for the optimal x).

1. Consider the linear programming problem:

Choose $x_1, x_2 \geq 0$ to solve max y subject to $-x_1 + 2x_2 \leq 5$
$3x_1 + x_2 \leq 3$,
where the objective function y is a function of x_1 and x_2 to be specified.

(a) Graph the feasible region. Put x_1 on the vertical axis and x_2 on the horizontal axis.
[When x_1 is on the vertical axis and x_2 is on the horizontal axis, the feasible region is a quadrilateral with the west and south edges going along the x_1 and x_2 axes, and with a point sticking out to the east at $(x_1,x_2) = (1/7, 18/7)$. It's the intersection of the nonnegative quadrant $(x_1,x_2 \geq 0)$, the area northwest of the line from $(x_1,x_2) = (-5,0)$ to $(0,5/2)$, and the area southwest of the line from $(x_1,x_2) = (1,0)$ to $(0,3)$. The corners, starting from the southwest and going clockwise, are $(0,0)$, $(1,0)$, $(1/7,18/7)$, $(0,5/2)$.]



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(b) Solve the problem graphically when:

(i) $y = x_2$.

[The solution when $y = x_2$ is at the point in the feasible region farthest to the east: the intersection of the two lines $-x_1 + 2x_2 = 5$ and $3x_1 + x_2 = 3$, easily found by algebra to be $(x_1,x_2) = (1/7, 18/7)$.]

(ii) $y = x_1 + x_2$.

[The solution when $y = x_1 + x_2$ is at the point in the feasible region farthest to the northeast (because the slope of the objective function contour, -1, is between the slopes of the constraints at this corner): again the intersection of the two lines at $(x_1,x_2) = (1/7, 18/7)$.]

(iii) $y = x_1 - x_2$.

[The solution when $y = x_1 - x_2$ is at the point in the feasible region farthest to the northwest (because the slope of the objective function contour, 1, is between the slopes of the constraints at this corner): $(x_1,x_2) = (1,0)$.]

(c) Identify the corners of the feasible region. For each corner, give an example of a linear objective function y (a linear function of x_1 and x_2) such that the solution of the problem occurs at (and only at) that corner.

[$(x_1,x_2) = (0, 0)$, for which the objective function $y = -x_1 - x_2$ will work. $(x_1,x_2) = (0, 5/2)$, for which the objective function $y = -x_1 + x_2$ will work. $(x_1,x_2) = (1/7, 18/7)$, for which the objective function $y = x_1 + x_2$ will work. $(x_1,x_2) = (1, 0)$, for which the objective function $y = x_1$ will work.]

(d) Now solve each of the problems in part (b) using Excel. Compare your answers to the graphical solutions. Are there any differences? Explain.

[The answers are the same, except that if there were multiple solutions (which there aren't here) Excel would not give you all of them: the particular solution the Excel finds will depend on how you entered the data.]

(e) Now multiply each of the objective functions in part (b) by 3. Solve the new problem (graphically or using Excel, whichever you prefer). How do the optimal x's and values change?

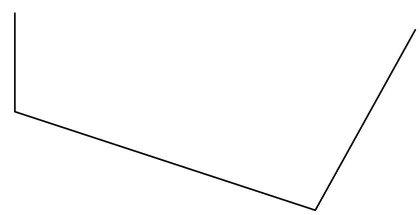
[The optimal x's don't change because the only way to max 3f(x) subject to x in S is to max f(x) subject to x in S. The optimal values are multiplied by 3.]

(Note that parts (e), (f), and (g) are independent. For example, when you do part (f) do not multiply the objective functions by 3: leave them as they were originally. Thinking should allow you to do (e)-(h) with little computation. But even if you don't see why, you should be able to do these parts and use them to understand the ideas they get at. Please be sure to compare the answers and comment on the changes as requested.)

