Correlated Equilibrium and Higher Order Beliefs about Play

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Abstract

We study a refinement of correlated equilibrium in which players' actions are driven by their beliefs and higher order beliefs about the play of the game (beliefs over what other players will do, over what other players believe others will do, etc.). For any finite, complete-information game, we characterize the behavioral implications of this refinement with and without a common prior, and up to any a priori fixed depth of reasoning. In every finite game "most" correlated equilibrium distributions are consistent with this refinement; as a consequence, this refinement gives a classification of "most" correlated equilibrium distributions based on the maximum order of beliefs used by players in the equilibrium. On the other hand, in a generic two-player game any non-degenerate mixed-strategy Nash equilibrium is not consistent with this refinement.

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1 Introduction

There are two views on correlation in non-cooperative game theory. The classical view introduced by Aumann (1974) relies on external, payoff-irrelevant signals: players display correlated behavior because they condition their actions on correlated signals.

Recently, Brandenburger and Friedenberg (2008) have introduced an *intrinsic* view of correlation. According to this view, the correlation in players' actions only comes from correlation in their beliefs and higher order beliefs about the play of the game. This view of correlation is intrinsic, because the correlation in actions does not come from outside signals, but instead from players' beliefs inside the game.

Motivated by this intrinsic approach to correlation, we study the behavioral implications of equilibrium under intrinsic correlation; that is, we ask: which predictions of conventional, extrinsic correlated equilibria can be replicated by intrinsic correlated equilibria?

More precisely, for an arbitrary complete-information game, we work with a correlating device that consists of individual states and beliefs over the states. A strategy maps individual states to actions. We assume that players are rational, so we study equilibrium strategy which prescribes an optimal action given the belief about others' actions at an individual state. We further assume that players' actions are driven by their beliefs and higher order beliefs about the play of the game; formally, we restrict an equilibrium strategy to take the same action on individual states that generate the same hierarchy of beliefs about play. Our main results characterize the set of actions that can be played in equilibrium given this restriction.

Higher order beliefs about play have the interpretation as beliefs about players' motivations. A player's first order belief about play is a belief on other players' actions, i.e., a theory on their behaviors. His second order belief about play is a joint belief on other players' actions and on their first order beliefs about play. Thus, the second order belief posits a connection between other players' first order beliefs and their actions, which is a theory on their motivations: why do they take a particular course of action? Intrinsic correlation requires beliefs about motivations (as understood in this way) to determine the actions. Since it is natural for players to strategize based on these beliefs, we are led to analyze an equilibrium solution concept under the assumption of intrinsic correlation.

There is another reason to study intrinsic correlation. The conventional view of correlated equilibrium relies on the incompleteness in the model of a game, as the equilibrium requires signals which are not a part of the game's basic description (players, actions, and payoffs). Therefore, it would seem that once the game is fully specified there would be no room for correlated equilibrium. The study of intrinsic correlation clarifies that this is not the case: some correlated equilibria (the intrinsic ones) do not need to appeal to any incompleteness of the basic description. Even if there is nothing beyond the basic description, players' beliefs and higher order beliefs about each other's actions can serve as correlating devices for intrinsic correlated equilibrium. First order beliefs about actions are fundamental to a game, because they are needed for players to play optimal actions. And once the first order beliefs about actions are admitted in the game, it is natural to admit second and higher order beliefs about actions, since typically an action can be rationalized or motivated by multiple first order beliefs.

Our main finding is an iterative procedure: given any positive integer l and player i, the procedure finds the set of actions W_i^l that are rationalized by a unique l-th order belief about play. We prove that if (and only if) in an equilibrium player i's actions are driven by his l-th order beliefs about play, then all actions in W_i^l must have distinct first-order rationalizing beliefs about play. This characterization is useful because the first-order rationalizing beliefs are easy to determine, and by construction any action in any W_i^l must have a unique first-order rationalizing belief. And by letting l tend to infinity we get the behavioral implications of equilibrium whose actions are driven by players' hierarchies of beliefs about play, i.e., intrinsic correlated equilibrium.

As in Aumann (1974) we distinguish between subjective and objective correlated equilibrium, and variations of the previous iterative procedure work for both cases. When players' beliefs and higher order beliefs about the current play are shaped by a previous interaction or by a common event, these beliefs are likely to be consistent with each other (in the sense of Feinberg (2000), that disagreeing over these beliefs cannot be common knowledge). In this case to study intrinsic correlation, objective correlated equilibrium is the appropriate solution concept. On the other hand, if players' beliefs come from pure introspection, then they need not necessarily be consistent with each other, so it is appropriate to use subjective correlated equilibrium.

We study in detail the distribution on action profiles played by intrinsic objective correlated equilibrium. Our main conclusion is that "most" correlated equilibrium distributions are intrinsic.[1] As a consequence, we get a classification of "most" correlated equilibrium distributions based on the maximum order of beliefs used by players in the equilibrium: for every $l \geq 1$, let C^l be the set of correlated equilibrium distributions in which the actions are

[1] As a counterpoint, we also prove that in a generic two-player game, any (non-degenerate) mixed-strategy Nash equilibrium is not an intrinsic correlated equilibrium distribution.

driven by players' l-th order beliefs about play. This classification raises some new questions: for example, for a given l (in particular, a small l), how prevalent are the correlated equilibrium distributions in the set C^l? And how would the prevalence vary as we vary l? We believe that these questions are not merely mathematical curiosities, and answering them will inform us of the "intrinsic" complexity of correlated equilibria. On this direction we prove that the set C^l is always convex if and only if $l \geq 2$, but obviously a lot remains to be done.

We are directly inspired by Brandenburger and Friedenberg (2008); in Section 5.1 we carefully compare the solution concepts of the two papers. The essential methodological difference between the two papers is that Brandenburger and Friedenberg focus on rationalizability, while we focus on equilibrium. In terms of results, we go beyond Brandenburger and Friedenberg (2008) in three dimensions. First, we provide a characterization of the solution that only depends on the payoff structure of the game and that is independent of any type structure, which provides a partial answer to an open question posed by Brandenburger and Friedenberg (see Section 5.1). Second, we characterize the solution when players' actions are driven by their l-th order beliefs about play, for any $l \geq 1$. And third, we examine the implications of a common prior.

A contemporaneous and independent paper by Peysakhovich (2009) shows that actions played under an objective correlated equilibrium must be consistent with the intrinsic correlation solution of Brandenburger and Friedenberg (2008); this provides another partial answer to Brandenburger and Friedenberg's open question. Incidentally, Peysakhovich's main result has a natural interpretation in our formulation when we allow for private randomization by players contingent on their beliefs about play; we discuss this in detail in Section 5.2.

Our paper is also related to a recent literature on redundant types and solution concepts: Ely and Peski (2006), Dekel, Fudenberg and Morris (2007), Liu (2009), Sadzik (2009), and Battigalli, Di Tillio, Grillo and Penta (2011). The main difference is that authors in this literature study incomplete-information games, with type structure and redundant types on the payoff states, while we focus on complete-information games, with type structure and redundant types on the actions played in the game. It is an interesting future direction to consider the implications of our results for incomplete-information games with type structure on both actions and payoff states.

The paper proceeds as follows. In the next section we specify our formulation. Section 3 characterizes the behavioral implications of intrinsic subjective correlated equilibrium, and Section 4.1 characterizes the behavioral implications of intrinsic objective correlated equilibrium. Section 4.2 shows that "most" correlated equilibrium distributions are intrinsic while any non-degenerate mixed-strategy Nash equilibrium is not, and discusses a classification of correlated equilibrium distributions based on the maximum order of beliefs used by players in the equilibrium. Section 5 relates our solution concept to that in Brandenburger and Friedenberg (2008), shows a private-randomization extension of our result based on Peysakhovich (2009), and discusses the case of infinite games.

2 Formulation

We work with an arbitrary finite game of complete information: (N, A, u), where N is a finite set of players ($|N| \geq 2$), $A = \prod_{i \in N} A_i$ a finite set of action profiles, and $u = (u_i)_{i \in N}$ players' payoff functions: $u_i : A \to \mathbb{R}$ for each $i \in N$.

A *correlating device* is a tuple $(\Omega, P_i, \beta_i)_{i \in N}$, where $\Omega = \prod_{i \in N} \Omega_i$ is a finite or countably infinite set of states, and $P_i \in \Delta(\Omega)$ denotes player i's *prior belief* over Ω. [2] Each Ω_i denotes the set of player i's *individual states*. The private information at an individual state ω_i is summarized by $\beta_i(\omega_i)$, which is player i's *posterior belief* conditional on ω_i: $\beta_i : \Omega_i \to \Delta(\Omega_{-i})$. We require the posterior belief to be consistent with the prior belief, so β_i must satisfy the Bayes rule whenever possible: $\beta_i(\omega_i)(\omega_{-i}) = P_i(\omega_i, \omega_{-i})/P_i(\{\omega_i\} \times \Omega_{-i})$ whenever $P_i(\{\omega_i\} \times \Omega_{-i}) > 0$.

A player chooses an action contingent on his individual state; thus, a (pure) *strategy* is a function $\sigma_i : \Omega_i \to A_i$.

We write $\beta = (\beta_i)_{i \in N}$, $\sigma = (\sigma_i)_{i \in N}$ and $P = (P_i)_{i \in N}$. If P_i is the same for every player (the case of a common prior), we abuse the notation a bit by setting $P = P_i$ as well. Finally, if a property of ω_i holds whenever $P_i(\{\omega_i\} \times \Omega_{-i}) > 0$, we say that it holds P_i-almost surely.

