Robust Mechanisms for the Financing of Public Goods*

Benjamin Brooks Songzi Du

March 30, 2023

Abstract

We propose a novel *proportional cost-sharing mechanism* for funding public goods with interdependent values. In this mechanism, the agents simultaneously submit "bids," the expenditure on the public good is an increasing function of the sum of the bids, and each agent is responsible for the fraction of the expenditure proportional to their bid. The proportional cost-sharing mechanism provides a non-trivial guarantee for social welfare, regardless of the structure of the agents' information and the equilibrium that is played, as long as the social value for the public good is sufficiently large. Moreover, this guarantee is shown to be unimprovable in environments where the designer knows a lower bound on the social value. These mechanisms guarantee the entire efficient surplus when the social value becomes large. When there are two agents, our model can be reinterpreted as one of bilateral trade.

KEYWORDS: Mechanism design, information design, public good, interdependent values, Bayes correlated equilibrium.

JEL CLASSIFICATION: C72, D44, D82, D83.

*Brooks: Department of Economics, University of Chicago, babrooks@uchicago.edu; Du: Department of Economics, University of California San Diego, sodu@ucsd.edu.

1 Introduction

Consider a society that is deciding how much resources to expend on a public good. The good can be produced in continuous quantity and at linear cost up to a maximum amount, and valuations are linear and increasing in the expenditure. The agents who make up the society can form a social contract, consisting of a mechanism that will determine how much to spend on the good and each agent's share of the expense. The agents possess private information about their respective valuations of the good, which may differ across agents. The agents must agree to the proposed mechanism after they are endowed with private information. This gives rise to a free-rider problem, wherein the agents may behave as if their value for the good is lower than it truly is, in order to reduce their share of the expense. The question is: What kind of mechanism should the society implement, in order to maximize their joint welfare?

This public expenditure problem has been studied in various forms going back to Samuelson (1954). The formulation closest to ours is that of Güth and Hellwig (1986), who assume that each agent knows their own value (and only their own value) for the public good and values are independently distributed. In general, the exact solution of the Bayesian mechanism design problem is sensitive to assumptions about the distribution of values and the higher-order beliefs held by the agents, which are typically modeled as an information structure (i.e., a Harsanyi type space). However, it seems hard to say which information structure is empirically relevant, especially if we allow for the possibility that signals may be correlated and values may be interdependent, meaning that one agent's private signal may be informative about another agent's value.

It is our opinion that in real-world applications, mechanism designers will only be able to describe relatively coarse features of the environment, like the average value of the agents, or the range of possible values. In particular, a mechanism designer may be unwilling or unable to commit to a particular information structure as the correct description of what the agents know. Thus, a useful formulation of the mechanism design problem will consider the performance of a mechanism across a range of information structures that are consistent with those coarse features.

This paper contributes such a theory. In our baseline model, the mechanism designer specifies a lower bound on the social value of the good (i.e., the sum of the agents' values). We later consider a richer version of the model where the designer further specifies an upper bound on the social value and a lower bound on the expected social value. We first describe the results for the baseline model, and then we will describe the extension.

The *welfare guarantee* of a mechanism defined to be its infimum social welfare across all information structures and equilibria. Importantly, subject to the restrictions in the previous paragraph, we allow for arbitrary (common prior) information, and there can be correlation in signals and interdependence in values. Loosely speaking, we compute mechanisms that maximize this guarantee. The solutions that we identify have the form of *proportional cost-sharing mechanisms*: Each agent's action is a non-negative number; the total expenditure is an increasing and concave function of the aggregate action; and each agent's share of the cost is proportional to their action. These mechanisms create a natural tradeoff for the agents that mitigates free riding: By reducing their action, agents can reduce their share of the expense, but they also reduce the total expenditure. To our knowledge, these mechanisms are new to the literature. The logic underlying these mechanisms is that they perfectly balance the *strategic virtual objective* across states and action profiles. This concept, and its connection to the informational-robustness of a mechanism, is discussed further in our literature review below.

In parallel, we also study the worst-case information structures against which the proportional cost-sharing mechanisms are implicitly guarding. The *welfare potential* of an information structure is its maximum welfare across all mechanisms and equilibria. We construct an information structure that has a potential equal to the guarantee of the proportional cost-sharing mechanism. In a sense that we will expand on shortly, these information structures minimize the potential. The information structure is of the following form: Perhaps not surprisingly, the social value of the good is equal to its lower bound with probability one, the agents' signals are non-negative real numbers, the density only depends on the sum of the signals, and each agent's expected value for the good is proportional to their signal.

In order to describe our results more precisely, we must address an important technical issue that arises in our model. The guarantee-maximizing mechanism has a continuum of actions and the potential-minimizing information has a continuum of signals. An obvious concern with such infinite objects is that the guarantee and potential might be either ill-defined or vacuous, because equilibria fail to exist for some or all information structures or mechanisms, respectively. A similar issue arises in our earlier work (Brooks and Du, 2021, 2023). We address this concern by proving that an equilibrium exists at the saddle point consisting of the guarantee-maximizing mechanism and potential-minimizing information structure. In fact, this equilibrium is simply the "truthful" or "obedient" strategies, where each agent plays an action equal to their signal. Thus, the solution we construct exhibits the "double revelation principle" that also appeared in our prior work on revenue maximization in common-value auctions (Brooks and Du, 2021). Moreover, the triple of the mechanism, information structure, and equilibrium we construct is a *strong maxmin solution*, as defined in (Brooks and Du, 2021): Fixing the mechanism, welfare can only go up as we vary the information structure and equilibrium, and fixing the information structure, welfare can only go down as we vary the mechanism and equilibrium.

The solution just described works as long as the minimum social value is sufficiently large. As long as the social value per unit expenditure is greater than one, it is socially efficient to produce the good. However, even when this is the case, it may still be impossible to implement efficient expenditure because of the free-rider problem, so that the maximum guarantee is less than the efficient surplus. In the case where the per capita social value is at least one, the solution is as just described. However, when the per-capital social value is less than one, so that the value of a dollar of public expenditure is worth less than a dollar to the average agent, we construct information structures for which the welfare potential is arbitrarily close to zero. That it should be so challenging to generate non-trivial guarantees is perhaps not surprising given the weakness of our assumptions on the information structure. Nonetheless, it is quite striking that even though there is common knowledge that the efficient outcome is full expenditure, there are information structures for which the free-rider problem is so severe that no mechanism and equilibrium can generate a non-negligible amount of surplus.

As mentioned above, we also consider the more general model where there is an upper bound on the social value and a lower bound on the expected social value. The value of this extension is that it allows for positive probability that the social value is arbitrarily small. The main finding is that proportional cost-sharing mechanisms continue to maximize the guarantee. If the expected social value per capita is less than one, the model again collapses, and the max guarantee and min potential are both zero. But as long as the expected social value per capita is greater than one, there are strong maxmin solutions with non-trivial guarantee. Moreover, a proportional cost-sharing mechanism is part of the solution and attains the guarantee. The difference with our baseline model is in the particular form of the total expenditure function.

Thus, while the exact optimal total expenditure rule depends on the particular assumptions we impose on fundamentals, the main takeaway from our model is that the class of proportional cost-sharing mechanisms provides unimprovable welfare guarantees, if we allow for relatively large uncertainty about the form of private information and the dispersion in values.

An important consideration is what happens if the assumptions about fundamentals are also misspecified. For example, what happens to the welfare guarantee of a guarantee-maximizing proportional cost-sharing mechanism if the social value ends up being significantly larger than the lower bound? We show that even if these assumptions are slightly misspecified, the mechanism will still provide a welfare guarantee that is close to the optimum. Moreover, Proposition 3 shows that as the social value grows large, the proportional cost-sharing mechanism guarantees a fraction of the efficient surplus that converges to one. Thus, when the social value is large, proportional cost-sharing mechanisms achieve approximately efficient outcomes.

Our work is connected to the literatures on the public expenditure problem and on informationally-robust mechanism design. Most of what is known about the public goods problem concerns information structures with private values. In this case, it is well-known that if there is no budget constraint, then the efficient outcome can be implemented in dominant strategies with the Vickrey-Clarke-Groves mechanism. Moreover, d'Aspremont and Gérard-Varet (1979) showed that it is still possible to achieve efficient outcomes with Bayes Nash implementation and ex post budget balance, as long as participation constraints are ex ante. The closest paper to ours appears to be Güth and Hellwig (1986), who study Bayes Nash implementation, ex ante budget balance, and interim participation constraints. They characterize social-welfare maximizing direct mechanisms, subject to ex ante budget balance. The mechanisms they describe have full expenditure if the sum of the agents' reported virtual values exceeds a cutoff. Otherwise, the expenditure is zero. The agents' interim payments are pinned down by the standard Mirleesian taxation principle. Importantly, the virtual values and the transfers depend on the value distribution, so in that sense, the designer needs to know a great deal about the environment in order to calibrate the mechanism.

We are unaware of other work that addresses optimal mechanism design for the public goods problem with interdependent values and correlated types. However, as we will expand on towards the end of the paper, the special case of our model with two agents can be reinterpreted as an instance of the bilateral trade problem, where there is an upper bound on the seller's value and a lower bound on the gains from trade. Our analysis is therefore also connected to the literature on bilateral trade, which has highlighted the inefficiency that results from budget balance and incentive and participation constraints (Myerson and Satterthwaite, 1983), as well as the potential welfare-reducing effects of interdependence in values and correlation in signals (Akerlof, 1970; Carroll, 2016). In the bilateral trade context, the proportional cost-sharing mechanism can be reinterpreted as a *proportional-price trading mechanism*: The buyer and seller submit non-negative numbers, trade occurs with a probability that depends on the aggregate signal, and the price is a weighted average of the lowest possible value of the buyer and the highest possible value of the seller. These weights are proportional to the actions of the seller and the buyer, respectively. Thus, by increasing their action, an agent can increase the probability of trade, but at the cost of moving the terms of trade in a direction that is unfavorable. Proposition 4 shows that proportional-price trading mechanisms provide optimal guarantees for gains from trade.

Within the robust mechanism design literature, our work is most closely related to the recent literature on maxmin mechanism design (Chung and Ely, 2007; Bergemann et al., 2016; Du, 2018; Brooks and Du, 2021, 2023). Most closely related is Brooks and Du (2023), who describe a general framework for informational robust optimal mechanism design. Setting aside technical differences regarding finite versus infinite mechanisms and information structures, our model is a special case of that of Brooks and Du (2023). That paper introduced the notions of strategic and informational virtual objectives which are the designer's objective plus an adjustment corresponding to local equilibrium constraints. In the case of the strategic virtual objective, this adjustment is the sum of the agents' gains from deviating to nearby actions, and in the case of the informational virtual objective, the adjustment is the sum of agents' gains from mimicking nearby types. Brooks and Du (2023) argued that the expectation (across payoff-relevant states) of the lowest (across actions) strategic virtual objective is a lower bound on a mechanism's guarantee. Similarly, the expectation (across signals) of the highest (across outcomes) informational virtual objective is an upper bound on the potential. The present paper applies this bounding methodology to the public goods problem: the proportional cost-sharing mechanisms maximize the expected lowest strategic virtual objective, and the worst-case information structures minimize the expected highest informational virtual objective. Some of our results are described in Brooks and Du (2023) at a high level as an application of the more general framework.

