Economics 109 Practice Problems 1, Vincent Crawford, Spring 2002

P1. Consider the following game. There are two piles of matches and two players. The game starts with Player 1 and thereafter the players take turns. When it is a player's turn, he can remove any number of matches from either pile. Each player is required to remove some number of matches if either pile has matches remaining, and he can only remove matches from a pile at a time. Whichever players removes the last match wins the game. Winning gives a player a payoff equal to 1, and losing gives a player a payoff equal to 0. Write down the strategic form of this game when the initial configuration of the piles has one match in one of the piles, and two in the other one.

P2. Consider the game Matching Pennies described and analyzed in class, but when the Row player R must go first and the Column player C has a spy who tells him, before C chooses his own action, whether the R is going to choose Heads or Tails. Assume that the spy can always predict R's choice correctly, and never lies. Also assume that both players know everything about the game, including the existence of the spy and exactly what information about R's choice the spy will give to C.
(a) Draw the game tree (or "extensive form") for this game. Indicate clearly each player's choices, the information he has when he makes them, and the payoffs that result from each combination of choices.
(b) Bearing in mind that a strategy must be a complete contingent plan for playing the game, list R's pure strategies. Then list C's pure strategies.
(c) Draw the payoff matrix of the game. (Please make R the Row player and C the Column player, and be sure to identify C's pure strategies clearly enough so that someone could tell from your description exactly what C is supposed to do in every possible situation.)
(d) Which, if any, of C's pure strategies are strictly dominated? Weakly dominated?
(e) Use your payoff matrix from (c) to identify which of Row's pure strategies become (weakly or strictly) dominated when C's dominated pure strategies are eliminated.
(f) Identify R's optimal strategy or strategies (all of them!) and his associated expected payoff. (There is more than one right way to do this. Pick the one that's easiest for you, but explain your argument.)
(g) Identify C's optimal strategy or strategies and his associated expected payoff, again explaining your argument.
(h) Show that any combination of R's and C's optimal strategies is in Nash equilibrium.
(i) How much is C's spy worth to him, in terms of expected payoff? (That is, compare his equilibrium expected payoffs with and without the spy)?

P3. Modify the original Battle of the Sexes game so that the man chooses his strategy first and the woman gets to observe his choice before choosing her strategy (and both know this). Give the game tree and payoff matrix that represent this game, and then find its Nash equilibrium or equilibria. How does your answer change (if at all) if the man chooses first but the woman does not get to observe his choice before choosing her strategy (and both know this)? (Hint: Is it possible, in this case, for the woman to base her choice on the man's choice?) How does your answer change (if at all) if the man goes first, the woman gets to observe his choice before making her own, but the man then gets to revise his choice if he wishes, and his decision to revise or not ends the game (and both know this)? Does the man's initial choice have any effect on the outcome in this case? Explain.

P4. Consider an ultimatum bargaining game with two players, R and C, and three possible contracts, A, B, and Z. The rules of bargaining allow R to propose one of these contracts, which C must then either accept or reject. If C accepts R's proposal, it determines the outcome; if he rejects it, the outcome is N (for "No Deal"). R's payoffs for the outcomes A, B, Z, and N are 1, 2, 3, and 0 respectively, and C's payoffs are 2, 1, -1, and 0 respectively. Clearly identifying each player's feasible pure strategies and making R the Row player, write the game tree and payoff matrix and identify the pure-strategy Nash equilibrium or equilibria, and the rollback or subgame-perfect equilibrium or equilibria, when:

(a) C can observe, before deciding whether to accept, which contract R has proposed

(b) C can observe, before deciding whether to accept, whether R has proposed Z or {either A or B}; but if R has proposed A or B, then C cannot tell which one. (For the game tree here, you don't have to re-draw the tree; just say how it must be changed from your tree for (a).)

(c) C cannot observe anything (except that R has proposed one of the three possible contracts) before deciding whether to accept. (Hint: Are there any rollback/subgame-perfect equilibria in which C accepts R's proposal with positive probability?)