Definition 1. An *a posteriori equilibrium* is a tuple (Ω, β, σ) such that for every player $i \in N$ and every individual state $\omega_i \in \Omega_i$, we have

$$\sum_{\omega_{-i} \in \Omega_{-i}} u_i(\sigma_i(\omega_i), \sigma_{-i}(\omega_{-i}))\beta_i(\omega_i)(\omega_{-i}) \geq \sum_{\omega_{-i} \in \Omega_{-i}} u_i(a_i, \sigma_{-i}(\omega_{-i}))\beta_i(\omega_i)(\omega_{-i}) \tag{1}$$

for all $a_i \in A_i$.

An *(objective) correlated equilibrium* is a tuple (Ω, P, σ) where P_i is the same for all players ($= P$), for every player i β_i is derived from P via Bayes rule P-almost surely, and (1) holds P-almost surely.

[2] For ease of exposition, we restrict to a countable set of states to avoid invoking measure theory.

An a posteriori equilibrium requires the optimality condition (1) to hold in the a posteriori stage, even after a probability-zero event. Aumann's (1974) *subjective correlated equilibrium*, similar to objective correlated equilibrium, instead maintains an ex ante perspective: the optimality condition (1) only needs to hold for ω_i such that $P_i(\{\omega_i\} \times \Omega_{-i}) > 0$.

Definition 2. A *type structure on actions* is a tuple $(T_i, \lambda_i)_{i\in N}$, where $\lambda_i : T_i \to \Delta(T_{-i}\times A_{-i})$ for each $i \in N$.

Fix an (a posteriori or correlated[3]) equilibrium (Ω, β, σ). The *equilibrium-induced type structure on actions* is the tuple $(T_i, \lambda_i)_{i\in N}$ such that $T_i = \Omega_i$, and for every $t_i^{\omega_i}$ (type in T_i corresponding to individual state $\omega_i \in \Omega_i$) we have:

$$\lambda_i(t_i^{\omega_i})(t_{-i}^{\omega_{-i}}, a_{-i}) := \begin{cases} \beta_i(\omega_i)(\omega_{-i}) & \text{if } a_{-i} = \sigma_{-i}(\omega_{-i}) \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

A type structure on actions $(T_i, \lambda_i)_{i\in N}$ is an implicit description of players' beliefs and higher order beliefs about each others' behavior in the game. To make the description explicit, let $\delta_i(t_i) = (\delta_i^1(t_i), \delta_i^2(t_i), \ldots)$ denote player i's *hierarchy of beliefs about actions (or play)* generated at type $t_i \in T_i$, where $\delta_i^l(t_i)$ is player i's *l-th order belief about actions (or play)* at t_i. The construction of δ_i^l is standard, and for completeness we briefly sketch one in the next page.

In this paper we study equilibrium in which players' actions are driven by their hierarchies of beliefs about play. Formally, for an a posteriori or correlated equilibrium, the hierarchies of beliefs about play are generated by the equilibrium-induced type structure.

Definition 3. An a posteriori equilibrium (Ω, β, σ) is *intrinsic* if the strategy function σ_i is constant on individual states with the same hierarchies of beliefs about play.

A correlated equilibrium (Ω, P, σ) is *intrinsic* if the strategy function σ_i is constant on individual states with the same hierarchies of beliefs about play, P-almost surely.

Naturally, in an intrinsic a posteriori or correlated equilibrium the players' actions may be driven solely by their l-th order beliefs about play:

Definition 4. In an a posteriori equilibrium (Ω, β, σ), players *condition their actions on their l-th order beliefs about play* if for every player i, the strategy function σ_i is constant on individual states with the same l-th order belief about play.

[3]In the case of correlated equilibrium, β_i is derived from the common prior P via Bayes rule P-almost surely.

In a correlated equilibrium (Ω, P, σ), *players condition their actions on their l-th order beliefs about play* if for every player i, the strategy function σ_i is constant on individual states with the same l-th order belief about play, P-almost surely.

For the sake of completeness, we sketch here a brief construction of hierarchies of beliefs about actions from a type structure on actions $(T_i, \lambda_i)_{i\in N}$, where $\lambda_i : T_i \to \Delta(T_{-i} \times A_{-i})$. Additional details may be found in Siniscalchi (2007).

For each $i \in N$, let $\mathcal{T}_i^1 := \Delta(A_{-i})$ be the set of player i's first order beliefs about actions. Let $\delta_i^1(t_i)$ be player i's first order belief about actions at type t_i, i.e.,

$$\delta_i^1(t_i) := \operatorname{marg}_{A_{-i}} \lambda_i(t_i),$$

where $\operatorname{marg}_{A_{-i}} \lambda_i(t_i)$ is the marginal distribution of $\lambda_i(t_i)$ on the set A_{-i}.

In general, fix an $l \geq 2$ and assume the induction hypothesis that players' $(l-1)$-th order beliefs about actions, $\delta_i^{l-1} : T_i \to \mathcal{T}_i^{l-1}$, are previously defined. Define $\mathcal{T}_i^l := \Delta(\mathcal{T}_{-i}^{l-1} \times A_{-i})$ to be the set of player i's l-th order beliefs about actions. Let $\delta_i^l(t_i)$ be player i's l-th order belief about actions at type t_i: $\delta_i^l(t_i)$ is the image measure of $\lambda_i(t_i)$ under the mapping $(\delta_j^{l-1}, \mathrm{id}_{A_j})_{j\neq i}$, or in other words $\delta_i^l(t_i) \in \Delta(\mathcal{T}_{-i}^{l-1} \times A_{-i})$, and for any measurable event $B \subseteq \mathcal{T}_{-i}^{l-1} \times A_{-i}$ we have:

$$\delta_i^l(t_i)(B) := \lambda_i(t_i)(((\delta_j^{l-1}, \mathrm{id}_{A_j})_{j\neq i})^{-1}(B)),$$

where id_{A_j} is the identity mapping on A_j.

3 Intrinsic A Posteriori Equilibrium

In this section we characterize the set of actions played by an intrinsic a posteriori equilibrium.

Let us first set up some notations. For a set of action profiles $Q = \prod_{i\in N} Q_i$, let

$$B_i^Q(a_i) := \{\mu \in \Delta(Q_{-i}) : a_i \text{ is optimal in } A_i \text{ for player } i \text{ under } \mu\}, \tag{3}$$

for every $i \in N$ and $a_i \in Q_i$. For any μ in $B_i^Q(a_i)$, we say that μ is a *rationalizing belief* of action a_i in Q_{-i}, and that μ *rationalizes* a_i.

Definition 5. A set of action profiles $Q = \prod_{i\in N} Q_i$ is a *best-response set* (BRS) if for every $i \in N$ and every $a_i \in Q_i$, there is a rationalizing belief of action a_i in Q_{-i}, i.e., $B_i^Q(a_i) \neq \emptyset$.

It is well-known that a set of action profiles $Q = \prod_{i\in N} Q_i$ is played by an a posteriori equilibrium (i.e., $Q_i = \sigma_i(\Omega_i)$ for every $i \in N$, for some a posteriori equilibrium (Ω, β, σ)) if and only if Q is a BRS (Brandenburger and Dekel (1987)). Our objective in this section is to characterize the refinement to BRS imposed by the intrinsicness criterion of Definition 3.

We now introduce a novel iterative construction on action sets that is intimately connected with the notion of intrinsicness. Fix a BRS Q. For every $i \in N$, define

$$\begin{aligned} W_i^1(Q) &:= \{a_i \in Q_i : B_i^Q(a_i) \text{ is a singleton}\}, \\ W_i^l(Q) &:= \{a_i \in W_i^{l-1}(Q) : B_i^Q(a_i)(W_{-i}^{l-1}(Q)) = 1\}, l \geq 2, \\ W_i(Q) &:= \bigcap_{l\geq 1} W_i^l(Q), \end{aligned} \tag{4}$$

where whenever $B_i^Q(a_i)$ is a singleton (i.e., $B_i^Q(a_i) = \{\mu\}$ for some belief μ), we write $B_i^Q(a_i)$ for the element contained within, (i.e., μ).

In the first line of (4), $W_i^1(Q)$ is the set of actions in Q_i that have a unique rationalizing belief in Q_{-i}. And inductively, $W_i^l(Q)$ is the subset of $W_i^{l-1}(Q)$ for which the unique rationalizing belief has a support contained in $W_{-i}^{l-1}(Q)$, $l \geq 2$.

Theorem 1-A. *Given a BRS $Q = \prod_{i\in N} Q_i$, Q is played by an intrinsic a posteriori equilibrium, if and only if for every $i \in N$, for any two distinct actions a_i and a_i' in $W_i(Q)$, we have $B_i^Q(a_i) \neq B_i^Q(a_i')$.*

Theorem 1-A says that in the context of a posteriori equilibrium, the intrinsicness assumption (that players' actions are driven by their hierarchies of beliefs about play) is characterized by an injectivity condition: actions in the subset $W_i(Q)$ must have distinct rationalizing beliefs. Moreover, if (and only if) players' actions are driven by their l-th order beliefs, then actions in the subset $W_i^l(Q)$ must have distinct rationalizing beliefs (see Theorem 1-B below).

Theorem 1 is a generalization of a previous result by Brandenburger and Friedenberg (2008, Proposition H.2), which states (in our language) that for any given BRS Q, if for every $i \in N$ we have $B_i^Q(a_i) \neq B_i^Q(a_i')$ for any two distinct actions a_i and a_i' in $W_i^1(Q)$, then there exists an intrinsic a posteriori equilibrium under which Q is played. In fact, we generalize Brandenburger and Friedenberg's result from first order belief to higher orders, in the following sense:

Theorem 1-B. *Fix an $l \geq 1$ and a BRS $Q = \prod_{i\in N} Q_i$. The set of action profiles Q is played by an a posteriori equilibrium in which players condition their actions on their l-th* *order beliefs about play, if and only if for every $i \in N$, for any two distinct actions a_i and a_i' in $W_i^l(Q)$, we have $B_i^Q(a_i) \neq B_i^Q(a_i')$.*

The proof of Theorem 1 can be found in Appendix A. Here we sketch the underlying idea. The iterative construction in equation (4) partitions an action set Q_i into disjoint subsets $Q_i \setminus W_i^1(Q)$, $W_i^1(Q) \setminus W_i^2(Q)$, $W_i^2(Q) \setminus W_i^3(Q)$, $W_i^3(Q) \setminus W_i^4(Q)$, ..., and $W_i(Q)$.