The rest of this paper is organized as follows. Section 2 describes the model. Section 3 analyzes the baseline model where there is only a lower bound on the social value. Section 4 analyzes the extension where there are lower and upper bounds and a known expected social value. Section 5 contains some additional results, including on robustness to misspecification of fundamentals and the connection to bilateral trade. Section 6 is a conclusion. Omitted proofs are in the Appendix.

2 Model

Society chooses how much resources to expend on a public good. The expenditure is denoted $E \in [0,1]$. The marginal value of expenditure to agent i is $\theta_i \in \mathbb{R}_+$. All of the expenditures must be raised from the N agents. Let $e_i \in \mathbb{R}$ denote the amount supplied by agent i. (Note that we allow expenditure to be negative, in which case an agent receives a subsidy from the rest of society.) Then budget balance requires that $E = \sum_i e_i \equiv \Sigma e$ (we write $\Sigma x \equiv \sum_i x_i$ for $x \in \mathbb{R}^k$ for some k). Agent i's payoff is $u_i = \theta_i E - e_i$. The mechanism designer's objective is to maximize social welfare, which is $\Sigma u = E(\Sigma\theta - 1)$. Full expenditure is socially efficient as long as $\Sigma\theta \geqslant 1$.

The agents' higher-order beliefs about θ are described by an *information structure* $I = (S, \rho, \eta)$, where S_i is a measurable set of signals for agent i, $S = \prod_{i=1}^N S_i$ is the set of signal profiles, and $\rho \in \Delta(S)$ is the joint distribution of signals, and $\eta : S \to \mathbb{R}_+^N$ is the interim expectation of θ conditional on the signal profile s. Due to the linearity of utility in θ, we can write the agent's interim expected utility of e at s as $\eta_i(s)E - e_i$, and the designer's interim welfare is $(\Sigma\eta(s) - 1)E$.

We will consider two kinds of assumption about the information structure. In the baseline model, there is a lower bound $\theta_L \geqslant 0$ on the social value, so that $\Sigma\eta(s) \geqslant \theta_L$ for all s. The baseline model is studied in Section 3. In Section 4, we consider a richer model where we further assume that the expected social value is at least $\widehat{\theta}$ and, moreover, there is an upper bound θ_H on the social value.

The agents interact through a *mechanism* $M = (A, e)$, where A_i is a measurable set of actions for agent i, $A = \prod_{i=1}^N A_i$ are the action profiles, $e : A \to \Omega$ where $\Omega = \{e \in \mathbb{R}^N : \Sigma e \in [0,1]\}$ is the set of outcomes.

A mechanism is *participation secure* if for every agent i, there exists an action $0 \in A_i$ such that $\theta_i E(0, a_{-i}) - e_i(0, a_{-i}) \geqslant 0$ for all $a_{-i} \in A_{-i}$ and $\theta \in \Theta$. Since $\theta_i = 0$ is a possibility, this participation security condition is equivalent to $e_i(0, a_{-i}) \leqslant 0$ for all $a_{-i} \in A_{-i}$.

A pair (M, I) of a mechanism and an information structure is a simultaneous-move Bayesian game, in which the behavioral strategies of agent i are mappings $b_i : S_i \to \Delta(A_i)$. We identify a profile b with the function $b : S \to \Delta(A)$ given by $b(ds|a) = \prod_i b_i(ds_i|s_i)$. Given a strategy profile b, the expected payoffs are

$$U_i(M, I, b) = \int_{\theta\in\Theta, s\in S, a\in A} (\eta_i(s)E(a) - e_i(a))b(da \mid s)\rho(ds)$$

The profile b is a *(Bayes Nash) equilibrium* if $U_i(b; M, I) \geqslant U_i(b_i', b_{-i}; M, I)$ for all i and b_i'. The objective of the designer is the expected *social welfare*

$$W(M, I, b) = \int_{s\in S, a\in A} (\Sigma\eta(s) - 1)E(a)b(da \mid s)\rho(ds).$$

For a mechanism M, define its *welfare guarantee* as the minimum expected welfare across all information structures I' and equilibria of (M, I').

For an information structure I, define its *welfare potential* as the maximum expected welfare across all mechanisms M' and equilibria of (M', I).

As in Brooks and Du (2021), we seek to characterize a *strong maxmin solution* for the mechanism designer. Formally, a tuple (M, I, b) is a ϵ-*strong maxmin solution* if

(i) b is an equilibrium of (M, I)

(ii) The welfare guarantee of M is at least $W(M, I, b) - \epsilon$;

(iii) The welfare potential of I is at most $W(M, I, b) + \epsilon$.

When $\epsilon = 0$, we call the tuple (M, I, b) a *strong maxmin solution*, M a *guarantee-maximizing mechanism*, and I a *potential-minimizing information structure*. The equilibrium b in Condition (i) is needed so that statements (ii) and (iii) about the guarantee and potential are not vacuous (because of equilibrium nonexistence). In the case of a strong maxmin solution, we say that its associated *value* is $W(M, I, b)$.

3 Known lower bound on the social value

3.1 Main result

For this section, we adopt the baseline assumption on the information structure, that the social value is at least θ_L. We construct a strong maxmin solution consisting of the following components. The guarantee-maximizing mechanism $\overline{M} = (\overline{A}, \overline{e})$ is a *proportional cost-sharing mechanisms*, where $\overline{A}_i = \mathbb{R}_+$,

$$\overline{e}_i(a) \equiv \begin{cases} \frac{a_i}{\Sigma a} \overline{E}(\Sigma a) & \text{if } \Sigma a > 0; \\ 0 & \text{if } \Sigma a = 0, \end{cases} \tag{1}$$

and $\overline{E}$ is the *total expenditure function*. When $\theta_L > N$, this is given by

$$\overline{E}(x) \equiv \begin{cases} \frac{\overline{\lambda}}{\theta_L - 1} \int_{y=0}^{x} \exp(y - x) \left(\frac{y}{x}\right)^{-(N-1)/(\theta_L - 1)} dy & x \leqslant \overline{x}, \\ 1 & x > \overline{x}. \end{cases} \tag{2}$$

The parameters $\overline{\lambda} > 0$ and $\overline{x} > 0$ are chosen so that $\overline{E}$ is continuously differentiable at $\overline{x}$, i.e., $\overline{E}(\overline{x}^-) \equiv \lim_{y \nearrow \overline{x}} \overline{E}(y) = 1$ and $\overline{E}'(\overline{x}^-) = 0$. The above $\overline{E}$ is well-defined if (and only if) $\theta_L > N$, since the exponent of y in the integral is $\frac{N-1}{\theta_L - 1} < 1$, so the integral is integrable around $y = 0$.

When $\theta_L \leqslant N$, then we define $\overline{E}(x) \equiv 0$ for all x, so $\overline{M}$ is the trivial mechanism where the expenditure is always zero.

The potential-minimizing information structure $\overline{I}$ has $\overline{S}_i = [\underline{x}/N, \infty)$, where $\underline{x} \geqslant 0$. The signals are distributed on $\overline{S}$ according to a density function $\overline{\rho}(\Sigma s)$ that depends only on the aggregate signals, where

$$\overline{\rho}(x) \equiv \begin{cases} \frac{\exp(x) x^{-(N-1)\theta_L/(\theta_L - 1)}}{\int_{y=\underline{x}}^{\overline{x}} \exp(y) y^{-(N-1)\theta_L/(\theta_L - 1)} \frac{y^{N-1}}{(N-1)!} dy} & x \in [\underline{x}, \overline{x}], \\ 0 & x > \overline{x}, \end{cases} \tag{3}$$

where $\overline{x}$ is the same as that in mechanism $\overline{M}$. The interim expectation for θ_i is

$$\overline{\eta}_i(s) \equiv \begin{cases} \frac{s_i}{\Sigma s} \theta_L & \text{if } \Sigma s > 0; \\ \frac{1}{N} \theta_L & \text{if } \Sigma s = 0. \end{cases} \tag{4}$$

Finally, let $\overline{b}$ be the strategy profile on $(\overline{M}, \overline{I})$, where $\overline{b}_i(\{s_i\}|s_i) = 1$ for all s_i, i.e., $\overline{b}$ are the obedient/truthful strategies.

Theorem 1. *If $\theta_L > N$, then there exist parameters $(\overline{\lambda}, \overline{x})$ in $\overline{M}$ and $\underline{x} = 0$ in $\overline{I}$ such that $(\overline{M}, \overline{I}, \overline{b})$ is a strong maxmin solution. The welfare guarantee of $\overline{M}$ and the welfare potential of $\overline{I}$ are equal to $\overline{\lambda} > 0$.*

If $\theta_L \leqslant N$, then for any $\epsilon > 0$ there exists a $\underline{x} > 0$ such that $(\overline{M}, \overline{I}, \overline{b})$ is a ϵ-strong maxmin solution. The welfare guarantee of $\overline{M}$ is 0, and the welfare potential of $\overline{I}$ is less than ϵ.

Similar to Brooks and Du (2021), in the solution in Theorem 1 the truthful equilibrium $\overline{b}$ mediates between the optimal mechanism $\overline{M}$ and the worst-case information structure $\overline{I}$. Thus, the solution exhibits a "double revelation principle": $\overline{M}$ is a welfare-maximizing, incentive-compatible direct mechanism on $\overline{I}$, while $\overline{I}$ is a welfare-minimizing Bayes correlated equilibrium on $\overline{M}$.

Let $\overline{W} \equiv W(\overline{M}, \overline{I}, \overline{b})$ denote the welfare guarantee/potential of the solution described before Theorem 1. In Figure 1 we plot $\overline{W}$ as a fraction of the full surplus $1(\theta_L - 1)$ when $N = 2$. As θ_L becomes larger than N, we see that the guaranteed fraction of the full surplus quickly approaches one.

On the other hand, when $\theta_L \leqslant N$ we have $\overline{W} = 0$. For some intuition, consider the information structure I' where it is common knowledge that $\theta_i = \theta_L/N$ for every agent i. Given any participation-secure mechanism, every agent taking the safe action of $a_i = 0$ (resulting in $e_i \leqslant 0$) is always an equilibrium under I' since the individual value $\theta_i = \theta_L/N \leqslant 1$ is dominated by the cost of expenditure. Thus, the welfare guarantee of any participation-secure mechanism is zero. Of course, there exists a participation secure mechanism with a fully efficient equilibrium under I', so one naturally wonders if the low welfare guarantee is an artifact of our assumption about equilibrium selection. The import of Theorem 1 for $\theta_L \leqslant N$ is that in the information structure $\overline{I}$ the welfare potential is arbitrarily close to 0; hence the expected welfare under $\overline{I}$ is arbitrarily small at every participation-secure mechanism and for every equilibrium.

3.2 Lower bound on welfare guarantee of $\overline{M}$

In this and the next subsections we present the proof of Theorem 1, leaving some technical details to Appendix A.

Call a mechanism *smooth* if $A_i = \mathbb{R}_+$, $e_i(a)$'s are differentiable, and the partial derivatives of e_i are bounded. The welfare guarantee of a smooth mechanism can be understood through the notion of *strategic virtual welfare* (Brooks and Du, 2023), which is defined as

$$\lambda(\theta, a) = (\Sigma\theta - 1)E(a) + \sum_i \frac{\partial}{\partial a_i} \left(E(a)\theta_i - e_i(a)\right). \tag{5}$$

Let $\Theta = \{\theta \in \mathbb{R}_+^N | \Sigma\theta \geqslant \theta_L\}$.

