Supplemental Materials for "What is the Optimal Trading Frequency in Financial Markets?"

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1 A Model with Multiple Dividend Payment

In the model of Du and Zhu (2016), we have assumed that the asset pays a single liquidating dividend at an exponentially-distributed time. In this section we consider a more general multi-dividend model. We derive a linear equilibrium and show that our conclusions of optimal trading frequency in the single-dividend model generalizes to this multi-dividend model. That is, under scheduled information arrivals, the trading frequency is never higher than information arrival frequency. But for stochastic information arrivals, the optimal trading frequency can far exceed the information arrival frequency.

1.1 Model setup

The multi-dividend model is specified as follows.

1. Dividends are paid at times $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots$, which follow a (homogeneous) Poisson process with intensity $r > 0$. We set $\mathcal{T}_0 \equiv 0$.

2. Before the K-th dividend is paid, shocks to dividends come at news times $T_{K,0}, T_{K,1}, T_{K,2}, \ldots$. We assume $T_{K,0} = \mathcal{T}_{K-1}$ and either:

   (a) $T_{K,k} - \mathcal{T}_{K-1} = k\gamma$ for a deterministic constant γ (scheduled arrivals of news); or

   (b) $\{T_{K,k} - \mathcal{T}_{K-1}\}_{k\geq 1}$ is a Poisson process with intensity μ (stochastic arrivals of news).

   At time $T_{K,0} = \mathcal{T}_{K-1}$ and immediately after the $(K-1)$-th dividend is paid, the dividend and private values are "renewed:"

   $$D_{T_{K,0}} \sim \mathcal{N}(0, \sigma_D^2), \qquad w_{i,T_{K,0}} \sim \mathcal{N}(0, \sigma_w^2). \tag{1}$$

   Subsequently at time $T_{K,k}$, $k \geq 1$, the common dividend and the private values are shocked according to:

   $$D_{T_{K,k}} - D_{T_{K,k-1}} \sim \mathcal{N}(0, \sigma_D^2), \qquad w_{T_{K,k}} - w_{T_{K,k-1}} \sim \mathcal{N}(0, \sigma_w^2). \tag{2}$$

   Trader i observes his private values and receives signals $S_{i,T_{K,k}}$ on the dividend shocks:

   $$S_{i,T_{K,k}} = D_{T_{K,k}} - D_{T_{K,k-1}} + \epsilon_{i,T_{K,k}}, \text{ where } \epsilon_{i,T_{K,k}} \sim \mathcal{N}(0, \sigma_\epsilon^2). \tag{3}$$

   If $T_{K,k'}$ is the last news time before the dividend is paid ($T_{K,k'} \leq \mathcal{T}_K$ and $T_{K,k'+1} > \mathcal{T}_K$), then each trader i receives $v_{i,T_{K,k'}} \equiv D_{T_{K,k'}} + w_{i,T_{K,k'}}$ per each unit of his asset at time $\mathcal{T}_K$.

3. Before the K-th dividend is paid, traders trade at time $\mathcal{T}_{K-1}, \mathcal{T}_{K-1}+\Delta, \mathcal{T}_{K-1}+2\Delta, \ldots$ If the K-th dividend is not yet paid at the t-th double auction ($\mathcal{T}_K \geq \mathcal{T}_{K-1}+t\Delta$), then let $z_{i,\mathcal{T}_{K-1}+t\Delta}$ be trader i's inventory before trading in the t-th double auction; his post-trading inventory is

$$z_{i,\mathcal{T}_{K-1}+(t+1)\Delta} = z_{i,\mathcal{T}_{K-1}+t\Delta} + x^*_{i,\mathcal{T}_{K-1}+t\Delta}. \tag{4}$$

If the t-th double auction is the last one before the K-th dividend is paid ($\mathcal{T}_{K-1}+t\Delta \leq \mathcal{T}_K < \mathcal{T}_{K-1}+(t+1)\Delta$), then $z_{i,\mathcal{T}_K} = z_{i,\mathcal{T}_{K-1}+(t+1)\Delta}$.

Here the trading time starts at the dividend time $\mathcal{T}_{K-1}$, instead of the next integer multiple of Δ. This assumption is made for analytical simplicity but is not critical for our results. Integer trading time can be incorporated by adding a discount factor $e^{-\beta((t'-t+1)\Delta-\tau)}$ in front of $\mathbb{E}[V_{i,0}(z_{i,t'\Delta}+x_{i,t'\Delta})]$ in Equation (5).