By construction, each action in $Q_i \setminus W_i^1(Q)$ is rationalized by an infinite number of first order beliefs about play (since $B_i^Q(a_i) \neq \emptyset$ is convex, it is either a singleton or an infinite set). Each action $a_i \in W_i^1(Q) \setminus W_i^2(Q)$ is rationalized by a unique first order belief about play (because $a_i \in W_i^1(Q)$); on the other hand, due to $a_i \notin W_i^2(Q)$, a_i is also rationalized by an infinite number of second order beliefs about play, because its unique first-order rationalizing belief places a positive probability on another action (of, say, player j) which can be rationalized by multiple first order beliefs of player j—mixing over these multiple rationalizations produces an infinite number of second order beliefs rationalizing a_i.

Likewise, each action in $W_i^2(Q) \setminus W_i^3(Q)$ is rationalized by a unique second order belief about play and by an infinite number of third order beliefs, each action in $W_i^3(Q) \setminus W_i^4(Q)$ is rationalized by a unique third order belief about play and by an infinite number of fourth order beliefs, and so on. Since Q_i is finite, we will never have any trouble finding distinct hierarchies of beliefs about play to rationalize actions in $Q_i \setminus W_i(Q)$.

On the other hand, by the argument above for every $l \geq 1$, each action a_i in $W_i(Q)$ is rationalized by a unique l-th order belief about play, which naturally projects down to the first-order rationalizing belief $B_i^Q(a_i)$. Therefore, the requirement that every player conditions his actions on his hierarchies of beliefs about play, together with the rationality condition that every player plays a best-response to his belief, translates into the requirement that each action a_i in $W_i(Q)$ has a distinct rationalizing belief $B_i^Q(a_i)$.

Example 1. Let us illustrate the characterization of intrinsic a posteriori equilibrium (Theorem 1) in the following game:

	x	y	z	w
a	1, 1	3, 0	0, 1	0, 1
b	3, 0	1, 1	0, 1	0, 1
c	0, 0	4, 0	1, 1	1, 1
d	4, 0	0, 0	1, 1	1, 1

Note that $\{a, b, c, d\} \times \{x, y, z, w\}$ is a BRS, so all actions can be played by an a posteriori equilibrium. We claim that actions a and b cannot be played under any intrinsic a posteriori equilibrium.

Set $Q_1 = \{a, b, c, d\}$ and $Q_2 = \{x, y, z, w\}$. We have the following sets of rationalizing beliefs:

$$\begin{aligned}
B_1^Q(a) &= \{1/2\,x + 1/2\,y\}, \\
B_1^Q(b) &= \{1/2\,x + 1/2\,y\}, \\
B_2^Q(x) &= \{a\}, \\
B_2^Q(y) &= \{b\}.
\end{aligned}$$

Therefore, we have $W_1^1(Q) = W_1(Q) = \{a, b\}$ and $W_2^1(Q) = W_2(Q) = \{x, y\}$. Clearly, B_1^Q is not injective on $W_1(Q)$, therefore by Theorem 1-A Q cannot be played by any intrinsic a posteriori equilibrium.

In fact, it is easy to see that for any $X = X_1 \times X_2 \subseteq \{a, b, c, d\} \times \{x, y, z, w\}$, if $a \in X_1$ then for X to be a BRS we must have $\{a, b\} \subseteq X_1$ and $\{x, y\} \subseteq X_2$, which again implies that X will fail the injectivity condition of Theorem 1-A. Therefore, action a is not played in any intrinsic a posteriori equilibrium, and likewise for action b.

4 Intrinsic Correlated Equilibrium

4.1 Characterization

We now turn our attention to an important special case of a posteriori equilibrium, namely correlated equilibrium in which players' posterior beliefs are consistent with a common prior. As discussed in the introduction, we can interpret the common prior as some previous interaction of players which has influenced their play in the current game. We can also view the study of intrinsic correlated equilibrium as a Bayesian exercise to understand whether the objective randomization explicit in a correlated equilibrium can be converted to players' beliefs and higher order beliefs about play that determine actions.

In this section we characterize the distributions of action profiles played by intrinsic correlated equilibria, in the following sense: a probability distribution of action profiles $\mu \in \Delta(A)$ is *played* or *obtained* by a correlated equilibrium (Ω, P, σ) if

$$\mu(a) = P(\{\omega \in \Omega : \sigma(\omega) = a\}) \tag{5}$$

for every $a \in A$. We call such distribution μ a *correlated equilibrium distribution* (CED). Clearly, $\mu \in \Delta(A)$ is a CED if and only if for every $i \in N$ and every action a_i in the support of μ (i.e., $\mu(\{a_i\} \times A_{-i}) > 0$), a_i is optimal for player i given the belief $\mu(\cdot \mid a_i)$.

As with a posteriori equilibrium, we start with an iterative construction. Fix a CED $\mu \in \Delta(A)$, and let Q_i be the support of the marginal distribution $\operatorname{marg}_{A_i} \mu$ for every $i \in N$. For every $i \in N$, define:

$$\begin{aligned} Y_i^1(\mu) &:= \{a_i \in Q_i : \ \mu(\cdot \mid a_i) \text{ is an extreme point of } B_i^Q(a_i)\}, \\ Y_i^l(\mu) &:= \{a_i \in Y_i^{l-1}(\mu) : \mu(Y_{-i}^{l-1}(\mu) \mid a_i) = 1\}, l \geq 2, \\ Y_i(\mu) &:= \bigcap_{l \geq 1} Y_i^l(\mu), \end{aligned} \tag{6}$$

where an extreme point of a convex set is one that cannot be written as a strict convex combination of other points in the set, and $B_i^Q(a_i)$ is the set of beliefs that rationalize action a_i (equation (3)).

We say that a CED is *intrinsic* if it is obtained by an intrinsic correlated equilibrium (via equation (5)).

Theorem 2-A. *A CED $\mu \in \Delta(A)$ is intrinsic if and only if for every $i \in N$, for any two distinct actions a_i and a_i' in $Y_i(\mu)$, we have $\mu(\cdot \mid a_i) \neq \mu(\cdot \mid a_i')$.*

Theorem 2-A shows that intrinsic CED is characterized by the injectivity of the posterior $\mu(\cdot \mid a_i)$ over actions in $Y_i(\mu)$. Moreover, if (and only if) for an intrinsic CED μ the players' actions are driven by their l-th order beliefs about play, then actions in $Y_i^l(\mu)$ must have distinct posteriors (see Theorem 2-B below). Comparing with the iterated construction associated with a posteriori equilibrium, we see that the common prior assumption is "manifested" as an extreme point requirement in first step of the iterations in (6).

The intuition for the extreme point requirement in (6) is as follows. The set $Y_i^1(\mu)$ identifies actions (in the support of $\operatorname{marg}_{A_i} \mu$) that are rationalized by a unique first order belief about play, given the common prior requirement. A posterior $\mu(\cdot \mid a_i)$ that is a non-extreme point of $B_i^Q(a_i)$ can be "split" into two distinct beliefs in $B_i^Q(a_i)$, so action a_i can be rationalized by these two beliefs, thus such a_i does not belong in $Y_i^1(\mu)$. This is analogous to $W_i^1(Q)$ for a posteriori equilibrium (equation (4)): when $B_i^Q(a_i)$ is not a singleton, a_i can be rationalized by any two beliefs in $B_i^Q(a_i)$. The difference is of course that for correlated equilibrium beliefs must come from a common prior, so not any two beliefs in $B_i^Q(a_i)$ can rationalize a_i—they must be two beliefs whose convex combination is equal to $\mu(\cdot \mid a_i)$.

The rest of the iterations in (6) are identical to those in (4) for a posterior equilibrium, and the idea underlying the injectivity condition of Theorem 2 is identical to that of Theorem 1 as well. See the discussion in page 9 for the intuition.

Theorem 2-B. *Fix an $l \geq 1$ and a CED $\mu \in \Delta(A)$. The distribution μ is played by a correlated equilibrium in which players condition their actions on their l-th order beliefs about play, if and only if for every $i \in N$, for any two distinct actions a_i and a_i' in $Y_i^l(\mu)$, we have $\mu(\cdot \mid a_i) \neq \mu(\cdot \mid a_i')$.*

The proof of Theorem 2 can be found in Appendix B.

Example 2 (Matching pennies)**.**

	a	b
a	1, -1	-1, 1
b	-1, 1	1, -1

The Nash equilibrium $\mu = (1/2\, a + 1/2\, b) \times (1/2\, a + 1/2\, b)$ is not an intrinsic CED: let $Q_1 = Q_2 = \{a, b\}$, then we have

$$\begin{aligned}
B_1^Q(a) &= \{p\, a + (1-p)\, b : 1/2 \leq p \leq 1\}, \\
B_1^Q(b) &= \{p\, a + (1-p)\, b : 0 \leq p \leq 1/2\}, \\
B_2^Q(a) &= \{p\, a + (1-p)\, b : 0 \leq p \leq 1/2\}, \\
B_2^Q(b) &= \{p\, a + (1-p)\, b : 1/2 \leq p \leq 1\}.
\end{aligned}$$

Thus, the belief $1/2\, a + 1/2\, b$ is an extreme point of both $B_i^Q(a)$ and $B_i^Q(b)$, $i \in \{1, 2\}$, so we have $Y_i^1(\mu) = Y_i(\mu) = \{a, b\}$. Clearly, we have $\mu(\cdot \mid a) = \mu(\cdot \mid b)$.