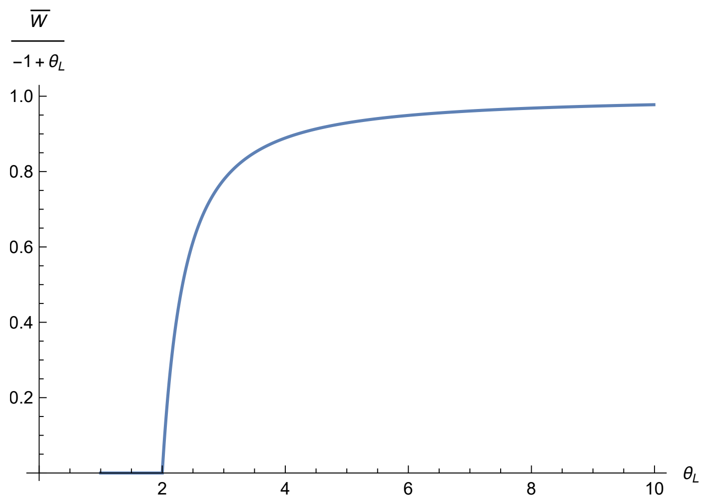

Figure 1: A plot of the optimal welfare guarantee as a fraction of the full surplus when $N = 2$.

Proposition 1. *Suppose the mechanism $M = (\mathbb{R}^N_+, e)$ is smooth. Then for any information structure $I = (S, \rho, \eta)$ and equilibrium b of (M, I), we have*

$$W(M, I, b) \geqslant \int \inf_{a \in \mathbb{R}^N_+} \lambda(\eta(s), a)\rho(ds) \geqslant \inf_{\theta \in \Theta, a \in \mathbb{R}^N_+} \lambda(\theta, a).$$

Proof. We have

$$\begin{aligned}
& W(M, I, b) \\
\geqslant & \int_{s,a} (\Sigma\eta(s) - 1)E(a)b(da \mid s)\rho(ds) \\
& + \sum_i \limsup_{\epsilon \to 0} \int_{s,a} \frac{(\eta_i(s)E(a_i + \epsilon, a_{-i}) - e_i(a_i + \epsilon, a_{-i})) - (\theta_i E(a) - e_i(a))}{\epsilon} b(da \mid s)\rho(ds) \\
= & \int_{s,a} (\Sigma\eta(s) - 1)E(a)b(da \mid s)\rho(ds) + \sum_i \int_{s,a} \frac{\partial(\eta_i(s)E(a) - e_i(a))}{\partial a_i} b(da \mid s)\rho(ds) \\
= & \int_{s,a} \lambda(\eta(s), a)b(da \mid s)\rho(ds) \\
\geqslant & \int_s \inf_a \lambda(\eta(s), a)\rho(ds) \\
\geqslant & \inf_{\theta,a} \lambda(\theta, a),
\end{aligned}$$

where interchanging $\limsup_{\epsilon \to 0}$ with the integral follows from the dominated convergence theorem. $\square$

For a proportional cost-sharing mechanism, we have $e_i(a) = (a_i/\Sigma a)E(\Sigma a)$. Then the strategic virtual welfare when $\Sigma a > 0$ depends only on the social value $\Sigma\theta$ and Σa and simplifies to

$$\lambda(\Sigma\theta, \Sigma a) = (\Sigma\theta - 1)E(\Sigma a) + (\Sigma\theta - 1)E'(\Sigma a) - \frac{(N-1)E(\Sigma a)}{\Sigma a}.$$

We construct $\overline{E}(x)$ so the strategic virtual welfare in the above equation satisfies

$$\lambda(\theta_L, \Sigma a) \begin{cases} = \overline{\lambda} & \Sigma a \leqslant \overline{x}, \\ > \overline{\lambda} & \Sigma a > \overline{x}, \end{cases} \tag{6}$$

for a constant $\overline{\lambda} > 0$.

The first case of (6) is a differential equation; given the initial condition $\overline{E}(0) = 0$, the unique solution is

$$\overline{E}(x) = \frac{\overline{\lambda}}{\theta_L - 1} \int_{y=0}^{x} \exp(y - x) \left(\frac{y}{x}\right)^{-(N-1)/(\theta_L - 1)} dy,$$

when $\theta_L > N$. We further impose $\overline{E}(\overline{x}) = 1$ and $E'(\overline{x}) = 0$; the existence of $(\overline{\lambda}, \overline{x})$ satisfying these conditions is established by Proposition 5 in Appendix A. These conditions imply the second case of (6), since $\lambda(\theta, a) = \overline{\lambda}$ when $\Sigma a = \overline{x}$ and $\lambda(\theta, a)$ is increasing in Σa when $\Sigma a > \overline{x}$. We further establish that the partial derivatives also exist when $\Sigma a = 0$, and hence $\overline{M}$ is smooth. Moreover, $\overline{E} + \overline{E}' \geqslant 0$, so that the minimum strategic virtual objective over $\theta \in \Theta$ is indeed attained when $\Sigma\theta = \theta_L$.

Therefore, Proposition 1 implies that the welfare guarantee of $\overline{M}$ is at least $\overline{\lambda}$ when $\theta_L > N$.

3.3 Upper bound on welfare potential of $\overline{I}$

We say an information structure is *smooth* if $S = [\underline{s}, \infty)^N$, the marginal distribution of signals has a density that is denoted (with slight abuse of notation) by $\rho(s)ds$, and the density $\rho(s)$ and interim value functions $\eta(s)$ have partial derivatives almost everywhere. For a smooth information structure, define the *informational virtual welfare*, for an outcome $e \in \Omega$ and a signal profile $s \in S$ at which ρ and η are differentiable:

$$\gamma(s, e) = \rho(s)(\Sigma\eta(s) - 1)\Sigma e - \sum_{i=1}^{N} \frac{\partial}{\partial s_i} \left(\rho(s)(\eta_i(s)\Sigma e - e_i)\right). \tag{7}$$

Because we allow $\rho(s)$ and $\eta(s)$ to be discontinuous at some s, it is convenient to also work with the discrete version of the informational virtual welfare: for $\epsilon > 0$,

$$\gamma(s, e; \epsilon) = \rho(s)(\Sigma\eta(s) - 1)\Sigma e - \sum_{i=1}^{N} \frac{\rho(s_i + \epsilon, s_{-i})(\eta_i(s_i + \epsilon, s_{-i})\Sigma e - e_i) - \rho(s)(\eta_i(s)\Sigma e - e_i)}{\epsilon}.$$

Proposition 2. *Suppose the information structure $I = ([\underline{s}, \infty)^N, \rho, \eta)$ is smooth. Then for any mechanism M and equilibrium b of (M, I), we have*

$$W(M, I, b) \leqslant \liminf_{\epsilon \to 0} \int_{s \in [\underline{s}, \infty)^N} \max_{e \in \Omega} \gamma(s, e; \epsilon) ds.$$

Proof. Without loss of generality, suppose $\underline{s} = 0$, so $S = \mathbb{R}_+^N$. Let $(\mathbb{R}_+^N, e^*)$ be a welfare-maximizing, incentive compatible, and individually rational direct mechanism on I. For any $\epsilon > 0$, we have

$$\begin{aligned}
&W(M, I, b)\\
\leqslant &\int_{s\in\mathbb{R}_+^N} (\Sigma\eta(s) - 1)E^*(s)\rho(s)ds\\
&- \sum_{i=1}^N \int_{s_i\in[\epsilon,\infty), s_{-i}\in\mathbb{R}_+^{N-1}} \frac{(\eta_i(s)E^*(s_i-\epsilon, s_{-i}) - e_i^*(s_i-\epsilon, s_{-i})) - (\eta_i(s)E^*(s) - e_i^*(s))}{\epsilon}\rho(s)ds\\
&+ \sum_{i=1}^N \int_{s_i\in[0,\epsilon), s_{-i}\in\mathbb{R}_+^{N-1}} \frac{\eta_i(s)E^*(s) - e_i^*(s)}{\epsilon}\rho(s)ds\\
= &\int_{s\in\mathbb{R}_+^N} (\Sigma\eta(s) - 1)E^*(s)\rho(s)ds\\
&+ \sum_{i=1}^N \int_{s\in\mathbb{R}_+^N} \frac{(\eta_i(s)E^*(s) - e_i^*(s))\rho(s) - (\eta_i(s_i+\epsilon, s_{-i})E^*(s) - e_i^*(s))\rho(s_i+\epsilon, s_{-i})}{\epsilon}ds\\
= &\int_{s\in\mathbb{R}_+^N} \gamma(s, e^*(s); \epsilon)ds\\
\leqslant &\int_{s\in\mathbb{R}_+^N} \max_{e\in\Omega} \gamma(s, e; \epsilon)ds
\end{aligned}$$

where we used the incentive compatibility condition for signals $s_i \in [\epsilon, \infty)$ (not wanting to misreport $s_i - \epsilon$), and individual rationality condition for signals $s_i \in [0, \epsilon)$. $\square$

We now prove Theorem 1. For a fixed $\epsilon > 0$, the informational virtual welfare of $\overline{I}$ is

$$\begin{aligned}
\gamma(s, e; \epsilon) &= \overline{\rho}(\Sigma s)(\overline{\theta} - 1)\Sigma e - \frac{1}{\epsilon}\sum_{i=1}^N \left(\overline{\rho}(\Sigma s + \epsilon)\left(\frac{s_i + \epsilon}{\Sigma s + \epsilon}\overline{\theta}\Sigma e - e_i\right) - \left(\overline{\rho}(\Sigma s)\left(\frac{s_i}{\Sigma s}\overline{\theta}\Sigma e - e_i\right)\right)\right)\\
&= \overline{\rho}(\Sigma s)(\overline{\theta} - 1)\Sigma e - \frac{1}{\epsilon}\sum_{i=1}^N \left(\overline{\rho}(\Sigma s + \epsilon)\frac{s_i + \epsilon}{\Sigma s + \epsilon} - \overline{\rho}(\Sigma s)\frac{s_i}{\Sigma s}\right)\overline{\theta}\Sigma e + \frac{\overline{\rho}(\Sigma s + \epsilon) - \overline{\rho}(\Sigma s)}{\epsilon}\Sigma e.
\end{aligned}$$

By Proposition 2, the welfare potential of $\overline{I}$ is at most

$$\int_{\Sigma s < \overline{x} - \epsilon} \max_{e\in\Omega} \gamma(s, e; \epsilon)ds + \int_{\Sigma s \geqslant \overline{x} - \epsilon} \max_{e\in\Omega} \gamma(s, e; \epsilon)ds.$$

By the dominated convergence theorem, the first term in the above upper bound converges to zero as $\epsilon \to 0$: $\max_{e\in\Omega} \gamma(s, e; \epsilon)$ tends to 0 when $\Sigma s \in (0, \overline{x})$ since by construction the limit satisfies

$$\gamma(s, e) = \left((\overline{\rho}(\Sigma s) - \overline{\rho}'(\Sigma s))(\overline{\theta} - 1) - \frac{(N-1)\theta_L\overline{\rho}(\Sigma s)}{\Sigma s}\right)\Sigma e = 0$$

for every $e \in \Omega$.