Traders have discount rate $\beta > 0$. A time discount is necessary for a model with infinitely many dividends. Trader i's conditional utility at time $t\Delta$ is (without loss of generality, suppose $\mathcal{T}_1 \geq t\Delta$):

$$\begin{aligned} &V_{i,t\Delta}(z_{i,t\Delta}) = \\ &\mathbb{E}\Bigg[\sum_{t'=t}^{\infty}\Bigg(\int_{\tau=(t'-t)\Delta}^{(t'-t+1)\Delta} re^{-r\tau}e^{-\beta\tau}\big(v_{i,t'\Delta}(z_{i,t'\Delta}+x^*_{i,t'\Delta}) + \mathbb{E}[V_{i,0}(z_{i,t'\Delta}+x^*_{i,t'\Delta})]\big)\,d\tau \\ &\qquad - \left(\int_{\tau=(t'-t)\Delta}^{(t'-t+1)\Delta} re^{-r\tau}\int_{s=(t'-t)\Delta}^{\tau} e^{-\beta s}\,ds\,d\tau + e^{-r(t'-t+1)\Delta}\int_{s=(t'-t)\Delta}^{(t'-t+1)\Delta} e^{-\beta s}\,ds\right)\frac{\lambda}{2}(z_{i,t'\Delta}+x^*_{i,t'\Delta})^2 \\ &\qquad - e^{-(r+\beta)(t'-t)\Delta}x^*_{i,t'\Delta}p^*_{t'\Delta}\Bigg)\Bigg| H_{i,t\Delta}\Bigg]. \end{aligned} \tag{5}$$

In the first term of the above equation we integrate over the payment time of the next dividend. If this dividend is paid in the interval $[t'\Delta, (t'+1)\Delta)$, then trader i has an expected continuation value of $\mathbb{E}[V_{i,0}(z_{i,t'\Delta}+x^*_{i,t'\Delta})]$, i.e., he "goes back" to time 0 with initial inventory $z_{i,t'\Delta}+x^*_{i,t'\Delta}$ and renewed realizations of dividend and private values. In the second term we integrate the discounted expected time during period t' when the dividend is yet to be paid. The third term is the expected payment during period t'.

By simplifying the integrals, we can rewrite the conditional utility as:

$$V_{i,t\Delta}(z_{i,t\Delta}) = \tag{6}$$
$$\mathbb{E}\left[\sum_{t'=t}^{\infty} e^{-(r+\beta)(t'-t)\Delta}\left(\frac{r(1-e^{-(r+\beta)\Delta})}{r+\beta}(v_{i,t'\Delta}(z_{i,t'\Delta}+x^*_{i,t'\Delta}) + \mathbb{E}[V_{i,0}(z_{i,t'\Delta}+x^*_{i,t'\Delta})]) - \frac{1-e^{-(r+\beta)\Delta}}{r+\beta}\frac{\lambda}{2}(z_{i,t'\Delta}+x^*_{i,t'\Delta})^2 - x^*_{i,t'\Delta}p^*_{t'\Delta}\right)\middle| H_{i,t\Delta}\right].$$

Note that the term $\mathbb{E}[V_{i,0}(z_{i,t'\Delta} + x_{i,t'\Delta})]$ in the above expression is the expectation over the new signals and new dividend, conditional on the inventory $z_{i,t'\Delta} + x_{i,t'\Delta}$. At the moment that the pending dividend is paid, signals regarding that dividend expire and new signals of the new dividend are yet to arrive.

1.2 Derivation of equilibrium strategy

For any K, traders face the same decision problem at time $\mathcal{T}_K + t\Delta$ as they do at time $t\Delta$, so it is natural that they use a strategy that is independent of $\mathcal{T}_K$. Without loss of generality, let us focus on $K = 0$, i.e., before the payment of the first dividend. To reduce algebra clutter, in this model we assume the total inventory $Z = 0$ to eliminate constant terms. For derivatives market, $Z = 0$ by definition.

Suppose that traders use the linear strategy:

$$x_{i,t\Delta}(p) = as_{i,t\Delta} - bp + dz_{i,t\Delta}, \tag{7}$$

where $s_{i,t\Delta}$ is the total signal as defined in the main text.

Since the continuation value at $t\Delta$ now contains a term $\mathbb{E}[V_{i,0}(z_{i,t'\Delta}+x^*_{i,t'\Delta})]$, we will first calculate its derivative with respect to $z_{i,0}$. Since $\mathbb{E}[V_{i,0}(z_{i,t'\Delta}+x^*_{i,t'\Delta})]$ is the expectation over the new signals and dividend, we take the expectation of (6) to pin down this derivative.