(f) Now multiply the second constraint of the problem by 12 (so that it becomes $36x_1 + 12x_2 \leq 36$). With the new constraint, solve each of the problems in part (b) again (graphically or using Excel). How do the optimal x's and values change?
[The constraint hasn't really changed, so the optimal x's and values don't change.]

(g) Now multiply the coefficient of x_1 in each constraint of the problem (except the nonnegativity constraints) and in each of the objective functions in part (b) by 12. With these changes, solve each of the problems in part (b) again (graphically or using Excel). How do the optimal x's and values change?
[The "real" optimal x_1 won't change, but the "nominal" solution for x_1 will be multiplied by 1/12. The real and nominal solutions for x_2 and the value don't change. Multiplying the coefficient of x_1 by 12 in each constraint and in the objective function—everywhere in the problem—is just like measuring x_1 in feet rather than inches. (Not inches rather than feet! Make sure you understand why: because "12 inches" turns into "1 foot".) Because x_1 is now expressed in feet rather than inches, the constraints and the objective function trade-off between x_1 and x_2 haven't really changed, but the nominal solution for x_1 must be translated into the new language. Given the translation, the value doesn't change.]

(h) Now repeat part (f), except this time multiply as in (f) but by -12 instead of 12.
[This doesn't change the location of the second constraint's boundary line, but it flips which side of it you are allowed to be on. The new feasible region is the intersection of the nonnegative quadrant ($x_1,x_2 \geq 0$), the area northwest of the line from $(x_1,x_2) = (-5,0)$ to $(0,5/2)$, and the area *northeast* of the line from $(x_1,x_2) = (1,0)$ to $(0,3)$.



(not drawn to scale)

This is unbounded to the north, so the problem might not have a solution for some objective functions (but might have a solution for others). You can check, graphically or using Excel, that when $y = x_2$ or $y = x_1 + x_2$, there is no solution, because the feasible region allows anything to the northwest of the line $-x_1 + 2x_2 = 5$; and when $y = x_1 - x_2$ there is again no solution, because the region allows any x_1 as long as x_2 is not too high.]

2. Reconsider the formulation example discussed in class: A UCSD degree … and your own MS-burger (pronounced "Messburger") franchise! You have three "profit centers":

- the MS-Burger ("a quarter of a quarter of a pound of USDA choice beef on a fresh-baked bun")
- the Beefburger ("for the total carnivore, a USDA choice beef pattie on a 'bun' also made entirely of beef"), and
- the Breadburger ("a fresh-baked bun in … what else? ... another fresh-baked bun")

Producing x_1 MS-burgers, x_2 Beefburgers, and x_3 Breadburgers yields total profit (taking costs into account) of $60x_1 + 50x_2 + 10x_3$.

MS-burger CEO Joel Watson sends you $b_1 > 0$ quarter-of-a-quarter-of-a-pound units of beef and $b_2 > 0$ buns every month; it takes one unit of beef and one bun to make an MS-burger, two units of beef (and no buns) to make a Beefburger, and two buns (and no beef) to make a Breadburger.

(a) Formulate the linear programming problem that determines the profit-maximizing use of your monthly supply of beef and buns, assuming that the x_i must be nonnegative, but ignoring integer restrictions. Do not assume that all the beef or all the buns must be used.

[Choose $x_1,x_2,x_3 \geq 0$ to solve max $60x_1 + 50x_2 + 10x_3$ subject to $x_1 + 2x_2 + 0x_3 \leq b_1$
$x_1 + 0x_2 + 2x_3 \leq b_2$]

(b) Write the dual of the problem.

[Choose $z_1,z_2 \geq 0$ to solve min $b_1z_1 + b_2z_2$ subject to $z_1 + z_2 \geq 60$
$2z_1 \geq 50$
$2z_2 \geq 10$]

(c) Interpret the dual.