(d) In which, if any, of the environments described in parts (a), (b), [and (c)] would both players benefit if contracts like Z were made legally unenforceable (so that if Z were proposed and accepted, the outcome would be N, not Z)? Explain.

P5. Two people, Rhoda ("R" for short) and Colin ("C") must decide independently whether to try to meet at the fights ("F") or the ballet ("B"). Before R and C decide where to try to meet, R (but not C) must announce her intentions, f or b, about where she plans to go. Then R and C choose, simultaneously and independently, between F and B. R is free to choose either F or B independent of her announcement f or b, but if she announces f but chooses B, or announces b but chooses F, she incurs a cost $c \geq 0$. If R and C both choose F, R's payoff is 2 less whatever cost she incurs, and C's payoff is 1. If R and C both choose B, R's payoff is 1 less whatever cost she incurs, and C's payoff is 2. If R chooses F but C chooses B, or vice versa, R's payoff is 0 less whatever cost she incurs, and C's payoff is 0. For example, if R announces f, chooses B, and C chooses B, R's payoff is 1-c. The structure of the game, including the announcement stage, is common knowledge.

(a) Clearly identifying players' decisions and information sets, draw the extensive form (game tree) for this game. (It's easier to draw the information sets if you put both of R's decisions first.)

(b) Identify the pure-strategy subgame-perfect equilibrium outcome(s) and payoffs when $c > 2$.

(c) Identify the pure-strategy subgame-perfect equilibrium outcome(s) and payoffs when $0 \leq c < 1$.

P6. (F'89 final) I have noticed that when two bicyclists meet going in opposite directions, they sometimes go into oscillations trying to avoid running into each other. (For example, one sometimes swerves to his right at the same time that the other swerves to his left, then they both swerve back in the opposite directions, and so on.) I have found that most of the time I can avoid these oscillations (and the ensuing crashes) by looking off in one direction and not meeting the other cyclist's eyes, so that he is not completely certain that I have seen him. (I usually look toward the side on which I hope to pass him, but it seems to work even when I look to the other side.) Explain this phenomenon as best you can, using whatever model or models you find helpful.

P7. (much too long for an exam question, but good practice) The government wishes to motivate two people, Algernon ("A" for short) and Bob ("B"), to register for the draft, but it has enough resources to prosecute only one person who fails to register. Assume that either person's payoff is 3 if he registers, and is therefore not prosecuted; 4 if he does not register, and is not prosecuted; and 0 if he does not register, and is prosecuted. Also assume that the structure of the game, including the rules described below, is common knowledge.

Suppose first that A and B must decide whether to register simultaneously, and the government announces that once A and B have had a chance to register, it will check up on them in alphabetical order (A first, then B), prosecuting only the first one (if any) who is found not to have registered.

(a) Treating only A and B (not the government) as players, write the extensive form (game tree) and normal form (payoff matrix, making A the Row player), clearly identifying players' strategies.
(b) Find A's and B's subgame-perfect equilibrium strategy profiles.
(c) Find A's and B's Nash equilibrium strategy profiles (all of them).
(d) Find A's and B's rationalizable strategies (all of them).

Now suppose that A and B must decide whether to register simultaneously, but the government instead checks up on A and B in random order (with each order equally likely), again prosecuting only the first one (if any) who is found not to have registered; everything else is the same as above.
(e) Answer part (a) again, for the new game.
(f) Answer part (b) again, for the new game.
(g) Answer part (c) again, for the new game.
(h) Answer part (d) again, for the new game.

Now suppose that A and B must decide whether to register sequentially, with A going first and B observing A's decision before making his own decision, and the government announces that once A and B have had a chance to register, it will check up on them in alphabetical order (A first, then B), prosecuting only the first one (if any) who is found not to have registered.

(i) Answer part (a) again, for the new game.
(j) Answer part (b) again, for the new game.
(k) Answer part (c) again, for the new game.
(l) Answer part (d) again, for the new game.

Finally, suppose that A and B must again decide whether to register sequentially, with A going first and B observing A's decision before making his own decision, but the government instead checks up on A and B in random order (with each order equally likely), prosecuting only the first one (if any) who is found not to have registered; everything else is the same as above.