Given the simplifying assumption $Z = 0$, we have $\mathbb{E}[D_{t\Delta} + w_{i,t\Delta} \mid z_{i,0}] = 0$, $\mathbb{E}[p^*_{t\Delta} \mid z_{i,0}] =$

0, $\frac{\partial p^*_{t\Delta}}{\partial z_{i,0}} = 0$, and $\mathbb{E}[z_{i,t\Delta} \mid z_{i,0}] = (1+d)^t z_{i,0}$. Therefore,

$$\begin{aligned}\frac{\partial\, \mathbb{E}[V_{i,0}(z_{i,0})]}{\partial z_{i,0}} =& \mathbb{E}\Bigg[\sum_{t=0}^{\infty}\Bigg(e^{-(r+\beta)t\Delta}\frac{r(1-e^{-(r+\beta)\Delta})}{r+\beta}(1+d)^{t+1}\frac{\partial\, \mathbb{E}[V_{i,0}(z_{i,(t+1)\Delta})]}{\partial z_{i,(t+1)\Delta}} \\ &- e^{-(r+\beta)t\Delta}\frac{1-e^{-(r+\beta)\Delta}}{r+\beta}(1+d)^{t+1}\lambda z_{i,(t+1)\Delta}\Bigg)\Bigg| z_{i,0}\Bigg] \\ =& \mathbb{E}\Bigg[\sum_{t=0}^{\infty} e^{-(r+\beta)t\Delta}\frac{r(1-e^{-(r+\beta)\Delta})}{r+\beta}(1+d)^{t+1}\frac{\partial\, \mathbb{E}[V_{i,0}(z_{i,(t+1)\Delta})]}{\partial z_{i,(t+1)\Delta}}\Bigg| z_{i,0}\Bigg] \\ &- \frac{\frac{1-e^{-(r+\beta)\Delta}}{r+\beta}(1+d)^2}{1-e^{-(r+\beta)\Delta}(1+d)^2}\lambda z_{i,0} \end{aligned} \tag{8}$$

We conjecture

$$\frac{\partial\, \mathbb{E}[V_{i,0}(z_{i,0})]}{\partial z_{i,0}} = h z_{i,0} \tag{9}$$

for some constant h. Substituting this expression back to (8), we get

$$h = \frac{\frac{1-e^{-(r+\beta)\Delta}}{r+\beta}(1+d)^2}{1-e^{-(r+\beta)\Delta}(1+d)^2}(rh-\lambda), \tag{10}$$

i.e.,

$$h = -\lambda\frac{(1-e^{-(r+\beta)\Delta})(1+d)^2}{r+\beta-r(1+d)^2-\beta e^{-(r+\beta)\Delta}(1+d)^2}. \tag{11}$$

Suppose $0 < 1+d < 1$ (which we verify later), then (8) defines a contraction mapping, so the above solution of $\frac{\partial\, \mathbb{E}[V_{i,0}(z_{i,0})]}{\partial z_{i,0}}$ is the unique fixed point for Equation (8).

Suppose $t\Delta \le \mathcal{T}_1$. Under the single deviation principal, trader i's first order condition at time $t\Delta$ is:

$$\begin{aligned}\mathbb{E}\Bigg[(n-1)b\Bigg(&\frac{1-e^{-(r+\beta)\Delta}}{r+\beta}\sum_{k=0}^{\infty}e^{-(r+\beta)k\Delta}(1+d)^k\left(rv_{i,(t+k)\Delta}+(rh-\lambda)(z_{i,(t+k)\Delta}+x^*_{i,(t+k)\Delta})\right) \\ &- p^*_{t\Delta} - \sum_{k=1}^{\infty}e^{-(r+\beta)k\Delta}(1+d)^{k-1}d\,p^*_{(t+k)\Delta}\Bigg) - x_{i,t\Delta}\ \Bigg|\ H_{i,t\Delta}\cup\{\textstyle\sum_{j\neq i}s_{j,t\Delta}\}\Bigg] = 0,\end{aligned} \tag{12}$$

where

$$\begin{aligned} z_{i,(t+k)\Delta} + x^*_{i,(t+k)\Delta} &= as_{i,(t+k)\Delta} - bp^*_{(t+k)\Delta} + (1+d)z_{i,(t+k)\Delta} \\ &= (as_{i,(t+k)\Delta} - bp^*_{(t+k)\Delta}) + (1+d)(as_{i,(t+k-1)\Delta} - bp^*_{(t+k-1)\Delta}) \\ &\quad + \cdots + (1+d)^{k-1}(as_{i,(t+1)\Delta} - bp^*_{(t+1)\Delta}) + (1+d)^k(x_{i,t\Delta} + z_{i,t\Delta}), \end{aligned} \tag{13}$$

$$p^*_{t\Delta} = \frac{a}{nb}\sum_{j=1}^{n} s_{j,t\Delta}. \tag{14}$$

Using the notation $\bar{s}_{t\Delta} = \sum_{1\le j\le n} s_{j,t\Delta}/n$, the first order condition can be rewritten as:

$$\begin{aligned}
&(n-1)b(1-e^{-(r+\beta)\Delta}) \\
&\cdot\Bigg[\frac{1}{1-e^{-(r+\beta)\Delta}(1+d)}\left(\frac{r}{r+\beta}\left(\alpha s_{i,t\Delta} + \frac{1-\alpha}{n-1}\sum_{j\neq i} s_{j,t\Delta}\right) - \frac{a}{b}\bar{s}_{t\Delta}\right) \\
&+\sum_{k=0}^{\infty}\frac{rh-\lambda}{r+\beta}e^{-(r+\beta)k\Delta}(1+d)^k\left(\frac{1}{-d}-\frac{(1+d)^k}{-d}\right)a(s_{i,t\Delta}-\bar{s}_{t\Delta}) \\
&+\frac{rh-\lambda}{(1-e^{-(r+\beta)\Delta}(1+d)^2)(r+\beta)}(x_{i,t\Delta}+z_{i,t\Delta})\Bigg] - x_{i,t\Delta} = 0,
\end{aligned} \tag{15}$$

where we have used the the identity:

$$1+\sum_{k=1}^{\infty} e^{-(r+\beta)k\Delta}(1+d)^{k-1}d = 1 + \frac{e^{-(r+\beta)\Delta}d}{1-(1+d)e^{-(r+\beta)\Delta}} = \frac{1-e^{-(r+\beta)\Delta}}{1-(1+d)e^{-(r+\beta)\Delta}}. \tag{16}$$

Rearranging the terms gives:

$$\begin{aligned}
&\left(1+\frac{(n-1)b(1-e^{-(r+\beta)\Delta})(\lambda-rh)}{(1-e^{-(r+\beta)\Delta}(1+d)^2)(r+\beta)}\right)x_{i,t\Delta} \\
=&(n-1)b(1-e^{-(r+\beta)\Delta}) \\
&\cdot\Bigg[\frac{1}{1-e^{-(r+\beta)\Delta}(1+d)}\left(\frac{r}{r+\beta}\frac{n\alpha-1}{n-1}s_{i,t\Delta} + \frac{r}{r+\beta}\frac{n-n\alpha}{n-1}\bar{s}_{t\Delta} - \frac{a}{b}\bar{s}_{t\Delta}\right) \\
&-\frac{(\lambda-rh)e^{-(r+\beta)\Delta}(1+d)}{(r+\beta)(1-(1+d)e^{-(r+\beta)\Delta})(1-(1+d)^2e^{-(r+\beta)\Delta})}a(s_{i,t\Delta}-\bar{s}_{t\Delta}) \\
&-\frac{\lambda-rh}{(1-e^{-(r+\beta)\Delta}(1+d)^2)(r+\beta)}z_{i,t\Delta}\Bigg].
\end{aligned} \tag{17}$$

On the other hand, our conjectured strategy implies:

$$x_{i,t\Delta} = a(s_{i,t\Delta}-\bar{s}_{t\Delta}) + dz_{i,t\Delta}. \tag{18}$$

Matching the coefficients of $s_{i,t\Delta}$, $\bar{s}_{i,t\Delta}$ and $z_{i,t\Delta}$ in (17) with those in (18) gives:

$$a = \frac{r}{r+\beta}b, \tag{19}$$

$$1 + \frac{(n-1)b(1-e^{-(r+\beta)\Delta})(\lambda - rh)}{(1-e^{-(r+\beta)\Delta}(1+d)^2)(r+\beta)}$$
$$= \frac{(1-e^{-(r+\beta)\Delta})(n\alpha - 1)}{1-e^{-(r+\beta)\Delta}(1+d)} - \frac{(n-1)b(1-e^{-(r+\beta)\Delta})e^{-(r+\beta)\Delta}(1+d)(\lambda - rh)}{(1-(1+d)e^{-(r+\beta)\Delta})(1-(1+d)^2e^{-(r+\beta)\Delta})(r+\beta)}, \tag{20}$$

$$\left(1 + \frac{(n-1)b(1-e^{-(r+\beta)\Delta})(\lambda - rh)}{(1-e^{-(r+\beta)\Delta}(1+d)^2)(r+\beta)}\right) d = -\frac{(n-1)b(1-e^{-(r+\beta)\Delta})(\lambda - rh)}{(1-e^{-(r+\beta)\Delta}(1+d)^2)(r+\beta)}. \tag{21}$$