The second term is equal to

$$\begin{aligned}
&\int_{\overline{x}-\epsilon\leqslant\Sigma s\leqslant\overline{x}} \max_{e\in\Omega}\left(\overline{\rho}(\Sigma s)(\theta_L-1)\Sigma e-\sum_{i=1}^{N}\frac{0-\overline{\rho}(\Sigma s)(\frac{\theta_L s_i}{\Sigma s}\Sigma e-e_i)}{\epsilon}\right)ds\\
=&\int_{x=\overline{x}-\epsilon}^{\overline{x}} \max_{\Sigma e\in[0,1]}\left(\overline{\rho}(x)(\theta_L-1)\Sigma e+\frac{\overline{\rho}(x)(\theta_L-1)\Sigma e}{\epsilon}\right)\frac{x^{N-1}}{(N-1)!}dx\\
=&\int_{x=\overline{x}-\epsilon}^{\overline{x}} \left(\overline{\rho}(x)(\theta_L-1)+\frac{\overline{\rho}(x)(\theta_L-1)}{\epsilon}\right)\frac{x^{N-1}}{(N-1)!}dx
\end{aligned}$$

which, as $\epsilon\to 0$, converges to

$$\overline{\rho}(\overline{x})(\theta_L-1)\frac{\overline{x}^{N-1}}{(N-1)!}=\frac{\exp(\overline{x})\overline{x}^{-(N-1)/(\theta_L-1)}}{\int_{y=\underline{x}}^{\overline{x}}\exp(y)y^{-(N-1)/(\theta_L-1)}dy}. \tag{8}$$

When $N<\theta_L$, we set $\underline{x}=0$. Then (8) is equal to $\overline{\lambda}$, since $\overline{E}(\overline{x})=1$ (cf. equation (2)). Thus, the welfare potential of $\overline{I}$ is at most $\overline{\lambda}$. Combined with Proposition 1, we conclude that the welfare guarantee of $\overline{M}$ is equal to the welfare potential of $\overline{I}$, both equal to $\overline{\lambda}$, provided that an equilibrium exists (so that both guarantee and potential are not vacuous).

Finally, in Proposition 6 in Appendix A, we show that the truthful strategy is an equilibrium of $(\overline{M},\overline{I})$. This step completes the proof that $(\overline{M},\overline{I},\overline{b})$ are a strong maxmin solution. The proof of Proposition 6 involves a fairly long and detailed calculation of interim utilities. The fact that truthful strategies are an equilibrium is both remarkable and rather mysterious. We comment further on this issue in Section 5 below.

When $N\geqslant\theta_L$, (8) converges to 0 as $\underline{x}\to 0$ for any fixed $\overline{x}$. Thus, the welfare potential of $\overline{I}$ tends to zero as $\underline{x}\to 0$. Since $\overline{M}$ is the trivial mechanism with no expenditure when $N\geqslant\theta_H$, the welfare guarantee of $\overline{M}$ is obviously zero, and $\overline{b}$ is obviously an equilibrium of $(\overline{M},\overline{I})$.

This concludes the proof of Theorem 1.

4 Lower bound on the expected social value

We next consider the richer model where we also have an upper bound θ_H on the social value and a lower bound $\widehat{\theta}$ on the expected social value. In other words, the possible information structures are those for which $\theta_L\leqslant\Sigma\eta(s)\leqslant\theta_H$ for all s and

$$\int_s \Sigma\eta(s)\rho(ds)\geqslant\widehat{\theta}.$$

The primary purpose of this exercise is to allow for positive probability that the social value is less than N. As we shall see, proportional cost-sharing mechanisms continue to provide unimprovable welfare guarantees.

When $\widehat{\theta}\leqslant N$, the potential-minimizing information structure from Theorem 1 is still feasible and still gives a welfare potential arbitrarily close to zero.

When $\widehat{\theta} > N$, however, we obtain non-trivial welfare guarantee. This is the case even though we may have $\theta_L \leqslant N$.

We will construct new strong maxmin solutions using the mechanisms and information structures from Theorem 1 as building blocks. Recall that in the solution from Theorem 1, the social value was equal to its lower bound with probability one. Let us now make explicit their dependence on the commonly known social value, which we denote by $\tilde{\theta}$. The proportional cost-sharing mechanism defined by equations (1) and (2) is denoted as $\overline{M}_{\tilde{\theta}}$, and the information structure defined by equations (3) and (4) is denoted by $\overline{I}_{\tilde{\theta},\underline{x}}$.

When $\widehat{\theta} > N$ and $\theta_L \leqslant N$, the guarantee maximizing mechanism turns out to be $\overline{M}_{\theta_H}$. In other words, we use the proportional cost-sharing mechanism *as if* there is a common knowledge that the social value is θ_H. The corresponding potential minimizing information structure is a public randomization between $\overline{I}_{N,\underline{x}}$ and $\overline{I}_{\theta_H,0}$, where $\underline{x}$ is close to zero, and the probability of $\overline{I}_{N,\underline{x}}$ is $\frac{\theta_H-\widehat{\theta}}{\theta_H-N}$; denote this "public mixture" information structure as

$$\tilde{I}_{\underline{x}} \equiv \frac{\theta_H-\widehat{\theta}}{\theta_H-N}\cdot\overline{I}_{N,\underline{x}} + \frac{\widehat{\theta}-N}{\theta_H-N}\cdot\overline{I}_{\theta_H,0}$$

That is, under $\tilde{I}_{\underline{x}}$, the agents receive a public signal telling them either the information structure is $I_{N,\underline{x}}$ or $I_{\theta_H,0}$, and then they receive private signals according to either $I_{N,\underline{x}}$ or $I_{\theta_H,0}$. In $I_{N,\underline{x}}$ (respectively, $I_{\theta_H,0}$) there is a common knowledge that $\Sigma\theta = N$ (respectively, $\Sigma\theta = \theta_H$); the probabilities on $I_{N,\underline{x}}$ and $I_{\theta_H,0}$ are such that the expected value of $\Sigma\theta$ is $\widehat{\theta}$.

When $\theta_L > N$, the guarantee maximizing mechanism $\widehat{M} = (\mathbb{R}_+^N, \widehat{e})$ is a proportional cost-sharing mechanism with

$$\widehat{E}(x) = \begin{cases} \frac{\lambda_L}{\theta_L-1}\int_{y=0}^{x}\exp(y-x)\left(\frac{y}{x}\right)^{-(N-1)/(\theta_L-1)}\,dy & x\in[0,x_L],\\ \widehat{E}(x_L)\exp(x_L-x)\left(\frac{x_L}{x}\right)^{-(N-1)/(\theta_H-1)} & \\ \qquad+\frac{\lambda_H}{\theta_H-1}\int_{y=x_L}^{x}\exp(y-x)\left(\frac{y}{x}\right)^{-(N-1)/(\theta_H-1)}\,dy & x\in(x_L,x_H],\\ 1 & x>x_H.\end{cases} \tag{9}$$

Equation (9) pastes together the expenditure functions from $\overline{M}_{\theta_L}$ and $\overline{M}_{\theta_H}$: below x_L the mechanism is as if there is a commonly knowledge that the social value is θ_L, and above x_L as if a commonly knowledge of θ_H. The equation and the parameters $(\lambda_L, x_L, \lambda_H, x_H)$ ensure smooth pasting at x_L and x_H; that is, $\widehat{E}(x)$ and $\widehat{E}'(x)$ are continuous at $x = x_L$ and at $x = x_H$.

Likewise, the potential minimizing information structure $\widehat{I} = (\mathbb{R}_+^N, \widehat{\rho}, \widehat{\eta})$ pastes together (though not smoothly) $\overline{I}_{\theta_L,0}$ and $\overline{I}_{\theta_H,0}$: the density $\widehat{\rho}(s)$ of the signal profile $s\in\mathbb{R}_+^N$ depends only on Σs and satisfies

$$\widehat{\rho}(x) = \begin{cases} \frac{\mu_L\exp(x)x^{-(N-1)\theta_L/(\theta_L-1)}}{\int_{y=0}^{x_L}\exp(y)y^{-(N-1)/(\theta_L-1)}/(N-1)!dy} & x\in[0,x_L),\\ \frac{\mu_H\exp(x)x^{-(N-1)\theta_H/(\theta_H-1)}}{\int_{y=x_L}^{x_H}\exp(y)y^{-(N-1)/(\theta_H-1)}/(N-1)!dy} & x\in[x_L,x_H),\\ 0 & x\geqslant x_H,\end{cases} \tag{10}$$

where

$$\mu_L = \frac{\widehat{\theta} - \theta_L}{\theta_H - \theta_L}, \quad \mu_H = \frac{\theta_H - \widehat{\theta}}{\theta_H - \theta_L}. \tag{11}$$

The interim value function is

$$\widehat{\eta}_i(s) = \begin{cases} \theta_L \frac{1}{N} & \text{if } \Sigma s = 0, \\ \theta_L \frac{s_i}{\Sigma s} & \text{if } \Sigma s \in (0, x_L), \\ \theta_H \frac{s_i}{\Sigma s} & \text{if } \Sigma s \in [x_L, x_H]. \end{cases} \tag{12}$$

Thus, whether Σs is below or above x_L reveals whether $\Sigma\theta$ is θ_L or θ_H, and conditional on either event $\widehat{I}$ is identical to $\overline{I}_{\theta_L,0}$ or $\overline{I}_{\theta_H,0}$.

Our result for this section is the following:

Theorem 2. *Suppose $\widehat{\theta} > N$ and $\theta_L \leqslant N$. For every $\epsilon > 0$, there exists an $\underline{x}$ so that $(\overline{M}_{\theta_H}, \tilde{I}_{\underline{x}}, b)$ is an ϵ-strong maxmin solution for some strategies b.*

Suppose $\theta_L > N$. Then there exist $(\lambda_L, x_L, \lambda_H, x_H)$ such that $(\widehat{M}, \widehat{I}, b)$ is a strong maxmin solution for some b.

The proof of Theorem 2 largely follows that of Theorem 1. We first establish a lower bound on welfare in a smooth mechanism, generalizing Proposition 1. The proof of the first part of the theorem is quite straightforward. The second half takes more effort, and involves a fixed point argument proving that there exist parameters so that the lower bound on the guarantee of $\widehat{M}$ coincides with the upper bound on the potential for $\widehat{I}$. These parameters are engineered to satisfy the complementary slackness property described below in Section 5.

Note that unlike in Theorem 1, we do not prove that the truthful strategies are an equilibrium of $(\widehat{M}, \widehat{I})$ (although we suspect that this is the case). Rather, we appeal to the existence theorem of Milgrom and Weber (1985) to obtain an equilibrium non-constructively.

Although we have presented our assumptions about fundamentals in terms of bounds on the social value, the baseline model effectively reduces to one in which the social value is equal to its lower bound. Similarly, the model of this section reduces to one where the social value is either θ_L or θ_H with known probabilities. We suspect that the analysis can be generalized to a model where the entire distribution of the social value is known, and that proportional cost-sharing mechanisms continue to be optimal. However, the optimal total expenditure function will have to satisfy a more complicated differential equation, in which social values are assortatively matched with aggregate actions. We leave this interesting extension to future work.

5 Discussion

5.1 Robustness to misspecification of fundamentals

The proportional cost-sharing mechanisms are parameterized by the total expenditure function $\overline{E}$. As we have shown, the guarantee maximizing expenditure function depends on the particular assumptions about fundamentals, whether it be the lower bound on the social value θ_L in the baseline model of Section 3, or the bounds θ_L and θ_H and the lower bound expectation $\widehat{\theta}$ in the extended model of Section 4.