But $(1/2\, a + 1/2\, b) \times (1/2\, a + 1/2\, b)$ is the unique CED of this game. Thus, this game has no intrinsic correlated equilibrium.

4.2 Implications of the Characterization

In this section we use the characterization of Theorem 2 to demonstrate a dichotomy: in every finite game, "most" correlated equilibrium distributions (CED) are intrinsic; however, in a generic two-person game, any non-degenerate mixed-strategy Nash equilibrium is not an intrinsic CED.

Let us begin with a natural decomposition of CED which will be useful for the study of intrinsicness. To fix ideas, the following distribution over action profiles has three *minimal* *components* which are themselves distributions: (a, a); (b, b); and $1/4\,(c, c) + 3/8\,(c, d) + 1/8\,(d, c) + 1/4\,(d, d)$,

	a	b	c	d
a	1/4	0	0	0
b	0	1/4	0	0
c	0	0	1/8	3/16
d	0	0	1/16	1/8

Formally, for a fixed CED $\mu \in \Delta(A)$, with support $Q_i = \mathrm{supp}(\mathrm{marg}_{A_i}\, \mu)$ for $i \in N$, let $S = \bigcup_{i \in N} Q_i$ be the set of all actions of all players in the support of μ. Two actions a^1 and a^k (of perhaps two different players i_1 and i_k) in S *communicate* if they are connected by a sequence of intermediate actions of which μ places positive probability for every consecutive pair: there exist $a^m \in A_{i_m}$, $2 \le m \le k-1$, such that $i_m \ne i_{m-1} \in N$ and $\mu(a^{m-1}, a^m) > 0$ for each $2 \le m \le k$.

It is readily checked that communication is an equivalence relation. Therefore, communication partitions S into equivalence classes: $S = \bigcup_{1 \le k \le r} S^k$. A *minimal component* of a CED μ is simply the restriction of μ to one of the equivalence classes S^k:

$$\mu^k(a) = \begin{cases} \mu(a)/\mu(\prod_{i \in N} S^k \cap A_i) & \text{if } a_i \in S^k \text{ for all } i \in N, \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that a minimal component of a CED is itself a CED. We call a CED that has only one minimal component *minimal*.

Proposition 1. *A CED is intrinsic if all of its minimal components are non-extreme points in the set of CED.*

Proof. Clearly, a CED is intrinsic if and only if every one of its minimal components is intrinsic.

Thus, without loss of generality, suppose that CED μ is minimal. Assume that μ is non-intrinsic. Thus, we have $Y_i(\mu) \ne \emptyset$ for all $i \in N$. Since μ is minimal, by an "infection" argument we must have $Y_i^1(\mu) = Q_i$, where Q_i is the support of μ in A_i.

Suppose that μ^1 and μ^2 are two CED's such that $\mu = \mu^1/2 + \mu^2/2$ and $\mathrm{supp}\,\mu^1 = \mathrm{supp}\,\mu^2 = \mathrm{supp}\,\mu = Q$. We will show that we must have $\mu^1 = \mu^2$; this implies that μ must be an extreme point in the set of CED, which proves the proposition.

Because $Y_i^1(\mu) = Q_i$, we must have $\mu^1(\cdot \mid a_i) = \mu^2(\cdot \mid a_i) = \mu(\cdot \mid a_i)$ for every $i \in N$ and $a_i \in Q_i$.

Suppose that $\mu^1 \neq \mu^2$; then there exists $a \in Q = \prod_{i \in N} Q_i$ such that $\mu^1(a) \neq \mu^2(a)$. Without loss of generality, suppose $\mu^1(a) < \mu^2(a)$. Because $\mu^1(\cdot \mid a_i) = \mu^2(\cdot \mid a_i)$ for every $i \in N$, we have that $\mu^1(b_{-i}, a_i) > 0 \Rightarrow \mu^1(b_{-i}, a_i) < \mu^2(b_{-i}, a_i)$ for every $i \in N$ and $b_{-i} \in Q_{-i}$. Because μ is minimal, so are μ^1 and μ^2; applying the reasoning in the last sentence to all actions in the support, we have that $\mu^1(b) > 0 \Rightarrow \mu^1(b) < \mu^2(b)$ for every $b \in Q$, which clearly cannot be. Thus, we must have $\mu^1 = \mu^2$. $\square$

Proposition 1 tells us that "most" of these CED are intrinsic, in the sense that "most" of the points in a non-degenerate convex set are not extreme points. The set of CED of a finite game is a convex polytope, so it has a finite number of extreme points; Proposition 1 thus says that any non-intrinsic CED has a minimal component which is one of these extreme CED.

As discussed in the introduction, given that "most" CED are intrinsic, Theorem 2-B provides a classification of these CED based on the maximum order of beliefs used by players in the equilibrium. Such classification raises questions about properties (e.g., how prevalent) of CED in each class. Here we settle their convexities:

Proposition 2. *Let C^l be the set of CED played by correlated equilibria in which players' actions are driven by their l-th order beliefs about play. The set C^l is always convex if and only if $l \geq 2$.*

The proof of the if direction is in Appendix C. The following example shows that C^1 needs not be convex.

Example 3.

	c	d
a	0, 1	2, 0
b	0, 2	1, 1

In Nash equilibria (a, c) and (b, c) of the game above, players condition their actions on their first order beliefs about play. However, players cannot condition only on their first order beliefs about play in any correlated equilibrium that obtains the convex combination $(p\,a + (1-p)\,b, c)$, because in any correlated equilibrium player 1 has a unique first order belief on the actions of player 2: c.

We now turn to an important special case of CED: Nash equilibrium, or CED obtained by independent randomization of players.

We say that a two-player game $(N = \{1, 2\}, A = A_1 \times A_2, u)$ is *generic* if for any $i \in \{1, 2\}$ and $x \in \Delta(A_i)$, we have

$$|\,\mathrm{BR}_j(x)| \leq |\,\mathrm{supp}(x)|, \tag{7}$$

where $j \neq i$, $\mathrm{supp}(x) := \{a_i \in A_i : x(a_i) > 0\}$ and $\mathrm{BR}_j(x) := \{a_j \in A_j : u_j(a_j, x) \geq u_j(a'_j, x)$ for all $a'_j \in A_j\}$. Von Stengel (2002), Theorem 2.10, has proved that two-player games that fail condition (7) are of Lebesgue measure 0. The genericity class given by condition (7) is well-known in the study of the Lemke-Howson (1964) algorithm for computing Nash equilibrium.

Proposition 3. *In a generic two-player game, any non-degenerate mixed-strategy Nash equilibrium is non-intrinsic.*

Intuitively, a Nash equilibrium does not have any variation in beliefs about the other players' actions (for any given player), i.e., no variation in the first order beliefs about play, which leads to the lack of variation in any higher order beliefs about play; on the other hand, a (non-pure) intrinsic correlated equilibrium requires variations in the hierarchies of beliefs about play to support distinct actions (of a given player) played in the equilibrium. Consequently, we see a conflict between intrinsicness and mixed-strategy Nash equilibrium.[4]

We do not know if Proposition 3 is true for n-player games, $n \geq 3$. Here is an example of a 3-player game with a unique Nash equilibrium which is in mixed strategy but nevertheless is intrinsic; this is in contrast to the matching pennies game in Example 2. Fix a 3-player game with rational payoffs whose unique Nash equilibrium contains irrational entries; the first example of such game dates back to Nash's original paper on Nash equilibrium. This Nash equilibrium must in mixed strategy, and cannot be an extreme point in the set of CED, because extreme points in the set of CED can be computed by linear programming, which always returns rational outputs given rational payoff inputs. Therefore, Proposition 1 implies that this unique mixed-strategy Nash equilibrium is an intrinsic CED.

[4] One proves Proposition 3 by showing that for any Nash equilibrium (x_1, x_2), condition (7) implies that x_i must be an extreme point of $B_j^A(a_j)$, for all $i \in \{1, 2\}$, $j \neq i$ and a_j in the support of x_j. This fact will be unsurprising to readers familiar with the Lemke-Howson algorithm, which consists of traversing the extreme points of sets $B_1^A(a_1)$ and $B_2^A(a_2)$ until settling on a pair that forms a Nash equilibrium. We omit the detail which can be found in a working paper version.

5 Discussion

5.1 Relation to Brandenburger and Friedenberg (2008)

Working with complete-information games (as we do), Brandenburger and Friedenberg study a refinement of correlated rationalizability where players' correlated beliefs satisfy conditions of *conditional independence* (CI) and *sufficiency* (SUFF).