Solving these equations gives:

$$d = -\frac{1}{2e^{-(r+\beta)\Delta}}\left((n\alpha - 1)(1-e^{-(r+\beta)\Delta}) + 2e^{-(r+\beta)\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-(r+\beta)\Delta})^2 + 4e^{-(r+\beta)\Delta}}\right), \tag{22}$$

$$b = \frac{(n\alpha - 1)(r+\beta)}{2(n-1)e^{-(r+\beta)\Delta}(\lambda - rh)} \tag{23}$$
$$\cdot \left((n\alpha - 1)(1-e^{-(r+\beta)\Delta}) + 2e^{-(r+\beta)\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-(r+\beta)\Delta})^2 + 4e^{-(r+\beta)\Delta}}\right).$$

Direct computations show that d satisfies:

$$\frac{(1-e^{-(r+\beta)\Delta})(1+d)^2}{1-e^{-(r+\beta)\Delta}(1+d)^2} = \frac{1+d}{n\alpha - 1}. \tag{24}$$

Therefore, (10) becomes:

$$h = \frac{1+d}{(n\alpha-1)(r+\beta)}(rh - \lambda), \tag{25}$$

so

$$h = \frac{-\lambda}{\frac{(n\alpha-1)(r+\beta)}{1+d} - r}, \tag{26}$$

and

$$\lambda - rh = \frac{\lambda}{1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}}. \tag{27}$$

Proposition 1. *Suppose that $n\alpha > 2$, which is equivalent to*

$$\frac{1}{n/2 + \sigma_\epsilon^2/\sigma_D^2} < \sqrt{\frac{n-2}{n}}\frac{\sigma_w}{\sigma_\epsilon}. \tag{28}$$

There exists a perfect Bayesian equilibrium in which at time $\mathcal{T}_K + t\Delta$, trader i submits the demand schedule

$$x_{i,t\Delta}(p) = b\left(\frac{r}{r+\beta}s_{i,\mathcal{T}_K+t\Delta} - p - \frac{(\lambda - rh)(n-1)}{(r+\beta)(n\alpha-1)}z_{i,\mathcal{T}_K+t\Delta}\right), \tag{29}$$

where

$$\begin{aligned} b = &\frac{(n\alpha-1)(r+\beta)}{2(n-1)e^{-(r+\beta)\Delta}(\lambda - rh)} \\ &\cdot\left((n\alpha-1)(1-e^{-(r+\beta)\Delta}) + 2e^{-(r+\beta)\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-(r+\beta)\Delta})^2 + 4e^{-(r+\beta)\Delta}}\right), \end{aligned} \tag{30}$$

$$d = -\frac{1}{2e^{-(r+\beta)\Delta}}\left((n\alpha-1)(1-e^{-(r+\beta)\Delta}) + 2e^{-(r+\beta)\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-(r+\beta)\Delta})^2 + 4e^{-(r+\beta)\Delta}}\right), \tag{31}$$

$$h = \frac{-\lambda}{\frac{(n\alpha-1)(r+\beta)}{1+d} - r}. \tag{32}$$

The period-t equilibrium price is

$$p^*_{t\Delta} = \frac{r}{(r+\beta)n}\sum_{i=1}^{n} s_{i,t\Delta}. \tag{33}$$

1.3 Optimal trading frequency

Let $z^*_{i,t\Delta}$ be the equilibrium inventory of trader i at time $t\Delta$. The equilibrium welfare conditional on the initial inventory is the fixed point $W(\cdot)$ that solves:

$$\begin{aligned} W(\{z_{i,0}\}) = \mathbb{E}\Bigg[\sum_{t=0}^{\infty}\frac{1-e^{-(r+\beta)\Delta}}{r+\beta}e^{-(r+\beta)t\Delta}\Bigg(&r\sum_{i=1}^{n} v_{i,t\Delta}z^*_{i,(t+1)\Delta} + rW(\{z^*_{i,(t+1)\Delta}\}) \\ &- \frac{\lambda}{2}\sum_{i=1}^{n}(z^*_{i,(t+1)\Delta})^2\Bigg)\Bigg|\{z_{i,0}\}\Bigg]. \end{aligned} \tag{34}$$

We first recall that

$$z^*_{i,(t+1)\Delta} = \sum_{t'=0}^{t}(1+d)^{t-t'}a(s_{i,t'\Delta} - \bar{s}_{t'\Delta}) + (1+d)^{t+1}z_{i,0}. \tag{35}$$

Because $z^*_{i,(t+1)\Delta}$ is squared in (34) and $\sum_{i=1}^{n}\mathbb{E}[v_{i,t\Delta}z^*_{i,(t+1)\Delta} \mid \{z_{i,0}\}]$ depends only on

$\sum_{i=1}^n (z_{i,0})^2$ as $\sum_{i=1}^n z_{i,0} = 0$, we conjecture that

$$W(\{z_{i,0}\}) = L_1 \sum_{i=1}^n (z_{i,0})^2 + L_2, \tag{36}$$

for constants L_1 and L_2.