A main virtue of the proportional cost-sharing mechanism is its robustness to misspecification to the information structure. But it is also robust to misspecification of the fundamentals, as we now explain. Proposition 1 shows that welfare is bounded below by the expectation (across θ) of the minimum (across a) of $\lambda(\theta, a)$. Since λ is continuous in θ, this lower bound will be weak-* continuous in the distribution of θ. Thus, the lower bound on welfare varies smoothly with fundamentals.[1]

Indeed, in our baseline model, we only assumed that there was a lower bound on the social value. What would happen to the welfare in the guarantee maximizing mechanism if the social value turned out to be greater than θ_L? By Proposition 1, the social welfare is at least the minimum value of $\lambda(\theta, a)$ over all a for any realized θ. The guarantee maximizing mechanism has the property that the strategic virtual objective

$$\lambda(\theta, a) = (\Sigma\theta - 1)(\overline{E}(\Sigma a) + \overline{E}'(\Sigma a)) - \frac{(N-1)E(\Sigma a)}{\Sigma a}$$

is equalized across all action profiles with $\Sigma a \leqslant \overline{x}$ when $\Sigma\theta = \theta_L$. But because the coefficient on $\Sigma\theta$ is $\overline{E} + \overline{E}'$, which is decreasing in Σa, we know that when $\Sigma\theta > \theta_L$, the minimum is attained at $\Sigma a = \overline{x}$, and hence the minimum strategic virtual objective is $(\Sigma\theta - 1) - (N-1)/\overline{x}$. So the ratio between the welfare guarantee and the efficient surplus is $\frac{(\Sigma\theta-1)-(N-1)/\overline{x}}{\Sigma\theta-1}$, which is strictly increasing in $\Sigma\theta$. As $\Sigma\theta$ grows large, this ratio tends to 1. We have therefore proven the following:

Proposition 3. *Let $\overline{M}$ be the guarantee-maximizing proportional cost-sharing mechanism when the lower bound on the social value is θ_L. The share of the efficient welfare guaranteed by $\overline{M}$ increases with $\Sigma\theta$ and converges to 1 as $\Sigma\theta$ tends to ∞.*

This result reinforces the main message of this paper: that society can obtain nontrivial welfare guarantees from proportional cost-sharing mechanisms, even if there is only relatively crude information about the informational environment and fundamentals.

5.2 On infinite actions/signals and equilibrium existence

As we mentioned in the introduction, a subtle technical issue is how to handle the possibility of non-existence of equilibrium when formulating the joint information/mechanism design problem. As far as we know, the guarantee of the proportional cost-sharing mechanism that we construct is strictly greater than that of any finite mechanism, and similarly, the potential of the information structures that we construct is lower than that of any finite information structure. But if we allow infinite mechanisms and infinite information structures, then there will be certain combinations for which no equilibria exist. This raises the possibility of other pathological strong maxmin solutions with different values, as we now explain.

[1]Similar observations about robustness to misspecification of fundamentals appeared in Du (2018), Brooks and Du (2021), and Brooks and Du (2023).

Considering the baseline model of Section 3, if a mechanism has an equilibrium on the information structure $\overline{I}$, then its guarantee must exist and be finite and, moreover, the guarantee must be less than that of $\overline{M}$. Similarly, if an information structure has an equilibrium on $\overline{M}$, then its potential is finite and is at least that of $\overline{I}$. Since these objects are relatively well-behaved, requiring that equilibria exist when paired with $\overline{I}$ and $\overline{M}$ seems to be a mild restriction on the mechanism and information structure, respectively, and within that class of strong maxmin solutions, we will have a unique value. Indeed, for the extended model in Section 4, we do not explicitly construct an equilibrium, but rather obtain one indirectly via an application of the existence theorem of Milgrom and Weber (1985), using the fact that $\overline{M}$ and $\overline{I}$ are analytically well-behaved.

Nonetheless, there remains a logical possibility of pathological mechanisms (e.g., integer games) that have no equilibria on any information structure, and therefore have infinite guarantee, or the somewhat less pathological but still disturbing possibility of mechanisms that do not have equilibria on $\overline{I}$, and therefore have strictly higher guarantees than $\overline{M}$.

In Brooks and Du (2021), we considered this issue in the context of revenue maximization in common value auctions. We introduced the notion of *finite approximability* of a strong maxmin solution, which is that there exist finite mechanisms and finite information structures whose guarantees and potentials, respectively, are arbitrarily close to the value of the solution. While we have not pursued that exercise in this paper, we have every reason to think that the solution $(\overline{M}, \overline{I}, \overline{b})$ is finitely approximable, using the same techniques as in Brooks and Du (2021). Moreover, as argued in that paper, all finitely approximable strong maxmin solutions must have the same value. This is a straightforward consequence of the existence of a Nash equilibrium when the mechanism and information structure are both finite.

In sum, while some care is needed when describing exact solutions with infinitely many actions and signals, we have every reason to think that the value of the strong maxmin solutions provided in this paper are the natural ones, and do not rely on a controversial use of equilibrium existence.

5.3 Complementary slackness and the double revelation principle

In Propositions 1, we argued that the expected lowest strategic virtual objective is a lower bound on the welfare guarantee of a smooth mechanism. The guarantee maximizing mechanism maximizes this lower bound. Similarly, in Proposition 2, we argued that an upper bound on the welfare potential is the expected highest informational virtual objective. The potential minimizing information minimizes this upper bound. These two bounding programs can be viewed as infinite dimensional linear programming problems.

In Brooks and Du (2023), we studied discrete analogues of these programs, where the number of actions and signals is finite, and we take the limit as that number goes to infinity. In the discrete setting, the bounding programs are “almost” a dual pair, in the following sense: The derivatives in the strategic and informational virtual objective are discrete and local upward. The dual of the program of minimizing the expected highest informational virtual objective has the form of maximizing an expected lowest strategic virtual objective, but where the local derivatives point down instead of up. In the dual pairing, the likelihood of (s, θ) is the Lagrange multiplier on the constraint that the minimum strategic virtual objective is at most that obtained at the value profile θ and a particular action profile equal to s. Similarly, the likelihood of an outcome e given the action a is the Lagrange multiplier on the constraint that the maximum informational virtual objective be at least that attained at the outcome e and signal profile equal to a.

In the continuum limit, we might expect the different between discrete upwards and discrete downwards to vanish, so that the two programs become an exact dual pair. We have yet to find a general and precise formulation of this duality in the continuum limit. Nonetheless, the strong maxmin solution we described exhibits the structure that one would expect of a saddle point for a linear programming problem. In particular, it satisfies a form of *complementary slackness*.

Under the potential-minimizing information structure, a profile (s, θ) has positive likelihood only if the associated actions $a = s$ in the guarantee-maximizing mechanism minimize the strategic virtual objective at θ. In the baseline model with $\theta_L > N$, all signals with $\Sigma s \leqslant \overline{x}$ have positive likelihood under $\overline{I}$, and the minimizers of the strategic virtual objective in $\overline{M}$ are those for which $\Sigma a \leqslant \overline{x}$. Similarly, in the enriched model when $\theta_L > N$, under $\widehat{I}$, there is positive likelihood of (s, θ) with $\Sigma\theta = \theta_L$ only if $\Sigma s \in [0, x_L]$, and positive likelihood of with $\Sigma\theta = \theta_H$ only if $\Sigma s \in [x_L, x_H]$. This exactly accords with the minimizers of the strategic virtual objective in $\widehat{M}$: If $\Sigma\theta = \theta_L$, then the minimizers are non-negative action profiles with $\Sigma a \in [0, x_L]$, and if $\Sigma\theta = \theta_H$, the minimizers satisfy $\Sigma a \in [x_L, x_H]$.

Similarly, under the guarantee-maximizing mechanism, an outcome has positive likelihood under an action profile a only if it maximizes the associated informational virtual objective for the potential-minimizing information structure. Since in both versions of the model the informational virtual objective is zero for all outcomes and s with $\Sigma s \in [0, x_H)$, we can have any interior expenditure and shares for action profiles with $\Sigma a < x_H$. For $\Sigma s \geqslant x_H$, the informational virtual objective for full expenditure is positive, but it does not depend on the particular shares. This accords with the fact that $\overline{E}(x)$ is strictly between 0 and 1 when $x \in [0, x_H)$ but equal to 1 for $x \geqslant x_H$.

A similar form of complementary slackness was manifest in our solution of the common value optimal auctions problem in Brooks and Du (2021). We suspect that the phenomenon is more general. In fact, in deriving the strong maxmin solution, we conjectured that complementary slackness would be satisfied, and used that ansatz to engineer the particular form of the information structure and mechanism and the cutoffs.

Finally, another striking feature of the strong maxmin solution for the baseline model is that the truthful/obedient strategies are an equilibrium of the Bayesian game consisting of the guarantee maximizing mechanism and the potential minimizing information. This "double revelation principle" also appeared in Brooks and Du (2021). While we have only proven this for the baseline model, we have every reason to think that it is also true in the enriched model. The detailed calculations needed to verify optimality of truthtelling in the baseline model are quite involved, and extending them to the richer solution described in Section 4 is a non-trivial task. That is why we proved equilibrium existence for the richer model using the non-constructive existence theorem of Milgrom and Weber (1985). Again, we suspect it is a general phenomenon of continuous strong maxmin solutions, and that it can be proved via a higher level argument that remains to be discovered.

5.4 Connection to bilateral trade

When there are two agents, our model can be reinterpreted as one of bilateral trade, as we now explain. There is a seller $i = 1$ who owns a single unit of a good that can be traded to a buyer $i = 2$. The seller's value for the good is v_1, which is between 0 and $v_H > 0$ (where the high value is a normalization), and the buyer's value v_2 is known to be at least zero. Moreover, there is common knowledge that $v_2 - v_1 \geqslant g$, meaning that there is a lower bound g on the gains from trade. The outcome is simply a likelihood of trade $q \in [0, 1]$ and a price at which the buyer and seller trade p. We assume $g > 0$, so that the gains from trade are strictly positive, and the unique efficient outcome is for the agents to always trade. Note that if $g > v_H$, then it is common knowledge that the buyer's value is greater than the seller's, and we can implement efficient trade with, e.g., a posted price of $(v_H + g)/2$. So, to keep things non-trivial, we assume that $g < v_H$.

We now map this into the the baseline model of the public goods problem. No expenditure is equivalent to no trade, in which case both seller's and buyer's net payoffs are zero. Full expenditure with agent 1 paying all of the expense should be identified with an outcome in the bilateral trade setting where trade occurs with probability one and the buyer's lowest possible payoff is zero, meaning that trade occurs at a low price of $p = g$. In this case, the seller's payoff is $g - v_1$ and the buyer's payoff is $v_2 - g \geqslant v_1 + g - g = v_1 \geqslant 0$. On the other hand, full expenditure with agent 2 paying all of the expense should be identified with an outcome where trade occurs with probability one and the seller's lowest possible payoff is zero, meaning that trade occurs at a high price of $p = v_H$. The seller's net payoff is therefore $v_H - v_1 \geqslant v_H - v_H = 0$.