Roughly, in a type structure on actions $(T_i, \lambda_i)_{i \in N}$, where $\lambda_i : T_i \to \Delta(T_{-i} \times A_{-i})$ for each $i \in N$, a type t_i satisfies *conditional independence* (CI) if the belief about others' actions in $\lambda_i(t_i)$ is independent conditional on other players' hierarchies of beliefs about play. And a type t_i satisfies *sufficiency* (SUFF) if the belief in $\lambda_i(t_i)$ is such for any player $j \neq i$, player j's action is influenced only by player j's hierarchy of beliefs about play.[5]

Brandenburger and Friedenberg study the set of actions that are consistent with epistemic conditions of CI, SUFF, and rationality and common beliefs of rationality (RCBR):

$$\begin{aligned} X_i := \{a_i \in A_i :\ & \text{there exist } (T_j, \lambda_j)_{j \in N} \text{ such that at every type CI and SUFF hold,} \\ & \text{and } t_i \in T_i \text{ such that } (a_i, t_i) \in \mathrm{Rat}_i(\lambda)\}, \end{aligned}$$

where $\mathrm{Rat}_i(\lambda)$ is the set of states of player i at which RCBR hold:

$$\begin{aligned} \mathrm{Rat}_i^1(\lambda) &:= \{(t_i, a_i) \in T_i \times A_i : a_i \text{ is optimal in } A_i \text{ for player } i \text{ under } \operatorname{marg}_{A_{-i}} \lambda_i(t_i)\}, \\ \mathrm{Rat}_i^l(\lambda) &:= \{(t_i, a_i) \in \mathrm{Rat}_i^{l-1}(\lambda) : \lambda_i(t_i)(\mathrm{Rat}_{-i}^{l-1}(\lambda)) = 1\}, l \geq 2, \\ \mathrm{Rat}_i(\lambda) &:= \bigcap_{l \geq 1} \mathrm{Rat}_i^l(\lambda) \end{aligned}$$

A precise characterization of the set $X = \prod_{i \in N} X_i$, in terms of actions and payoffs of the game and independent of type structure, is an open question posed by Brandenburger and Friedenberg. Our Theorem 1-A (the characterization of intrinsic a posteriori equilibrium) provides a partial answer: the injectivity condition of Theorem 1-A is a sufficient condition for a set of action profiles to be a subset of X. A contemporaneous and independent paper by Peysakhovich (2009) provides another partial answer; we will discuss Peysakhovich's result in Section 5.2.

Let us now discuss the relationship between intrinsic a posteriori equilibrium and epis-

[5]Because of constraint in space we omit Brandenburger and Friedenberg's formal definitions of CI and SUFF, although we present an equivalent definition in the paragraphs below to illustrate the difference to our approach.

temic conditions of CI, SUFF and RCBR. We first argue that intrinsic a posteriori equilibrium is characterized by RCBR plus a *determinism* condition on type structure $(T_i, \lambda_i)_{i \in N}$, where $\lambda_i : T_i \to \Delta(T_{-i} \times A_{-i})$:

$$\begin{aligned}
&\text{There exist maps } \sigma_i : T_i \to A_i, i \in N, \\
&\text{measurable w.r.t. hierarchies of beliefs about play,} \\
&\text{such that for every player } i \in N \text{ and type } t_i \in T_i, \\
&\lambda_i(t_i)(a_{-i} \mid t_{-i}) = 1_{\sigma_{-i}(t_{-i})}(a_{-i}), \text{ whenever } \lambda_i(t_i)(t_{-i}) > 0.
\end{aligned} \tag{determinism}$$

The determinism condition says that conditional on a hierarchy of beliefs about play, a deterministic action of the player will be played, and this is commonly believed by all players. In particular, every strategic uncertainty in the game is traced back to players' hierarchies of beliefs about play — uncertainty about play is a manifestation of uncertainty about beliefs.

It is clear that determinism condition plus RCBR is equivalent to intrinsic a posteriori equilibrium. In fact, determinism is implicit in the very idea of equilibrium; see Brandenburger (2010) for a thoughtful discussion of this point.

For comparison, conditions CI and SUFF on all types are equivalent to the following condition:

$$\begin{aligned}
&\text{For every player } i \in N \text{ and type } t_i \in T_i, \\
&\text{there exist maps } \sigma_j : T_j \to \Delta(A_j), j \neq i, \\
&\text{measurable w.r.t. hierarchies of beliefs about play,} \\
&\text{such that } \lambda_i(t_i)(a_{-i} \mid t_{-i}) = \prod_{j \neq i} \sigma_j(t_j)(a_j), \text{ whenever } \lambda_i(t_i)(t_{-i}) > 0.
\end{aligned} \tag{CI + SUFF}$$

Notice that (CI + SUFF) differs are from determinism in two places, or in other words there are two sources of indeterminism in (CI + SUFF): first, the maps σ_j between types and actions (of player j) may depend on the belief $\lambda_i(t_i)$ and need not be consistent across types and players; and second, the map σ_j itself may be stochastic. Thus, (CI + SUFF) says that a player attributes other players' actions, up to some idiosyncratic randomization, to their hierarchies of beliefs about play, but such attribution needs not be consistent across players or across types (i.e., states of mind) of a same player.

If we relax determinism so that the action is stochastic conditional on a hierarchy of beliefs about play (but still insisting that such link between actions and beliefs is consistent across types and players), this would still have strictly more refined behavioral implications than (CI + SUFF). The game in Example 1 is an illustration: since there are two players, (CI + SUFF) has no bite; and recall that all actions are rationalizable. Therefore, every action is consistent with RCBR, CI and SUFF. However, one can show that actions x and y of player 2 cannot be played under RCBR and determinism, even if we allow $\sigma_i : T_i \to \Delta(A_i)$ in the definition of determinism.[6]

5.2 Private Randomization

Let us relax the restriction to pure strategies (contingent on beliefs and higher order beliefs about play).

Consider the case of a common prior. Fix a correlating device (Ω, P), where $\Omega = \prod_{i \in N} \Omega_i$ and $P \in \Delta(\Omega)$. Players are now allowed to use private randomization: $\sigma_i : \Omega_i \to \Delta(A_i)$.

The definition of correlated equilibrium (Definition 1) still applies without change.

For every player i and individual state $\omega_i \in \Omega_i$, the first order belief about play at ω_i, $\delta_i^1(\omega_i) \in \Delta(A_{-i})$, is now as follows:

$$\delta_i^1(\omega_i)(a_{-i}) = \sum_{\omega_{-i} \in \Omega_{-i}} P(\omega_{-i} \mid \omega_i) \prod_{j \neq i} \sigma_j(\omega_j)(a_j),$$

for every $a_{-i} \in A_{-i}$, whenever $P(\omega_i) > 0$.

The tuple (Ω, P, σ) *obtains* a distribution $\mu \in \Delta(A)$, where for every $a \in A$,

$$\mu(a) = \sum_{\omega \in \Omega} P(\omega) \prod_{i \in N} \sigma_i(\omega_i)(a_i).$$

The following theorem is a reinterpretation of Peysakhovich (2009)'s main result; details of the proof can be found in his paper.

Theorem (Peysakhovich)**.** *For any CED $\mu \in \Delta(A)$, there exists a correlated equilibrium (Ω, P, σ) that obtains μ such that players condition their randomized actions only on their first order beliefs about play.*

Therefore, we have a trade-off between using mixed strategies and conditioning on higher order beliefs about play. On the one hand, every CED can be obtained from a correlated

[6] Details can be found in a previous working paper version.

equilibrium in which every player plays randomized actions contingent on his first order beliefs about play. On the other hand, "most" CED (see Proposition 1) can be obtained from correlated equilibrium in which every player plays pure actions contingent on his higher order beliefs about play; that is, the player does not randomize, but he might have to rely on more refined information, i.e., his higher order beliefs.

5.3 Infinite Games

The assumption that the game is finite is crucial to our techniques in this paper. Yet we conjecture that under some regularity conditions variants of our characterizations would work for infinite games, for the reason that in general the set of beliefs $\Delta(A_{-i})$ is of strictly higher cardinality than the action set A_i, if all A_i's are of the same cardinality. On the other hand, conditions such as the set of rationalizing beliefs $B_i^Q(a_i)$ being a singleton set and beliefs being extreme points of $B_i^Q(a_i)$ would probably have to be refined for infinite games, because we believe such conditions would only be necessary but not sufficient. We leave the work on infinite games to future research.

APPENDIX

We will only write the proofs for Theorem 1-A and 2-A. Adapting the proofs to the finite-order cases (1-B and 2-B) is immediate.

A Proof of Theorem 1-A

A.1 Only If:

Fix an intrinsic a posteriori equilibrium (Ω, β, σ). Let $Q_i = \sigma_i(\Omega_i)$ for each $i \in N$. We will show that for any $a_i \neq a_i' \in W_i(Q)$, we have $B_i^Q(a_i) \neq B_i^Q(a_i')$.

Let $(T_i, \lambda_i)_{i \in N}$ be the (Ω, β, σ)-induced type structure as in Definition 2, where $T_i = \Omega_i$, and $\lambda_i : T_i \to \Delta(T_{-i} \times A_{-i})$ is defined in equation (2). For simplicity, let us use ω_i to denote both an individual state in Ω_i and the equivalent type in T_i. For each $\omega_i \in \Omega_i = T_i$, let $\delta_i^l(\omega_i)$ denote player i's l-th order belief about action at individual state/type ω_i (see the paragraphs following Definition 4).

The following lemma, which is essentially Proposition 11.1 in Brandenburger and Friedenberg (2008), demonstrates the connection between the action set $W_i^l(Q)$ and player i's equilibrium-induced l-th order beliefs about actions.

Lemma 1. *For any $l \geq 1$, $i \in N$, and $a_i \in W_i^l(Q)$, there is exactly one l-th order belief from Ω_i being mapped by σ_i to a_i; that is, if $\sigma_i(\omega_i) = \sigma_i(\omega_i') = a_i$, then $\delta_i^l(\omega_i) = \delta_i^l(\omega_i')$.*

Proof. If $\sigma_i(\omega_i) = a_i \in W_i^1(Q)$, then clearly $\delta_i^1(\omega_i) = \mathrm{marg}_{A_{-i}} \lambda_i(\omega_i) = B_i^Q(a_i)$. Thus the lemma is true when $l = 1$.