Substituting this conjecture into (34) and matching the coefficients, we get:

$$L_1 = \sum_{t=0}^{\infty} \frac{1-e^{-(r+\beta)\Delta}}{r+\beta} e^{-(r+\beta)t\Delta} \left(rL_1(1+d)^{2(t+1)} - \frac{\lambda}{2}(1+d)^{2(t+1)} \right) \tag{37}$$

$$= \frac{(1-e^{-(r+\beta)})(1+d)^2}{(1-e^{-(r+\beta)\Delta}(1+d)^2)(r+\beta)} \left(rL_1 - \frac{\lambda}{2} \right) = \frac{1+d}{(n\alpha-1)(r+\beta)} \left(rL_1 - \frac{\lambda}{2} \right), \tag{38}$$

i.e.,

$$L_1 = h/2 = \frac{-\lambda/2}{\frac{(n\alpha-1)(r+\beta)}{1+d} - r}. \tag{39}$$

$$L_2 = \mathbb{E}\left[\sum_{t=0}^{\infty} \frac{1-e^{-(r+\beta)\Delta}}{r+\beta} e^{-(r+\beta)t\Delta} \left(r \sum_{i=1}^n v_{i,t\Delta} z^*_{i,(t+1)\Delta} + rL_1 \sum_{i=1}^n ((z^*_{i,(t+1)\Delta})^2 - (1+d)^{2(t+1)}(z_{i,0})^2) + rL_2 - \frac{\lambda}{2} \sum_{i=1}^n ((z^*_{i,(t+1)\Delta})^2 - (1+d)^{2(t+1)}(z_{i,0})^2) \right) \middle| \{z_{i,0}\} \right], \tag{40}$$

i.e.,

$$L_2 = \frac{r+\beta}{\beta} \mathbb{E}\left[\sum_{t=0}^{\infty} \frac{1-e^{-(r+\beta)\Delta}}{r+\beta} e^{-(r+\beta)t\Delta} \left(r \sum_{i=1}^n v_{i,t\Delta} z^*_{i,(t+1)\Delta} - \frac{\lambda - rh}{2} \sum_{i=1}^n (z^*_{i,(t+1)\Delta})^2 \right) \middle| \{z_{i,0}\} \right] + \frac{r+\beta}{\beta} \frac{1+d}{(n\alpha-1)(r+\beta)} \frac{\lambda-rh}{2} \sum_{i=1}^n (z_{i,0})^2. \tag{41}$$

Define

$$z^e_{i,\tau} = \frac{r(n\alpha-1)}{(\lambda-rh)(n-1)} \left(s_{i,\tau} - \frac{1}{n} \sum_{j=1}^n s_{j,\tau} \right). \tag{42}$$

Under this definition, we have:

$$z^*_{i,(t+1)\Delta} - z^e_{i,t\Delta} = (1+d)(z^*_{i,t\Delta} - z^e_{i,t\Delta}). \tag{43}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=0}^{\infty}(1-e^{-(r+\beta)\Delta})e^{-(r+\beta)t\Delta}\left(\sum_{i=1}^{n}v_{i,t\Delta}z^{*}_{i,(t+1)\Delta}-\frac{\lambda-rh}{2r}\sum_{i=1}^{n}(z^{*}_{i,(t+1)\Delta})^2\right)\middle|\{z_{i,0}\}\right] \qquad (44)\\
=&\mathbb{E}\left[\sum_{t=0}^{\infty}(1-e^{-(r+\beta)\Delta})e^{-(r+\beta)t\Delta}\left(\sum_{i=1}^{n}v_{i,t\Delta}z^{e}_{i,t\Delta}-\frac{\lambda-rh}{2r}\sum_{i=1}^{n}(z^{e}_{i,t\Delta})^2\right)\middle|\{z_{i,0}\}\right]\\
&-\mathbb{E}\left[\sum_{t=0}^{\infty}(1-e^{-(r+\beta)\Delta})e^{-(r+\beta)t\Delta}\frac{\lambda-rh}{2r}(z^{e}_{i,t\Delta}-z^{*}_{i,(t+1)\Delta})^2\middle|\{z_{i,0}\}\right]\\
=&\mathbb{E}\left[\sum_{t=0}^{\infty}(1-e^{-(r+\beta)\Delta})e^{-(r+\beta)t\Delta}\frac{\lambda-rh}{2r}\sum_{i=1}^{n}(z^{e}_{i,t\Delta})^2\middle|\{z_{i,0}\}\right]\\
&-\left(\mathbb{E}\left[\sum_{t=0}^{\infty}(1-e^{-(r+\beta)\Delta})e^{-(r+\beta)t\Delta}\frac{\lambda-rh}{2r}\frac{1+d}{n\alpha-1}\sum_{i=1}^{n}(z^{e}_{i,t\Delta})^2\middle|\{z_{i,0}\}\right]+\frac{\lambda-rh}{2r}\frac{1+d}{n\alpha-1}\sum_{i=1}^{n}(z_{i,0})^2\right),
\end{aligned}$$