To find the right conversion of units, note that the highest payoff for buyer and seller is v_H (trade at the highest price and lowest value for the seller, trade at the lowest price and highest value for the buyer). Thus, we should identify $v_H = \alpha\theta_L$ for some scaling parameter $\alpha > 0$. Also, the gains from trade $v_2 - v_1 \geqslant g$, which should be identified with the net social value from public expenditure, so that $g = \alpha(\theta_L - 1)$. Note that this is consistent with the aforementioned parametric restriction that $g < v_H$. We obtain:

$$\alpha = v_H - g;$$
$$\theta_L = \frac{v_H}{v_H - g}.$$

It is clear that the condition $\theta_L > 2$ is equivalent to $g > v_H/2$. In that case, we define the *proportional pricing mechanism* $\tilde{M}$ as follows: Seller and buyer report non-negative real numbers a_1 and a_2, respectively; trade occurs with probability $\overline{E}(a_1 + a_2)$, where $\overline{E}$ is defined according to (2) with $\theta_L = v_H/(v_H - g)$; and trade occurs at a price equal to

$$\overline{p}(a_1, a_2) \equiv g\frac{a_1}{a_1 + a_2} + v_H\frac{a_2}{a_1 + a_2}.$$

In addition, we define the information structure $\tilde{I}$ in which buyer and seller receive non-negative signals whose density is given by $\overline{\pi}$ (again when $\theta_L = v_H/(v_H - g)$); conditional on the signal profile s, the seller's value is v_H with likelihood $s_1/(s_1+s_2)$, and is zero otherwise; and the buyer's value is equal to the seller's value plus g.

As an immediate consequence of Theorem 1, we obtain the following result for the bilateral trade problem:

Proposition 4. *If $g > v_H/2$, then $(\tilde{M}, \tilde{I}, \bar{b})$ is a strong maxmin solution. The guarantee for gains from trade is equal to the welfare guarantee of the proportional cost-sharing mechanism scaled by α.*

If $g \leqslant v_H/2$, then for every $\epsilon > 0$, there exists an information structure for which the probability of trade is at most ϵ in any mechanism and equilibrium.

Importantly, in the case $g \leqslant v_H/2$, trade is impossible even though it is common knowledge that the gains from trade are strictly positive. This is a dramatic strengthening of the result of Akerlof (1970) and Carroll (2016), who only give conditions under which the market may break down under posted prices. In fact, the condition $g \leqslant v_H/2$ is precisely the condition for which efficient trade is impossible with a posted price in the lemons information structure (where the buyer has no information and the seller knows the value). But in the information structures we construct, there is no trade at all, no matter which mechanism is used and which equilibrium is played.

Clearly, it is possible to further weaken the assumptions on fundamentals in the bilateral trade model along the lines of Section 4. A strong assumption that is harder to dispense with is the upper bound on the seller's value. It is our hope that the methodology developed in this paper and in Brooks and Du (2023) can be fruitfully applied to richer versions of the bilateral trade problem. We hope to pursue this topic in future work.

5.5 Alternative mechanisms

In this section we numerically compare the guarantee-maximizing proportional cost-sharing mechanism with some alternative mechanisms.

First, consider the *unilateral mechanism* where each agent is responsible for $1/N$ of the public good. The agents simultaneously choose from $A_i = \{0, 1/m, 2/m, \ldots, 1\}$, where the integer $m \geqslant 1$ is a parameter, and $e_i(a) = a_i/N$.

Second, consider the *linear proportional cost-sharing mechanism*: The agents simultaneously choose from $A_i = \{0, 1/m, 2/m, \ldots, 1\}$, and

$$e_i(a) = \begin{cases} a_i & \Sigma a \leqslant 1, \\ \frac{a_i}{\Sigma a} & \Sigma a > 1. \end{cases}$$

Thus, the total expenditure function is $E(\Sigma a) = \Sigma a$ for $\Sigma a \leqslant 1$ and $E(\Sigma a) = 1$ for $\Sigma a > 1$.

Surprisingly, when $N = 2$ numerical simulations suggest that the welfare guarantees of these two mechanisms are independent of the parameter m in both the baseline and extended models. These welfare guarantees are plotted in Figures 2 and 3, along with that of the guarantee-maximizing proportional cost-sharing mechanisms. We see that the unilateral mechanism generally offers poor welfare guarantees; this is not surprising since in the unilateral mechanism the expenditure contributed by an individual agent is at most

1/2 even if his value is equal to the social value. The welfare guarantees of the linear proportional cost-sharing mechanism are significantly better and come close to the optimal guarantees in the extended model. In the baseline model with a moderate θ_L, the guarantee-maximizing mechanism still significantly outperforms the linear proportional cost-sharing mechanism.

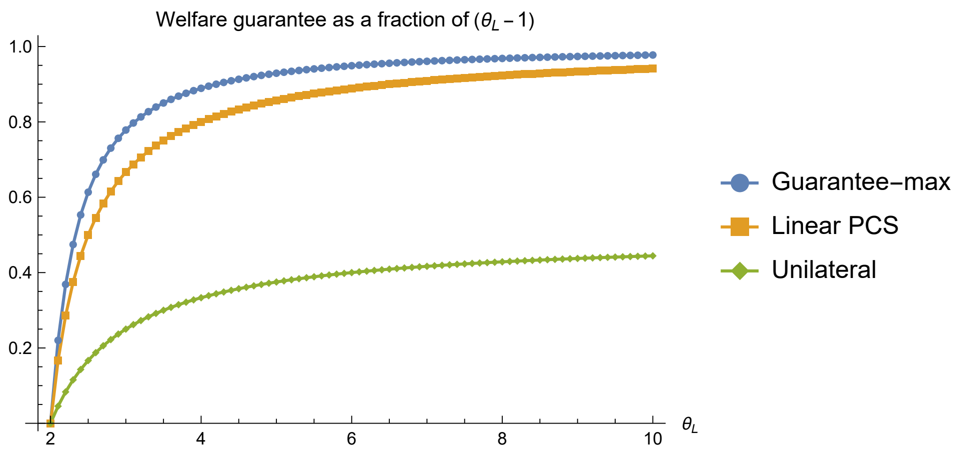

Figure 2: Welfare guarantees of the various mechanisms in the baseline model when $N = 2$.

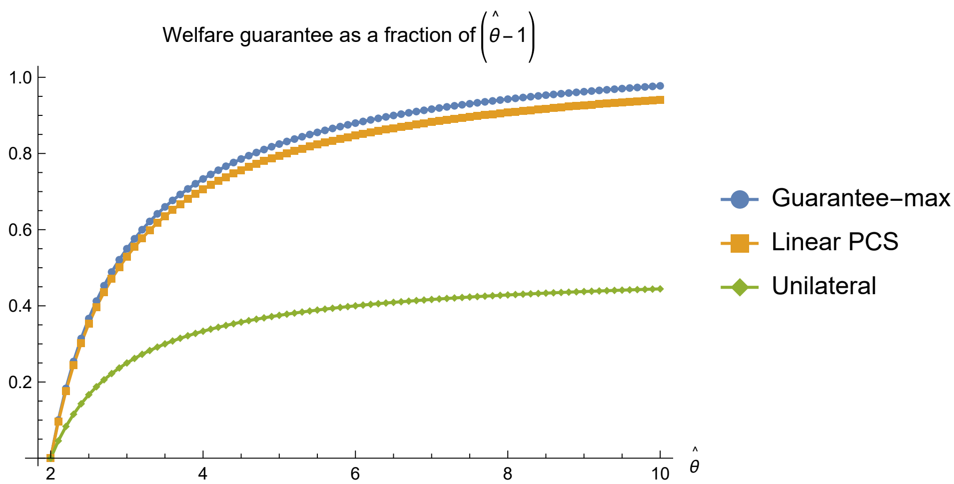

Figure 3: Welfare guarantees of the various mechanisms in the extended model with $\theta_L = 0$, $\theta_H = 10$, and $N = 2$.

A takeaway from the numerical simulations is that the linear proportional-cost sharing mechanism offers good welfare guarantees and is a good candidate for practical applications.

6 Conclusion

This paper has used an informationally-robust welfare criterion to derive new proportional cost-sharing mechanisms for the public goods problem, as well as the proportional-price trading mechanism for the bilateral trade problem. These mechanisms are shown to provide unimprovable guarantees for welfare, even with minimal assumptions about the nature of private information and the social value of the good. These mechanisms seem simple enough that they could actually be implemented in practice.[2] They are parameterized only by the total expenditure rule, and the performance of the mechanism is robust to misspecification of the fundamentals. Moreover, they completely mitigate the free-rider problem in the limit when the social value is large.

An important limitation of our analysis is that the social value has to be relatively large in order for the guarantee to be non-trivial. In particular, the ratio of per capita social value to each dollar spent must be greater than one. This condition seems more likely to be satisfied when the number of agents is also relatively small. A potential application could be to climate change mitigation policies, and to the design of a treaty that would assign country-level greenhouse gas emissions reduction targets. Individual countries can refuse to participate in the treaty at the interim stage, but once it is signed, there is an enforcement mechanism that compels countries to meet their assigned targets. As long as the social value of mitigating climate change is relatively large and the number of countries involved is relatively small, our results show that proportional cost-sharing mechanisms provide non-trivial welfare guarantees.

The fact that the guarantee is zero when the social value per capita is small is, in a sense, a consequence of the weakness of our assumptions: For the potential minimizing information structures, no mechanism can achieve positive welfare, no matter what equilibrium is played. Thus, to obtain non-trivial guarantees in such cases, it seems necessary to restrict the degree of heterogeneity in values across agents. Changes to other aspects of the model may be needed as well, as the following example shows. Suppose that the social value is greater than one, so that production is efficient, but the social value per capita is less than one. Further suppose that all agents have the same expected value. Then any participation secure mechanism will have an equilibrium in which all of the agents play their participation secure actions: An agent who deviates from this strategy profile will have to cover the entire cost of the public good, which is less than its private value to the deviator. As a result, under this information structure, any budget balanced and participation secure mechanism will have a welfare guarantee of zero. Thus, in generalizing the theory to the case where the social value is low, it may also be necessary to modify the participation constraint, the budget constraint, or the equilibrium selection rule.

[2]At no point in our analysis did we explicitly rule out more complicated mechanisms. For example, since the information structure is common knowledge among the agents, it is in principle possible for the designer to ask the agents to report that common knowledge, and then run the social welfare maximizing mechanism for the true information structure. But such a mechanism cannot improve on proportional cost-sharing mechanisms in the worst case. And focusing on the smooth mechanisms and strategic virtual objective led us to lower dimensional solutions.

References

Akerlof, G. A. (1970): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *The Quarterly Journal of Economics*, 488–500.

Bergemann, D., B. Brooks, and S. Morris (2016): "Informationally Robust Optimal Auction Design," Tech. rep., Princeton University and the University of Chicago and Yale University, working paper.

Brooks, B. and S. Du (2021): "Optimal auction design with common values: An informationally robust approach," *Econometrica*, 89, 1313–1360.

——— (2023): "On the Structure of Informationally Robust Optimal Mechanisms," Tech. rep., The University of Chicago and University of California-San Diego, working paper.

Carroll, G. (2016): "Informationally robust trade and limits to contagion," *Journal of Economic Theory*, 166, 334–361.