Now suppose $l \geq 2$, and that the lemma is true for $l-1$. Let $\sigma_i(\omega_i) = \sigma_i(\omega_i') = a_i \in W_i^l(Q)$. Then, $\mathrm{marg}_{A_{-i}} \lambda_i(\omega_i) = \mathrm{marg}_{A_{-i}} \lambda_i(\omega_i') = B_i^Q(a_i)$ because we have $a_i \in W_i^1(Q)$. If it holds that $B_i^Q(a_i)(a_{-i}) > 0$, that $\lambda_i(\omega_i)(\omega_{-i}, a_{-i}) > 0$, and that $\lambda_i(\omega_i')(\omega_{-i}', a_{-i}) > 0$, then we must have $\sigma_{-i}(\omega_{-i}) = \sigma_{-i}(\omega_{-i}') = a_{-i}$ (by the construction of λ_i) and $a_{-i} \in W_{-i}^{l-1}(Q)$ (by the construction of $W_i^l(Q)$); and by the induction hypothesis, $\delta_j^{l-1}(\omega_j) = \delta_j^{l-1}(\omega_j')$ for every $j \neq i$. Therefore, the image measure of $\lambda_i(\omega_i)$ equals the image measure of $\lambda_i(\omega_i')$, under the mapping $(\delta_j^{l-1}, \mathrm{id}_{A_j})_{j \neq i}$; in other words, we have $\delta_i^l(\omega_i) = \delta_i^l(\omega_i')$. □

Corollary 1. *For every $i \in N$ and $\mu \in \Delta(W_{-i}(Q))$, there can be at most one hierarchy of beliefs from Ω_i having first order belief μ; that is, if $\delta_i^1(\omega_i) = \mu = \delta_i^1(\omega_i')$, then $\delta_i^l(\omega_i) = \delta_i^l(\omega_i')$ for every $l \geq 1$.*

Proof. Suppose that $\mu \in \Delta(W_{-i}(Q))$ and $\mathrm{marg}_{A_{-i}} \lambda_i(\omega_i) = \mu = \mathrm{marg}_{A_{-i}} \lambda_i(\omega_i')$. If it holds that $\mu(a_{-i}) > 0$, that $\lambda_i(\omega_i)(\omega_{-i}, a_{-i}) > 0$, and that $\lambda_i(\omega_i')(\omega_{-i}', a_{-i}) > 0$, then we must have $\sigma_{-i}(\omega_{-i}) = \sigma_{-i}(\omega_{-i}') = a_{-i} \in W_{-i}(Q)$, and by the previous lemma $\delta_j^l(\omega_j) = \delta_j^l(\omega_j')$ for every $j \neq i$ and $l \geq 1$. Thus, $\delta_i^l(\omega_i) = \delta_i^l(\omega_i')$ for every $l \geq 1$. □

Now, for any $i \in N$ and $a_i \neq a_i' \in W_i(Q)$, by the assumption of $Q_i = \sigma_i(\Omega_i)$, there exist $\omega_i, \omega_i' \in \Omega_i$ such that $\sigma_i(\omega_i) = a_i$ and $\sigma_i(\omega_i') = a_i'$. Individual states ω_i and ω_i' must have distinct hierarchies of beliefs about play, by the intrinsicness of the equilibrium (Ω, β, σ). Also, we have $\mathrm{marg}_{A_{-i}} \lambda_i(\omega_i) = B_i^Q(a_i)$ and $\mathrm{marg}_{A_{-i}} \lambda_i(\omega_i') = B_i^Q(a_i')$; and clearly $B_i^Q(a_i)(W_{-i}(Q)) = B_i^Q(a_i')(W_{-i}(Q)) = 1$. Then we must have $B_i^Q(a_i) \neq B_i^Q(a_i')$, for otherwise the corollary above would imply that ω_i and ω_i' induce the same hierarchy of beliefs about play.

A.2 If:

Fix a BRS $Q = \prod_{i \in N} Q_i$ such that B_i^Q is injective on $W_i(Q)$. We will construct an intrinsic a posteriori equilibrium (Ω, β, σ) under which Q is played.

For each $i \in N$, let the set of individual states

$$\Omega_i = \{a_i(k) : a_i \in Q_i \setminus W_i(Q), k \in \{1, 2\}\} \cup W_i(Q),$$

where $a_i(1)$ and $a_i(2)$ are two distinct "copies" of a_i.

We define the strategy $\sigma_i : \Omega_i \to A_i$ as follows. For each $i \in N$, let $\sigma_i(a_i(1)) = \sigma_i(a_i(2)) = a_i$ for every $a_i \in Q_i \setminus W_i(Q)$; and let $\sigma_i(a_i) = a_i$ for every $a_i \in W_i$.

For every $i \in N$, set $\omega(a_i) = a_i(1)$ if $a_i \in Q_i \setminus W_i(Q)$; and set $\omega(a_i) = a_i$ if $a_i \in W_i(Q)$. The individual state $\omega(a_i)$ is the "default" state associated with action a_i.

With states and strategies defined, the next step is to construct the the posterior belief function $\beta_i : \Omega_i \to \Delta(\Omega_{-i})$. This is accomplished in several steps, each of which corresponds to an iteration of $W_i^l(Q)$.

Step 1:

For each $a_i \in Q_i \setminus W_i^1(Q)$, fix $\nu(a_i, 1) \neq \nu(a_i, 2) \in B_i^Q(a_i) \setminus B_i^Q(W_i^1(Q))$ such that they are all distinct from each other; in other words,

$$|\{\nu(a_i, k) : a_i \in Q_i \setminus W_i^1(Q), k \in \{1, 2\}\}| = 2|Q_i \setminus W_i^1(Q)|.$$

This is possible because $Q_i \setminus W_i^1(Q)$ and $B_i^Q(W_i^1(Q))$ are finite sets, but $B_i^Q(a_i)$ is infinite for any $a_i \in Q_i \setminus W_i^1(Q)$.

For $a_i \in Q_i \setminus W_i^1(Q)$ and $k \in \{1, 2\}$, let

$$\beta_i(a_i(k))(\omega_{-i}) = \begin{cases} \nu(a_i, k)(a_{-i}) & \omega_j = \omega(a_j) \text{ for every } j \neq i \\ 0 & \text{otherwise} \end{cases}$$

for every $\omega_{-i} \in \Omega_{-i}$.

Clearly, each individual state $a_i(k)$, $a_i \in Q_i \setminus W_i^1(Q)$ and $k \in \{1, 2\}$, induces a distinct first order belief about play.

Step l: $(2 \leq l \leq L = \min\{l \geq 1 : W^l(Q) = W(Q)\})$

For each $a_i \in W_i^{l-1} \setminus W_i^l(Q)$, choose a $c(a_i) \in W_m^{l-2} \setminus W_m^{l-1}$, $m \neq i$, (where $W_m^0 = Q_m$) such that $B_i^Q(a_i)(c(a_i)) > 0$; such $c(a_i)$ exists by constructions of $W_i^l(Q)$'s, and $c(a_i)$'s can be chosen so that $B_i^Q(a_i) = B_i^Q(a_i') \Rightarrow c(a_i) = c(a_i')$. And choose $\kappa(a_i, 1) \neq \kappa(a_i, 2) \in [0, 1]$ such that for any $a_i \neq a_i' \in W_i^{l-1}(Q) \setminus W_i^l(Q)$ with $B_i^Q(a_i) = B_i^Q(a_i')$, we have that $\kappa(a_i, 1)$, $\kappa(a_i', 1)$, $\kappa(a_i, 2)$ and $\kappa(a_i', 2)$ are all distinct.

For $a_i \in W_i^{l-1} \setminus W_i^l(Q)$ and $k \in \{1,2\}$, let

$$\beta_i(a_i(k))(t_{-i}) = \begin{cases} B_i^Q(a_i)(a_{-i}) & \omega_j = \omega(a_j), j \neq i, \text{ and } a_m \neq c(a_i) \\ \kappa(a_i,k) B_i^Q(a_i)(a_{-i}) & \omega_j = \omega(a_j), j \notin \{i,m\}, \text{ and } \omega_m = c(a_i)(1) \\ (1-\kappa(a_i,k)) B_i^Q(a_i)(a_{-i}) & \omega_j = \omega(a_j), j \notin \{i,m\}, \text{ and } \omega_m = c(a_i)(2) \\ 0 & \text{otherwise} \end{cases}$$

for every $\omega_{-i} \in \Omega_{-i}$. Essentially, what we are doing here is to introduce heterogeneity, through $\kappa(a_i,k)$, in beliefs about others' $(l-1)$-th order beliefs (i.e., in the l-th order beliefs) among individual states $a_i(k)$ that have the same $(l-1)$th order beliefs about play.

By induction on l, it is easy to see that each $a_i(k)$, $a_i \in W_i^{l-1}(Q) \setminus W_i^l(Q)$ and $k \in \{1,2\}$, induces a distinct l-th order belief about play.

Final Step:

Finally, for $a_i \in W_i(Q)$, let

$$\beta_i(a_i)(\omega_{-i}) = \begin{cases} B_i^Q(a_i)(a_{-i}) & \omega_j = \omega(a_j) \text{ for every } j \neq i \\ 0 & \text{otherwise} \end{cases}$$

for every $\omega_{-i} \in \Omega_{-i}$.

By assumption, each $a_i \in W_i(Q)$ has a distinct first order belief.

B Proof of Theorem 2-A

B.1 Only if

Fix a correlated equilibrium (Ω, P, σ). Let $\mu \in \Delta(A)$ be the distribution played by the equilibrium. Without loss suppose that P places positive probability on all elements of Ω. And set Q_i to be the support of μ in A_i. We prove the following analogue of Lemma 1. The proof of the only if part of Theorem 2-A follows from Lemma 2 exactly as the only if part of Theorem 1-A follows from Lemma 1.