where the first equality follows from Lemma 1 of Du and Zhu (2016), and the second equality follows from Lemma 2 of Du and Zhu (2016) and from

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\alpha s_{i,t\Delta}+\frac{1-\alpha}{n-1}\sum_{j\neq i}s_{j,t\Delta}\right)z^{e}_{i,t\Delta}-\frac{\lambda-rh}{r}(z^{e}_{i,t\Delta})^2=\sum_{i=1}^{n}\left(\alpha s_{i,t\Delta}+\frac{1-\alpha}{n-1}\sum_{j\neq i}s_{j,t\Delta}-\frac{\lambda-rh}{r}z^{e}_{i,t\Delta}\right)z^{e}_{i,t\Delta}\\
&=\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{n}s_{j,t\Delta}\right)z^{e}_{i,t\Delta}=0.
\end{aligned}$$

Therefore,

$$L_2=\frac{\lambda-rh}{2\beta}\left(1-\frac{1+d}{n\alpha-1}\right)\sum_{t=0}^{\infty}(1-e^{-(r+\beta)\Delta})e^{-(r+\beta)t\Delta}\sum_{i=1}^{n}\mathbb{E}[(z^{e}_{i,t\Delta})^2], \qquad (45)$$

and (using Equation (27))

$$\begin{aligned}
W(\{z_{i,0}\})=&\frac{-\lambda/2}{\frac{(n\alpha-1)(r+\beta)}{1+d}-r}\sum_{i=1}^{n}(z_{i,0})^2 \qquad (46)\\
&+\frac{\lambda/(2\beta)}{1-\frac{(1+d)r}{(n\alpha-1)(r+\beta)}}\left(1-\frac{1+d}{n\alpha-1}\right)\sum_{t=0}^{\infty}(1-e^{-(r+\beta)\Delta})e^{-(r+\beta)t\Delta}\sum_{i=1}^{n}\mathbb{E}[(z^{e}_{i,t\Delta})^2].
\end{aligned}$$

1.3.1 Scheduled arrivals of information

Suppose that new information arrives at time $0,\gamma,2\gamma,\ldots$. For simplicity of notation, we use the shorthand W for $\mathbb{E}[W(\{z_{i,0}\})]$. We also write $W(\Delta)$ to emphasize the dependence of W

on Δ.

Proposition 2. *For any integer $l \geq 1$, we have $W(\gamma/l) < W(\gamma)$. If the traders are ex-ante symmetric, i.e., $z_{i,0} = 0$ for every trader i, then as $n \to \infty$, the optimal $l^* \to 1$, where l^* maximizes $W(l\gamma)$ over integer $l \geq 1$.*

Proof. Let $\Delta = \gamma/l$ where $l \geq 1$ is an integer. We have:

$$\begin{aligned}
\sum_{i=1}^{n} \mathbb{E}[(z^e_{i,k\gamma})^2] &= (k+1)\sum_{i=1}^{n} \mathbb{E}[(z^e_{i,0})^2] = (k+1)\sum_{i=1}^{n} \mathbb{E}\left[\frac{r^2(n\alpha-1)^2}{(\lambda - rh)^2(n-1)^2}\left(s_{i,0} - \frac{1}{n}\sum_{j=1}^{n} s_{j,0}\right)^2\right] \\
&= (k+1)\left(1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}\right)^2 \sum_{i=1}^{n} \mathbb{E}\left[\frac{r^2(n\alpha-1)^2}{\lambda^2(n-1)^2}\left(s_{i,0} - \frac{1}{n}\sum_{j=1}^{n} s_{j,0}\right)^2\right] \\
&\equiv (k+1)\left(1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}\right)^2 \sigma_z^2, \qquad (47)
\end{aligned}$$

where in the second line we have used Equation (27), and in the last line we define σ_z^2 as in the main text.