Chung, K.-S. and J. C. Ely (2007): "Foundations of Dominant-Strategy Mechanisms," *The Review of Economic Studies*, 74, 447–476.

d'Aspremont, C. and L.-A. Gérard-Varet (1979): "Incentives and incomplete information," *Journal of Public economics*, 11, 25–45.

Du, S. (2018): "Robust mechanisms under common valuation," *Econometrica*, 86, 1569–1588.

Güth, W. and M. Hellwig (1986): "The private supply of a public good," *Journal of Economics*, 46, 121–159.

Milgrom, P. R. and R. J. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 1089–1122.

——— (1985): "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research*, 10, 619–632.

Myerson, R. B. and M. A. Satterthwaite (1983): "Efficient mechanisms for bilateral trading," *Journal of economic theory*, 29, 265–281.

Samuelson, P. A. (1954): "The pure theory of public expenditure," *The review of economics and statistics*, 387–389.

A Proof of Theorem 1

Lemma 1. *For $\alpha \in (0,1)$ and $x > 0$, define*

$$g(x) = \exp(-x)x^{\alpha} \int_{y=0}^{x} \exp(y)y^{-\alpha}\, dy. \tag{13}$$

Then we have

$$g'(x) = \alpha g(x)/x - g(x) + 1. \tag{14}$$

Moreover, we have $\lim_{x\to 0} g(x) = 0$, $\lim_{x\to 0} g(x)/x = \lim_{x\to 0} g'(x) = 1/(1-\alpha)$, and $\lim_{x\to\infty} g(x) = 1$.

Proposition 5. *Suppose $\overline{E}(x) = \frac{\overline{\lambda}}{\theta_L - 1} g(x)$ for $\alpha = \frac{N-1}{\theta_L-1} \in (0,1)$. There exist $(\overline{\lambda}, \overline{x})$ such that $\overline{E}(\overline{x}) = 1$ and $\overline{E}'(\overline{x}) = 0$.*

Proof. Set

$$\overline{\lambda} = \frac{\theta_L - 1}{g(\overline{x})}.$$

This ensures that $\overline{E}(\overline{x}) = 1$. The condition $\overline{E}'(\overline{x}) = 0$ is equivalent to $g'(\overline{x}) = 0$. We now show that there exists a $\overline{x}$ such that $g'(\overline{x}) = 0$. Since $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to\infty} g(x) = 1$ by Lemma 1, it suffices to show that $g(x) > 1$ for some $x \in (0, \infty)$.

Define

$$h(x) = \int_{y=0}^{x} \exp(y)y^{-\alpha} dy - \exp(x)x^{-\alpha}.$$

We have

$$h'(x) = \alpha \exp(x) x^{-\alpha-1},$$

which tends to infinity as $x \to \infty$. Hence, we have that $h(x)$ tends to infinity $x \to \infty$. This implies that $h(x) > 0$ and $g(x) > 1$ when x is sufficiently large. $\square$

Proposition 6. *$\overline{b}$ is an equilibrium in $(\overline{M}, \overline{I})$.*

Proof. The interim utility of agent i from playing s_i' when the signal is s_i is

$$\begin{aligned} U_i(s_i, s_i') &= \int_{y=0}^{\infty} \left(\frac{s_i}{s_i + y}\theta_L - \frac{s_i'}{s_i' + y} \right) \overline{E}(s_i' + y)\overline{\rho}(s_i + y) \frac{y^{N-2}}{(N-2)!}\, dy \\ &= C \int_{y=0}^{\overline{x} - s_i} \left(\frac{s_i}{s_i + y}\theta_L - \frac{s_i'}{s_i' + y} \right) g(s_i' + y) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy, \end{aligned}$$

where $C > 0$, $\alpha = \frac{N-1}{\theta_L - 1}$, $g(x)$ is defined by equation (13) when $x \leqslant \underline{x}$, and $g(x) = g(\overline{x})$ when $x \geqslant \overline{x}$. We will show $U_i(s_i, s_i) \geqslant U_i(s_i, s_i')$ for all $s_i \in (0, \overline{x}]$ and $s_i' \geqslant 0$, by showing that $\partial U_i(s_i, s_i')/\partial s_i'$ is non-negative for $s_i' \leqslant s_i$ and non-positive for $s_i' \geqslant s_i$.

We calculate:

$$
\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i')\\
=&\int_{y=0}^{\overline{x}-s_i}\left(\left(\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}\right)g'(s_i'+y)-\frac{y}{(s_i'+y)^2}g(s_i'+y)\right)\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy.
\end{aligned}
$$

Let us simplify the integrand when $s_i' \leqslant s_i$ using the differential equation (14):

$$
\begin{aligned}
&\left(\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}\right)g'(s_i'+y)-\frac{y}{(s_i'+y)^2}g(s_i'+y)\\
=&\left(\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha g(s_i'+y)}{s_i'+y}-g(s_i'+y)+1\right)-\frac{y}{(s_i'+y)^2}g(s_i'+y) \qquad (15)\\
=&\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}+\left(\left(\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha}{s_i'+y}-1\right)-\frac{y}{(s_i'+y)^2}\right)g(s_i'+y).
\end{aligned}
$$

Therefore, for $s_i' \leqslant s_i$:

$$
\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i')\\
=&\int_{y=0}^{\overline{x}-s_i}\left(\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}\right)\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy\\
&+\underbrace{\int_{y=0}^{\overline{x}-s_i}\left(\left(\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha}{s_i'+y}-1\right)-\frac{y}{(s_i'+y)^2}\right)g(s_i'+y)\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy}_{A}.
\end{aligned}
$$

Note that the term A implicitly involves a double integral, though the function g, which we now aim to simplify. According to Mathematica, we have

$$
\begin{aligned}
&\int\left(\left(\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha}{s_i'+y}-1\right)-\frac{y}{(s_i'+y)^2}\right)\exp(s_i-s_i')(s_i'+y)^{\alpha}(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy\\
=&\frac{\exp(s_i-s_i')}{N-1}y^{N-1}(s_i+y)^{-\frac{(N-1)\theta_L}{\theta_L-1}}(s_i'+y)^{\frac{N-\theta_L}{\theta_L-1}}(N-1+s_i'+y-(s_i'+y)\theta_L). \qquad (16)
\end{aligned}
$$

Therefore, integrating by parts, we have

$$
\begin{aligned}
A=&\frac{\exp(s_i-s_i')}{N-1}y^{N-1}(s_i+y)^{-\frac{(N-1)\theta_L}{\theta_L-1}}(s_i'+y)^{\frac{N-\theta_L}{\theta_L-1}}(N-1+s_i'+y-(s_i'+y)\theta_L)\int_{z=0}^{y+s_i'}\exp(z)z^{-\alpha}\,dz\Bigg|_{y=0}^{\overline{x}-s_i}\\
&-\int_{y=0}^{\overline{x}-s_i}\frac{\exp(s_i-s_i')}{N-1}y^{N-1}(s_i+y)^{-\frac{(N-1)\theta_L}{\theta_L-1}}(s_i'+y)^{\frac{N-\theta_L}{\theta_L-1}}(N-1+s_i'+y-(s_i'+y)\theta_L)\exp(s_i'+y)(s_i'+y)^{-\alpha}\,dy.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i')\\
=&\frac{\exp(s_i - s_i')}{N-1}(\overline{x}-s_i)^{N-1}\overline{x}^{-\frac{(N-1)\theta_L}{\theta_L-1}}(s_i'+\overline{x}-s_i)^{\frac{N-\theta_L}{\theta_L-1}}(N-1-(s_i'+\overline{x}-s_i)(\theta_L-1))\int_{z=0}^{\overline{x}-s_i+s_i'}\exp(z)z^{-\alpha}\,dz\\
&+\int_{y=0}^{\overline{x}-s_i}\left(\frac{s_i}{s_i+y}\theta_L-\frac{s_i'}{s_i'+y}-\frac{y}{s_i'+y}\left(1-\frac{s_i'+y}{\alpha}\right)\right)\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy\\
=&\exp(s_i-s_i')(\overline{x}-s_i)^{N-1}\overline{x}^{-\frac{(N-1)\theta_L}{\theta_L-1}}(s_i'+\overline{x}-s_i)^{\frac{N-\theta_L}{\theta_L-1}}\left(1-\frac{s_i'+\overline{x}-s_i}{\alpha}\right)\int_{z=0}^{\overline{x}-s_i+s_i'}\exp(z)z^{-\alpha}\,dz\\
&+\int_{y=0}^{\overline{x}-s_i}\left(\frac{s_i}{s_i+y}\theta_L-1+\frac{y}{\alpha}\right)\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy.
\end{aligned}
\tag{17}
$$

We next argue that (17) is zero when $s_i' = s_i$, i.e., that the first-order condition $\partial U_i(s_i, s_i)/\partial s_i' = 0$ is satisfied. This is equivalent to showing that

$$
\begin{aligned}
&-(\overline{x}-s_i)^{N-1}\overline{x}^{-N}\left(1-\frac{\overline{x}}{\alpha}\right)\int_{z=0}^{\overline{x}}\exp(z)z^{-\alpha}\,dz\\
=&\int_{y=0}^{\overline{x}-s_i}\left(\frac{s_i}{s_i+y}\theta_L-1+\frac{y}{\alpha}\right)\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy \qquad (18)\\
=&\int_{z=s_i}^{\overline{x}}\left(\frac{s_i}{z}\theta_L-1+\frac{z-s_i}{\alpha}\right)\exp(z)z^{-\alpha-(N-1)}(z-s_i)^{N-2}\,dz.
\end{aligned}
$$

By construction, we have $g'(\overline{x}) = 0$, so by equation (14),

$$
\exp(\overline{x})\overline{x}^{-\alpha} = \left(1-\frac{\alpha}{\overline{x}}\right)\int_{z=0}^{\overline{x}}\exp(z)z^{-\alpha}\,dz.
$$

Substituting this into (18) gives

$$
\begin{aligned}
\left(1-\frac{s_i}{\overline{x}}\right)^{N-1}\exp(\overline{x})\overline{x}^{-\alpha} =&\alpha\int_{z=s_i}^{\overline{x}}\left(\frac{s_i}{z}\theta_L-1+\frac{z-s_i}{\alpha}\right)\exp(z)z^{-\alpha-(N-1)}(z-s_i)^{N-2}\,dz\\
=&\int_{z=s_i}^{\overline{x}}\left(\frac{\frac{s_i}{z}(\alpha+N-1)-\alpha}{z-s_i}+1\right)\exp(z)z^{-\alpha}\left(1-\frac{s_i}{z}\right)^{N-1}dz\\
=&\int_{z=s_i}^{\overline{x}}\left(-\frac{\alpha}{z}+\frac{s_i(N-1)}{z(z-s_i)}+1\right)\exp(z)z^{-\alpha}\left(1-\frac{s_i}{z}\right)^{N-1}dz,
\end{aligned}
\tag{19}
$$

using $\alpha\theta_L = \alpha + N - 1$. But the above equation holds since

$$
\begin{aligned}
&\int_{z=s_i}^{\overline{x}}\frac{\alpha}{z}\exp(z)z^{-\alpha}\left(1-\frac{s_i}{z}\right)^{N-1}dz\\
=&\ -z^{-\alpha}\exp(z)\left(1-\frac{s_i}{z}\right)^{N-1}\Bigg|_{z=s_i}^{\overline{x}}+\int_{z=s_i}^{\overline{x}}z^{-\alpha}\frac{d}{dz}\left(\exp(z)\left(1-\frac{s_i}{z}\right)^{N-1}\right)dz\\
=&-\overline{x}^{-\alpha}\exp(\overline{x})\left(1-\frac{s_i}{\overline{x}}\right)^{N-1}+\int_{z=s_i}^{\overline{x}}z^{-\alpha}\exp(z)\left(1-\frac{s_i}{z}\right)^{N-1}\left(1+\frac{(N-1)z}{z-s_i}\frac{s_i}{z^2}\right)dz,
\end{aligned}
$$

using integration by parts.