(m) Answer part (a) again, for the new game.
(n) Answer part (b) again, for the new game.
(o) Answer part (c) again, for the new game.
(p) Answer part (d) again, for the new game.

P8. Modify the payoff matrix of Battle of the Sexes to reflect the assumption that, other things equal, the man prefers to be with the woman, but the woman prefers to avoid the man more than she prefers the ballet to the fights. (Choose whatever payoff numbers you like, as long as the order is correct.) Then find the Nash equilibrium or equilibria of the modified game.

P9. (fairly hard) Consider a two-person normal-form (payoff-matrix) game in which each player has a finite number of pure strategies.
(a) Prove that if the game can be reduced to a single strategy combination by iterated deletion of *weakly or strictly* dominated strategies (or a mixture of both), then that strategy combination is a Nash equilibrium in the original game.
(b) Prove that if the game can be reduced to a single strategy combination by iterated deletion of only *strictly* dominated strategies, then that strategy combination is the *unique* Nash equilibrium in the original game.
(c) Prove that if the game has no strictly dominated strategies (even if dominated only by mixed strategies), then all of each player's pure strategies are rationalizable.

P10. (a) What strategy would you play, against one randomly selected member of this class, in the Stag Hunt game presented in class. Explain.
(b) What strategy would you play if the game were modified to include (simultaneously) everyone in the class, with the cooperation of everyone required for success in hunting a stag, but each guaranteed success (still with lower payoff) if he chooses to hunt rabbits alone. Explain.

P11. Consider a Cournot duopoly game in which two firms, 1 and 2, simultaneously choose the quantities they will sell, q_1 and q_2, with the goal of maximizing their profits. Each firm's cost is c per unit sold, and the price each firm receives per unit sold, given q_1 and q_2, is $P(q_1, q_2) = a – b (q_1 + q_2)$, where $a > 0$ and $b > 0$. The structure of the game is common knowledge.
(a) Write firm i's profit-maximization problem and use the first- and second-order conditions to derive its best-response function, expressing its optimal output q_i as a function of firm j's output q_j.
(b) Identify the Nash equilibrium strategy profile(s) in this game.
(c) Which values of q_i are rational for firm i, for some feasible value of q_j?
(d) Which values of q_j are rational for firm j, for some value of q_i that is consistent with firm i being rational?
(e) Identify each firm's rationalizable strategies in this game. (Hint: This can be done using algebra, but I think it's easier to use your answer to (a) and a graphical argument.)
(f) Answer parts (a), (b), and (e) (yes, (e)) again for the analogous game with three firms, 1, 2, and 3, with unit cost c, outputs q_1, q_2, and q_3, and price per unit sold $P(q_1, q_2, q_3) = a – b (q_1 + q_2 + q_3)$.

P12. Find all pure strategy Nash equilibria in each of the following simultaneous-move games. In each case, comment on whether the prediction of the theory is reasonable. Where there are multiple equilibria, comment on how one might get selected.

(i)

		Bill: Orange	Bill: Crimson
Ann	Orange	1, 1	0, 0
Ann	Crimson	0, 0	1, 1

(ii)

		Bill: Left	Bill: Right
Ann	Up	100, 0.1	99, -0.1
Ann	Down	101, 0.1	0, -0.1

(iii)

Charles

C1

		Bill: B1	Bill: B2
Ann	A1	2,2,2	2,3,2
Ann	A2	3,2,2	1,1,0

C2

		Bill: B1	Bill: B2
Ann	A1	2,2,3	0,1,1
Ann	A2	1,0,1	1,1,1

In games (i) and (ii), the payoffs in each cell are in the order (Ann, Bill). In game (iii) the payoffs are in the order (Ann, Bill, Charles), and Charles's decision is to choose matrix C1 or matrix C2.

P13. Determine the Row and Column players' optimal strategies in the following zero-sum two-person games:

(a)

	L	C	R	FR
T	2	-3	-1	1
M	-1	1	-2	2
B	-1	2	-1	3

(b)

	L	C	R	FR
T	3	-3	-2	-4
M	-4	-2	-1	1
B	1	-1	2	0