Lemma 2. *For any $l \geq 1$, $i \in N$ and $a_i \in Y_i^l(\mu)$, there is exactly one l-th order belief from Ω_i being mapped by σ_i to a_i; that is, if $\sigma_i(\omega_i) = \sigma_i(\omega_i') = a_i$, then $\delta_i^l(\omega_i) = \delta_i^l(\omega_i')$.*

Proof. Suppose $l = 1$. Fix $i \in N$ and $a_i \in Y_i^1(\mu)$. If there exist $\omega_i, \omega_i' \in \Omega_i$ such that $\delta_i^1(\omega_i) \neq \delta_i^1(\omega_i') \in \Delta(A_{-i})$ but $\sigma_i(\omega_i) = \sigma_i(\omega_i') = a_i$ (and for simplicity, assume that $\sigma_i^{-1}(a_i) =$

$\{\omega_i, \omega_i'\}$), then because we have a common prior, the posterior $\mu(\cdot \mid a_i)$ *must* be a strict convex combination of the first order beliefs $\delta_i^1(\omega_i)$ and $\delta_i^1(\omega_i')$. This contradicts $\mu(\cdot|a_i)$ being an extreme point of $B_i^Q(a_i)$, because the optimality condition for correlated equilibrium (condition (1)) implies that $\delta_i^1(\omega_i)$ and $\delta_i^1(\omega_i')$ are in $B_i^Q(a_i)$.

The inductive step is the same as that in Lemma 1 and does not use the common prior assumption. □

B.2 If

Suppose a correlated equilibrium distribution $\mu \in \Delta(A)$ is given such that for every $i \in N$ and for any two distinct $a_i \neq a_i' \in Y_i(\mu)$, we have that $\mu(\cdot \mid a_i) \neq \mu(\cdot \mid a_i')$. We will construct an intrinsic correlated equilibrium (Ω, P, σ) that obtains μ. For each $i \in N$ let Q_i be the support of $\operatorname{marg}_{A_i} \mu$. Our construction is to split each action $a_i \in Q_i \setminus Y_i(\mu)$ into two copies (and making each copy an individual state with a distinct hierarchy of beliefs about play) repeatedly using Lemma 3, whose proof we defer until the end of this section.

Lemma 3. *Fix a finite and non-empty $X = \prod_{i \in N} X_i$ and a $\mu \in \Delta(X)$ such that $\mu(x_i) > 0$ for every $i \in N$ and $x_i \in X_i$. And fix $(Z_i)_{i \in N}$, where $Z_i \subseteq X_i$, and $\{(\nu(x_i, 1), \nu(x_i, 2))\}_{x_i \in Z_i, i \in N}$, where $\nu(x_i, 1), \nu(x_i, 2) \in \Delta(X_{-i})$, such that for every $i \in N$ and $x_i \in Z_i$, we have $\mu(\cdot \mid x_i) = \kappa(x_i)\nu(x_i, 1) + (1 - \kappa(x_i))\nu(x_i, 2)$ for some $\kappa(x_i) \in (0, 1)$.*

Let $\tilde{X} = \prod_{i \in N} \tilde{X}_i$, $\tilde{X}_i = \{x_i(k) : x_i \in Z_i, k \in \{1, 2\}\} \cup (X_i \setminus Z_i)$ (where $x_i(1)$ and $x_i(2)$ are two distinct copies of x_i). Define $f_i : \tilde{X}_i \to X_i$ such that $f_i(x_i) = x_i$ for $x_i \notin Z_i$, and $f_i(x_i(1)) = f_i(x_i(2)) = x_i$ for $x_i \in Z_i$; define $f : \tilde{X} \to X$ and $f_{-i} : \tilde{X}_{-i} \to X_{-i}$ in the obvious way.

Then, there exists a $\tilde{\mu} \in \Delta(\tilde{X})$ such that $\tilde{\mu}(f^{-1}(x)) = \mu(x)$ for each $x \in X$, and $\tilde{\mu}(f_{-i}^{-1}(x_{-i}) \mid x_i(k)) = \nu(x_i, k)(x_{-i})$ for every $i \in N$, $x_i \in Z_i$, $k \in \{1, 2\}$ and $x_{-i} \in X_{-i}$. Furthermore, if for every $i \in N$ and $x_i \in Z_i$, $\nu(x_i, 1)$ and $\nu(x_i, 2)$ have the same support as $\mu(\cdot \mid x_i)$, then for every $i \in N$, $x_i \in Z_i$ and $x_{-i} \in \tilde{X}_{-i}$, $\tilde{\mu}(x_i(1), x_{-i}) > 0$ if and only if $\tilde{\mu}(x_i(2), x_{-i}) > 0$ (if and only if $\mu(x_i, f_{-i}(x_{-i})) > 0$).

Step 1:

For each $i \in N$ and $a_i \in Q_i \setminus Y_i^1(\mu)$, choose $\nu(a_i, 1) \neq \nu(a_i, 2) \in B_i^Q(a_i)$ such that $\mu(\cdot \mid a_i) = \nu(a_i, 1)/2 + \nu(a_i, 2)/2$ and that $\nu(a_i, 1)$ and $\nu(a_i, 2)$ have the same support as $\mu(\cdot \mid a_i)$. This is possible by construction of $Y_i^1(\mu)$. Furthermore, we can choose $\nu(a_i, k)$'s in a way such that for every $i \in N$:

$$|\{\nu(a_i, k) : a_i \in Q_i \setminus Y_i^1(\mu), k \in \{1,2\}\}| = 2|Q_i \setminus Y_i^1|$$

and

$$\{\nu(a_i, k) : a_i \in Q_i \setminus Y_i^1(\mu), k \in \{1,2\}\} \cap \{\mu(\cdot \mid a_i) : a_i \in Y_i^1(\mu)\} = \emptyset.$$

Now, apply Lemma 3 to μ, Q, $(Q_i \setminus Y_i^1(\mu))_{i \in N}$ and $\{(\nu(a_i, 1), \nu(a_i, 2))\}_{a_i \in Q_i \setminus Y_i^1(\mu), i \in N}$ to obtain $\Omega^1 = \prod_{i \in N} \Omega_i^1$ (where $\Omega_i^1 = \{a_i(k) : a_i \in Q_i \setminus Y_i^1(\mu), k \in \{1,2\}\} \cup Y_i^1$), $P^1 \in \Delta(\Omega^1)$ and $f_i^1 : \Omega_i^1 \to Q_i, i \in N$, with properties stated in the lemma. These properties imply that (Ω^1, P^1, f^1) is a correlated equilibrium that obtains μ, and that each $a_i(j)$, $a_i \in Q_i \setminus Y_i^1(\mu)$ and $j \in \{1,2\}$, has a distinct first order belief about play.

Step l: $(2 \leq l \leq L = \min\{l \geq 1 : Y^l(\mu) = Y(\mu)\})$

Suppose that $\Omega^{l-1} = \prod_{i \in N} \Omega_i^{l-1}$ (where $\Omega_i^{l-1} = \{a_i(k) : a_i \in Q_i \setminus Y_i^{l-1}(\mu), k \in \{1,2\}\} \cup Y_i^{l-1}(\mu)$), $P^{l-1} \in \Delta(\Omega^{l-1})$ and $f_i^{l-1} : \Omega_i^{l-1} \to \Omega_i^{l-2}, i \in N$, (let $\Omega_i^0 = Q_i$) are obtained from Lemma 3 in the previous step.

For each $i \in N$ and $a_i \in Y_i^{l-1}(\mu) \setminus Y_i^l(\mu)$, choose a $c(a_i) \in Y_j^{l-2}(\mu) \setminus Y_j^{l-1}(\mu)$, $j \neq i$, (let $Y_j^0(\mu) = Q_j$) such that $\mu(c(a_i) \mid a_i) > 0$; such $c(a_i)$ exists by construction of $Y_i^l(\mu)$'s, and $c(a_i)$'s can be chosen so that $\mu(\cdot \mid a_i) = \mu(\cdot | a_i') \Rightarrow c(a_i) = c(a_i')$. For each $\omega_{-(i,j)} \in \Omega_{-(i,j)}^{l-1} = \prod_{k \notin \{i,j\}} \Omega_k^{l-1}$, we have $P^{l-1}(\omega_{-(i,j)}, c(a_i)(1), a_i) > 0$ if and only if $P^{l-1}(\omega_{-(i,j)}, c(a_i)(2), a_i) > 0$ (by Lemma 3); and $P^{l-1}(\{c(a_i)(1), c(a_i)(2)\} \times \{a_i\} \times \Omega_{-(i,j)}^{l-1}) = \mu(c(a_i), a_i) > 0$. Let

$$\nu(a_i, 1)(\omega_{-i}) = \begin{cases} P^{l-1}(\omega_{-i} \mid a_i) & P^{l-1}(t_{-i} \mid a_i) = 0 \text{ or } \omega_j \notin \{c(a_i)(1), c(a_i)(2)\} \\ P^{l-1}(\omega_{-(i,j)}, c(a_i)(1) \mid a_i) - \kappa(a_i) & P^{l-1}(\omega_{-i} \mid a_i) > 0 \text{ and } \omega_j = c(a_i)(1) \\ P^{l-1}(\omega_{-(i,j)}, c(a_i)(2) \mid a_i) + \kappa(a_i) & P^{l-1}(\omega_{-i} \mid a_i) > 0 \text{ and } \omega_j = c(a_i)(2) \end{cases},$$

and

$$\nu(a_i, 2)(\omega_{-i}) = \begin{cases} P^{l-1}(\omega_{-i} \mid a_i) & P^{l-1}(t_{-i} \mid a_i) = 0 \text{ or } \omega_j \notin \{c(a_i)(1), c(a_i)(2)\} \\ P^{l-1}(\omega_{-(i,j)}, c(a_i)(1) \mid a_i) + \kappa(a_i) & P^{l-1}(\omega_{-i} \mid a_i) > 0 \text{ and } \omega_j = c(a_i)(1) \\ P^{l-1}(\omega_{-(i,j)}, c(a_i)(2) \mid a_i) - \kappa(a_i) & P^{l-1}(\omega_{-i} \mid a_i) > 0 \text{ and } \omega_j = c(a_i)(2) \end{cases},$$

for every $\omega_{-i} \in \Omega_{-i}^{l-1}$, where $\kappa(a_i) > 0$ is sufficiently small so that $\nu(a_i, 1)$ and $\nu(a_i, 2)$ has the same support as $\mu^{l-1}(\cdot \mid a_i)$. Notice that $\nu(a_i, 1)/2 + \nu(a_i, 2)/2 = P^{l-1}(\cdot \mid a_i)$.