[The dual variables are the prices of inputs. For example, z_1, the dual variable associated with beef, is the value, measured in the units of the objective function, of an additional unit of beef. The constraints of the dual guarantee that it is at least as profitable to sell the inputs at their dual prices as to use them in the franchise. Each control variable in the dual is associated with a corresponding constraint in the primal, and each control variable in the primal is associated with a corresponding constraint in the dual: z_1 goes with the b_1 constraint, and x_1 with the "60" constraint.]

(d) Setting $b_1 = 20$ and $b_2 = 30$, solve the primal and the dual using Excel.

[Solution of primal (via Excel): $x_1 = 20$, $x_2 = 0$, $x_3 = 5$, $60x_1 + 50x_2 + 10x_3 = 1250$.
Solution of dual (via Excel): $z_1 = 55$, $z_2 = 5$, $20z_1 + 30z_2 = 1250$.]

(e) Compare your answers for the primal and the dual and confirm the conclusion of the duality theorem of linear programming and all complementary slackness conditions.
[You can check on your spreadsheet that:
(i) The values of the primal and dual are the same.
(ii) If a primal variable is positive, the associated dual constraint must bind (hold with equality).
(iii) If a primal constraint is not binding, the associated dual variable must be zero.
(iv) If a dual variable is positive, the associated primal constraint must bind.
(v) If a dual constraint is not binding, the associated primal variable must be zero.]

3. You must assign three people, A, B, and C, to fill five jobs, 1, 2, 3, 4, and 5. Each person *must* be given either one or two jobs, but you are otherwise free to make the assignment in any way you like. The costs are given in the following table; if a person is assigned to two jobs, the total cost of that part of the assignment is computed by adding the costs for the two jobs. c_{ij} is the cost of having worker i assigned to job j.

	1	2	3	4	5
A	4	9	3	5	3
B	3	6	2	6	1
C	1	7	7	3	4

(a) Show how to formulate this problem as a linear programming problem. (Hint: I found it helpful to create two mathematical duplicates of each worker (A1 and A2, and so on) and to use the variable x_{ij} to represent whether worker i is assigned to job j), with $x_{ij} = 1$ meaning that worker i is assigned to job j and $x_{ij} = 0$ meaning worker i is not assigned to job j.) Your formulation *must* include a definition of the variables and a clear statement (in both algebra and words) of the objective function and all relevant constraints.
[Choose the x_{ij} to solve Min $\Sigma c_{ij}x_{ij}$ where the summation is over all workers i and jobs j, subject to $\Sigma x_{ij} \geq 1$ for all j (so that at least one worker is assigned to each job) and $\Sigma x_{ij} \leq 1$ for all i (so that no worker is assigned more than one job, bearing in mind that duplicate workers are treated as different workers). The problem will necessarily leave exactly one worker-duplicate unemployed, and give jobs to both duplicates of two workers and one of the third. This satisfies the constraints.]

(b) Is a linear programming formulation fully appropriate? Comment on whether there are any important assumptions made in the formulating the problem as a linear program.
[No, in that the linear program allows x_{ij} to take values other than 0 or 1, which is not really feasible. But it turns out, as I'll explain later in the class, that that kind of potential infeasibility doesn't really matter for this kind of problem.]

(c) Use Excel to solve the problem.
[See the spreadsheet: A fills job 3, B fills jobs 2 and 5, C fills jobs 1 and 4; cost 14. The x_{ij} all come out (to a close approximation) 0 or 1, not a coincidence, as we'll see.]

(d) Now suppose that job 5 has been eliminated, but the rest of the problem is unchanged. Can this problem still be formulated as a linear programming problem? Explain why, or why not.
[It cannot validly be formulated as a linear programming problem. The problem is that now it's possible that the linear program will want to fill all four jobs with only two of the three workers, leaving the third with no job. This violates the constraint that each person must be given either one or two jobs. If you try it (the easiest way is by making all the costs of job 5 really big and then resolving the problem), you will find that the linear program assigns A only to job 5, violating the constraint that each person has at least one job. The actual solution must be feasible. What it is must be determined some way other than linear programming.]