We can then simplify (46) to:

$$\begin{aligned}
W(\{z_{i,0}\}) = &\frac{-\lambda/2}{\frac{(n\alpha-1)(r+\beta)}{1+d} - r}\sum_{i=1}^{n}(z_{i,0})^2 \qquad (48) \\
&+ \frac{\lambda}{2\beta}\left(1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}\right)\left(1 - \frac{1+d}{n\alpha-1}\right)\sum_{k=0}^{\infty}(1 - e^{-(r+\beta)\gamma})e^{-(r+\beta)k\gamma}(k+1)\sigma_z^2.
\end{aligned}$$

Since $1+d$ is decreasing in Δ, it is easy to see that

$$\frac{1}{\frac{(n\alpha-1)(r+\beta)}{1+d} - r}$$

is decreasing in Δ, and that

$$\left(1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}\right)\left(1 - \frac{1+d}{n\alpha-1}\right)$$

is increasing in Δ. Therefore, $W(\{z_{i,0}\})$ above is increasing in $\Delta = \gamma/l$ for positive integer l.

Now suppose $\Delta = l\gamma$, $l \geq 1$. We have

$$\sum_{i=1}^{n} \mathbb{E}[(z^e_{i,t\Delta})^2] = (tl+1)\left(1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}\right)^2 \sigma_z^2. \qquad (49)$$

The above expression is same as Equation (50) below, for the case when news come at Poisson times, by setting $l = \mu\Delta$. The proof of the second part of this proposition follows from the proof of Proposition 3 below.

□

1.3.2 Stochastic arrivals of information

Suppose that new information arrives according to a Poisson process with intensity $\mu > 0$.

Proposition 3. *Suppose traders are ex-ante symmetric: $z_{i,0} = 0$ for every trader i.*

1. *The optimal Δ^* is strictly decreasing in the intensity μ from ∞ (as $\mu \to 0$) to 0 (as $\mu \to \infty$).*
2. *As $n \to \infty$, the optimal $\Delta^* \to 0$.*

Proof. Similar to (47), we have here:

$$\sum_{i=1}^{n} \mathbb{E}[(z^e_{i,t\Delta})^2] = (t\Delta\mu + 1)\left(1 - \frac{(1+d)r}{(n\alpha - 1)(r+\beta)}\right)^2 \sigma_z^2. \tag{50}$$

We can then simplify (46) to:

$$\begin{aligned}
W(\{z_{i,0}\}) &= \frac{-\lambda/2}{\frac{(n\alpha-1)(r+\beta)}{1+d} - r} \sum_{i=1}^{n} (z_{i,0})^2 \\
&\quad + \frac{\lambda}{2\beta}\left(1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}\right)\left(1 - \frac{1+d}{n\alpha - 1}\right)\sum_{t=0}^{\infty}(1 - e^{-(r+\beta)\Delta})e^{-(r+\beta)t\Delta}(t\Delta\mu + 1)\sigma_z^2. \\
&= \frac{-\lambda/2}{\frac{(n\alpha-1)(r+\beta)}{1+d} - r} \sum_{i=1}^{n} (z_{i,0})^2 \\
&\quad + \frac{\lambda}{2\beta}\left(1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}\right)\left(1 - \frac{1+d}{n\alpha - 1}\right)\left(\frac{\Delta\mu e^{-(r+\beta)\Delta}}{1 - e^{-(r+\beta)\Delta}} + 1\right)\sigma_z^2.
\end{aligned} \tag{51}$$

The first term vanishes since by assumption $z_{i,0} = 0$ for every i.

We first observe that for the ex-ante symmetric case optimizing W over Δ is equivalent to optimizing $\tilde{W}$ over Δ, where:

$$\tilde{W} \equiv \log\left(1 - \frac{1+d}{n\alpha - 1}\right) + \log\left(1 - \frac{(1+d)r}{(n\alpha-1)(r+\beta)}\right) + \log\left(\frac{\Delta\mu e^{-(r+\beta)\Delta}}{1 - e^{-(r+\beta)\Delta}} + 1\right). \tag{52}$$

For Part 1 of the proposition, we calculate:

$$\frac{\partial^2}{\partial\mu\partial\Delta}\log\left(\frac{\Delta\mu e^{-(r+\beta)\Delta}}{1-e^{-(r+\beta)\Delta}}+1\right)<0, \tag{53}$$

which implies that Δ^* must decrease with μ. As $\mu \to 0$, $\Delta^* \to \infty$ because $1+d$ decreases with Δ. As $\mu \to \infty$, $\Delta^* \to 0$ because $\frac{\Delta e^{-(r+\beta)\Delta}}{1-e^{-(r+\beta)\Delta}}$ decreases with Δ.

Part 2 of the proposition follows from the fact that $(1+d)/(n\alpha-1) \to 0$ as $n \to \infty$.

□

References

Du, S. and H. Zhu (2016): "What is the Optimal Trading Frequency in Financial Markets?" Working paper.