Furthermore, we can choose the $\kappa(a_i)$'s so that for any $a_i \neq a_i' \in Y_i^{l-1} \setminus Y_i^l(\mu)$ such that $\mu(\cdot \mid a_i) = \mu(\cdot|a_i')$, we have that $\nu(a_i,1), \nu(a_i,2), \nu(a_i',1)$ and $\nu(a_i',2)$ all differ from each other in their probabilities on $c(a_1)(1)$.

Now, apply Lemma 3 to P^{l-1}, Ω^{l-1}, $(Y_i^{l-1}(\mu) \backslash Y_i^l(\mu))_{i\in N}$ and $\{(\nu(a_i,1),\nu(a_i,2))\}_{a_i \in Y_i^{l-1}(\mu)\backslash Y_i^l(\mu), i\in N}$ to obtain $\Omega^l = \prod_{i\in N} \Omega_i^l$ (where $\Omega_i^l = \{a_i(k) : a_i \in Q_i \backslash Y_i^l(\mu), k \in \{1,2\}\} \cup Y_i^l(\mu))$, $P^l \in \Delta(\Omega^l)$ and $f_i^l : \Omega_i^l \to \Omega_i^{l-1}, i \in N$, with properties stated in the lemma. These properties imply that $(P^l, \Omega^l, f^1 \circ \cdots \circ f^l)$ is a correlated equilibrium that obtains μ, and that each $a_i(k)$, $a_i \in Y_i^{l-1}(\mu) \setminus Y_i^l(\mu)$ and $k \in \{1,2\}$, induces a distinct l-th order belief about play.

Finally:

Let $\Omega = \Omega^L$ $(\Omega_i = \Omega_i^L = \{a_i(k) : a_i \in Q_i \setminus Y_i(\mu), k \in \{1,2\}\} \cup Y_i(\mu))$, $P = P^L$, and $\sigma_i = f_i^1 \circ \ldots \circ f_i^L$. It is easy to see that that (Ω, P, σ) is an intrinsic correlated equilibrium that obtains μ.

Proof of Lemma 3. Without loss of generality suppose that $N = \{1, \ldots, n\}$.

Let $\mu^1 \in \Delta(\tilde{X}_1 \times \prod_{2\leq i\leq n} X_i)$ be such that

$$\mu^1(x_1(1), x_{-1}) = \mu(x_1)\kappa(x_1)\nu(x_1,1)(x_{-1})$$

and

$$\mu^1(x_1(2), x_{-1}) = \mu(x_1)(1-\kappa(x_1))\nu(x_1,2)(x_{-1}),$$

where $\mu(\cdot \mid x_1) = \kappa(x_1)\nu(x_1,1) + (1-\kappa(x_1))\nu(x_1,2)$, for each $x_1 \in Z_1$ and $x_{-1} \in X_{-1}$.

And let $\mu^1(x_1, x_{-1}) = \mu(x_1, x_{-1})$ for every $x_1 \notin Z_1$ and $x_{-1} \in X_{-1}$.

In general, for $2 \leq l \leq n$, let $\mu^l \in \Delta(\prod_{1\leq j\leq l} \tilde{X}_j \times \prod_{l+1\leq i\leq n} X_i)$ be such that for every $x_l \in Z_l$, $(x_1, \ldots, x_{l-1}) \in \prod_{1\leq i\leq l-1} \tilde{X}_i$ and $(x_{l+1}, \ldots, x_n) \in \prod_{l+1\leq i\leq n} X_i$:

$$\begin{aligned}\mu^l(x_1, \ldots, x_{l-1}, x_l(1), x_{l+1}, \ldots, x_n) =& \mu(x_l)\kappa(x_l)\frac{\mu^{l-1}(x_1, \ldots, x_{l-1}, x_l, \ldots, x_n)}{\mu(f_1(x_1), \ldots, f_{l-1}(x_{l-1}), x_l, \ldots, x_n)} \\ & \times \nu(x_l,1)(f_1(x_1), \ldots, f_{l-1}(x_{l-1}), x_{l+1}, \ldots, x_n)\end{aligned}$$

and

$$\begin{aligned}\mu^l(x_1, \ldots, x_{l-1}, x_l(2), x_{l+1}, \ldots, x_n) =& \mu(x_l)(1-\kappa(x_l))\frac{\mu^{l-1}(x_1, \ldots, x_{l-1}, x_l, \ldots, x_n)}{\mu(f_1(x_1), \ldots, f_{l-1}(x_{l-1}), x_l, \ldots, x_n)} \\ & \times \nu(x_l,2)(f_1(x_1), \ldots, f_{l-1}(x_{l-1}), x_{l+1}, \ldots, x_n),\end{aligned}$$

if $\mu(f_1(x_1), \ldots, f_{l-1}(x_{l-1}), x_l, \ldots, x_n) > 0$, and

$$\mu^l(x_1, \ldots, x_{l-1}, x_l(1), x_{l+1}, \ldots, x_n) = \mu^l(x_1, \ldots, x_{l-1}, x_l(2), x_{l+1}, \ldots, x_n) = 0$$

otherwise, where $\mu(\cdot \mid x_l) = \kappa(x_l)\nu(x_l, 1) + (1 - \kappa(x_l))\nu(x_l, 2)$.

And let

$$\mu^l(x_1, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_n) = \mu^{l-1}(x_1, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_n)$$

for every $x_l \notin Z_l$, $(x_1, \ldots, x_{l-1}) \in \prod_{1 \leq i \leq l-1} \tilde{X}_i$ and $(x_{l+1}, \ldots, x_n) \in \prod_{l+1 \leq i \leq n} X_i$.

It is easy to verify that $\tilde{\mu} = \mu^n$ satisfies the desired properties.

□

C Proof of Proposition 2, if direction

Fix an $l \geq 2$. Suppose that $\mu^1, \mu^2 \in \Delta(A)$ are CED's such that players condition their actions on their l-th order beliefs about play. By Theorem 2-B, this means that $\mu^k(\cdot \mid a_i)$ is injective on $Y_i^l(\mu^k)$, for $k = 1, 2$ and $i \in N$. Let $\gamma \in (0, 1)$ and $\mu = \gamma\mu^1 + (1 - \gamma)\mu^2$. We will show that $\mu(\cdot \mid a_i)$ is injective on $Y_i^l(\mu)$ as well.

For any $i \in N$, if $\mu^1(a_i) > 0$, $\mu^2(a_i) > 0$ and $\mu^1(\cdot \mid a_i) \neq \mu^2(\cdot \mid a_i)$, then $\mu(\cdot \mid a_i)$ is a strict convex combination of $\mu^1(\cdot \mid a_i)$ and $\mu^2(\cdot \mid a_i)$, so clearly $a_i \notin Y_i^1(\mu)$. Therefore, if $a_i \in Y_i^1(\mu)$, and $\mu^1(a_i) > 0$ (respectively, $\mu^2(a_i) > 0$), then we have that $\mu(\cdot \mid a_i) = \mu^1(\cdot \mid a_i)$ (respectively, $\mu(\cdot \mid a_i) = \mu^2(\cdot \mid a_i)$).

Let $Q_i^1 = \operatorname{supp}(\operatorname{marg}_{A_i} \mu^1)$ and $Q_i^2 = \operatorname{supp}(\operatorname{marg}_{A_i} \mu^2)$ for every $i \in N$. By the argument above, we have $Y_i^1(\mu) \cap Q_i^1 \subseteq Y_i^1(\mu^1)$ and $Y_i^1(\mu) \cap Q_i^2 \subseteq Y_i^1(\mu^2)$ for each $i \in N$. This implies that $Y_i^l(\mu) \cap Q_i^1 \subseteq Y_i^l(\mu^1)$ and $Y_i^l(\mu) \cap Q_i^2 \subseteq Y_i^l(\mu^2)$.

If $a_i \neq a_i' \in Y_i^l(\mu) \cap Q_i^1$, then $a_i \neq a_i' \in Y_i^l(\mu^1)$, and thus $\mu^1(\cdot \mid a_i) \neq \mu^1(\cdot \mid a_i')$. Therefore, we have $\mu(\cdot \mid a_i) \neq \mu(\cdot \mid a_i')$, by the reasoning in the second paragraph. And likewise for $a_i \neq a_i' \in Y_i^l(\mu) \cap Q_i^2$.

Now, suppose $a_i \neq a_i' \in Y_i^2(\mu)$ such that $a_i \in Q_i^1 \backslash Q_i^2$, $a_i' \in Q_i^2 \backslash Q_i^1$ and $\mu(\cdot \mid a_i) = \mu(\cdot \mid a_i')$. Then we have $\mu^1(\cdot \mid a_i) = \mu^2(\cdot \mid a_i')$. For any $a_j \in A_j$, $j \neq i$, such that $\mu^1(a_j \mid a_i) = \mu^2(a_j \mid a_i') > 0$, we have $a_j \in Y_j^1(\mu)$, which implies that $\mu(\cdot \mid a_j) = \mu^1(\cdot \mid a_j) = \mu^2(\cdot \mid a_j)$. But this implies that $\mu^1(a_i \mid a_j) = \mu(a_i \mid a_j) = \mu^2(a_i \mid a_j) > 0$, which contradicts $a_i \notin Q_i^2$. Thus, this case is impossible, and we are done.

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