Next, we argue that $\partial U_i(s_i, s_i)/\partial s_i'$ is decreasing in s_i' for $s_i' \leqslant s_i$. Defining $x = \overline{x} - (s_i - s_i')$, the derivative of (17) with respect to s_i' is

$$\begin{aligned}
&\frac{d}{dx}\left(\exp(\overline{x}-x)x^{\frac{N-\theta_L}{\theta_L-1}}\left(1-\frac{x}{\alpha}\right)\int_{z=0}^{x}\exp(z)z^{-\alpha}\,dz\right)\\
&=\exp(\overline{x}-x)x^{\frac{N-\theta_L}{\theta_L-1}}\left(-\left(1-\frac{x}{\alpha}\right)+\frac{N-\theta_L}{\theta_L-1}x^{-1}\left(1-\frac{x}{\alpha}\right)-\frac{1}{\alpha}\right)\int_{z=0}^{x}\exp(z)z^{-\alpha}\,dz\\
&\quad+\exp(\overline{x}-x)x^{\frac{N-\theta_L}{\theta_L-1}}\left(1-\frac{x}{\alpha}\right)\exp(x)x^{-\alpha}\\
&=\exp(\overline{x}-x)x^{\frac{N-\theta_L}{\theta_L-1}}\left(-2+\frac{x}{\alpha}+\frac{N-\theta_L}{(\theta_L-1)x}\right)\int_{z=0}^{x}\exp(z)z^{-\alpha}\,dz+\exp(\overline{x})x^{-1}\left(1-\frac{x}{\alpha}\right)\\
&=\exp(\overline{x})x^{-1}\left(-2+\frac{x}{\alpha}+\frac{\alpha-1}{x}\right)g(x)+\exp(\overline{x})x^{-1}\left(1-\frac{x}{\alpha}\right).
\end{aligned}$$

Now define

$$f(x)=\left(2-\frac{x}{\alpha}+\frac{1-\alpha}{x}\right)g(x)+\frac{x}{\alpha}-1$$

for all $x \in [0, \overline{x}]$. We calculate

$$\begin{aligned}
f(0)&=0,\\
f(\overline{x})&=\frac{2-\frac{\overline{x}}{\alpha}+\frac{1-\alpha}{\overline{x}}}{1-\frac{\alpha}{\overline{x}}}+\frac{\overline{x}}{\alpha}-1=\frac{1-\frac{\overline{x}}{\alpha}+\frac{1}{\overline{x}}}{1-\frac{\alpha}{\overline{x}}}+\frac{\overline{x}}{\alpha}=\frac{\frac{1}{\overline{x}}}{1-\frac{\alpha}{\overline{x}}}>0,
\end{aligned}$$

using the facts that $\lim_{x\to 0} g(x)/x = \frac{1}{1-\alpha}$ and $g'(\overline{x}) = 0$ and hence $g(\overline{x}) = \frac{1}{1-\alpha/\overline{x}} > 0$ from Lemma 1.

It therefore remains to show that $f(x) \geqslant 0$ for all $x \in (0, \overline{x})$. At any point where $f(x) = 0$, we must have

$$\begin{aligned}
g(x)&=\frac{1-x/\alpha}{2-x/\alpha+(1-\alpha)/x}\\
&=\frac{x(x-\alpha)}{x^2-2\alpha x-\alpha(1-\alpha)}\\
&=\frac{x(x-\alpha)}{(x-\alpha-\sqrt{\alpha})(x-\alpha+\sqrt{\alpha})}\equiv h(x).
\end{aligned}$$

So, it suffices to show that $x = 0$ is the unique point where the functions g and h intersect. Note that because $\alpha \in (0, 1)$, only one of the roots of the denominator of h is positive. We have to consider separately what happens for on either side of the positive root $\alpha + \sqrt{\alpha}$.

If $x < \alpha + \sqrt{\alpha}$, then the denominator of $h(x)$ is positive, and we clearly have that

$$h(x)=\frac{1-x/\alpha}{2-x/\alpha+(1-\alpha)/x}\leqslant\frac{1}{2+(1-\alpha)/x}\leqslant\frac{1}{1+(1-\alpha)/x}=\frac{x}{x+1-\alpha}\equiv\widehat{h}(x),$$

so it suffices to show that $g \geqslant \widehat{h}$, and they are equal only at zero. Note that

$$\widehat{h}'(x) = \frac{1-\alpha}{(x+1-\alpha)^2}.$$

Now, suppose that at $x > 0$ we have $g(x) = \widehat{h}(x)$. Then

$$\begin{aligned} g'(x) &= \left(\frac{\alpha}{x} - 1\right) \frac{x}{x+1-\alpha} + 1 \\ &= \frac{\alpha - x}{x+1-\alpha} + 1 \\ &= \frac{1}{x+1-\alpha} \\ &= \frac{x+1-\alpha}{1-\alpha} \widehat{h}'(x), \end{aligned}$$

so that $g'(x) \geqslant \widehat{h}'(x)$, and the inequality is strict of $x > 0$. Thus, by Lemma 2 of Milgrom and Weber (1982), $g(x) \geqslant \widehat{h}(x)$ for all x, and the inequality is strict if $x > 0$.

Now consider $x > \alpha + \sqrt{\alpha}$. Note that

$$\begin{aligned} h(\overline{x}) &= \frac{\overline{x}(\overline{x}-\alpha)}{\overline{x}^2 - 2\alpha\overline{x} - \alpha(1-\alpha)} \\ &= \frac{(\overline{x}-\alpha)^2}{\overline{x}^2 - 2\alpha\overline{x} - \alpha(1-\alpha)} \frac{\overline{x}}{\overline{x}-\alpha} \\ &= \frac{\overline{x}^2 - 2\alpha\overline{x} + \alpha^2}{\overline{x}^2 - 2\alpha\overline{x} + \alpha^2 - \alpha} \frac{\overline{x}}{\overline{x}-\alpha} \\ &> \frac{\overline{x}}{\overline{x}-\alpha} = g(\overline{x}). \end{aligned}$$

Moreover, we claim that $h'(x) < 0$ for $x > \alpha + \sqrt{\alpha}$, and since $g' > 0$, we cannot have $h(x) = g(x)$ in this range:

$$\begin{aligned} h'(x) &= \frac{(2x-\alpha)(x^2 - 2\alpha x - \alpha(1-\alpha)) - (x^2 - \alpha x)(2x - 2\alpha)}{(x^2 - 2\alpha x - \alpha(1-\alpha))^2} \\ &= \frac{2x^3 - 4\alpha x^2 - 2\alpha(1-\alpha)x - \alpha x^2 + 2\alpha^2 x + \alpha^2(1-\alpha) - 2x^3 + 2\alpha x^2 + 2\alpha x^2 - 2\alpha^2 x}{(x^2 - 2\alpha x - \alpha(1-\alpha))^2} \\ &= \frac{-2\alpha x - \alpha x^2 + 2\alpha^2 x + \alpha^2(1-\alpha)}{(x^2 - 2\alpha x - \alpha(1-\alpha))^2} \\ &= \frac{-\alpha\left[x^2 + 2(1-\alpha)x - \alpha(1-\alpha)\right]}{(x^2 - 2\alpha x - \alpha(1-\alpha))^2}. \end{aligned}$$

Thus, h' has the opposite sign as the term in brackets, which is clearly increasing in x, and is therefore minimized at $x = \alpha + \sqrt{\alpha}$. Plugging in, we get

$$\begin{aligned} &(\alpha + \sqrt{\alpha})^2 + 2(1-\alpha)(\alpha + \sqrt{\alpha}) - \alpha(1-\alpha) \\ &= \alpha^2 + 2\alpha\sqrt{\alpha} + \alpha + 2\alpha + 2\sqrt{\alpha} - 2\alpha^2 - 2\alpha\sqrt{\alpha} - \alpha + \alpha^2 \\ &= 2\alpha\sqrt{\alpha} + 2\alpha > 0. \end{aligned}$$

Thus, $h' < 0$, and we are done. This completes the proof that $f(0) = 0$, and $f(x) > 0$ for all $x \in [0, \overline{x}]$. This in turn completes the proof that $\partial U_i(s_i, s_i')/\partial s_i' \geqslant 0$ for $s_i' \leqslant s_i$.

Finally, we consider the case where $s_i' \geqslant s_i$, in which we have:

$$\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
&= \int_{y=0}^{\max(\overline{x}-s_i',0)} \left(\left(\frac{s_i}{s_i + y}\theta_L - \frac{s_i'}{s_i' + y} \right) g'(s_i' + y) - \frac{y}{(s_i' + y)^2} g(s_i' + y) \right) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
&\quad - \int_{y=\max(\overline{x}-s_i',0)}^{\overline{x}-s_i} \frac{y}{(s_i' + y)^2} g(\overline{x}) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy,
\end{aligned}$$

since $g'(x) = 0$ and $g(x) = g(\overline{x})$ for $x > \overline{x}$. Clearly, $\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \leqslant 0$ when $s_i' \geqslant \overline{x}$. So to show $\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \leqslant 0$ it suffices to consider only $s_i' \in [s_i, \overline{x}]$.

Applying equation (15), we get for $s_i' \in [s_i, \overline{x}]$:

$$\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
&= \int_{y=0}^{\overline{x}-s_i'} \left(\frac{s_i}{s_i + y}\theta_L - \frac{s_i'}{s_i' + y} \right) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
&\quad + \underbrace{\int_{y=0}^{\overline{x}-s_i'} \left(\left(\frac{s_i}{s_i + y}\theta_L - \frac{s_i'}{s_i' + y} \right) \left(\frac{\alpha}{s_i' + y} - 1 \right) - \frac{y}{(s_i' + y)^2} \right) g(s_i' + y) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy}_{B} \\
&\quad - \int_{y=\overline{x}-s_i'}^{\overline{x}-s_i} \frac{y}{(s_i' + y)^2} g(\overline{x}) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy,
\end{aligned}$$

where, by (16) we have

$$\begin{aligned}
B =& \frac{\exp(s_i - s_i')}{N-1} y^{N-1} (s_i + y)^{-\frac{(N-1)\theta_L}{\theta_L - 1}} (s_i' + y)^{\frac{N-\theta_L}{\theta_L-1}} (N - 1 + s_i' + y - (s_i' + y)\theta_L) \int_{z=0}^{y+s_i'} \exp(z) z^{-\alpha}\, dz \Bigg|_{y=0}^{\overline{x}-s_i'} \\
&- \int_{y=0}^{\overline{x}-s_i'} \frac{\exp(s_i - s_i')}{N-1} y^{N-1} (s_i + y)^{-\frac{(N-1)\theta_L}{\theta_L - 1}} (s_i' + y)^{\frac{N-\theta_L}{\theta_L-1}} (N - 1 + s_i' + y - (s_i' + y)\theta_L) \exp(s_i' + y)(s_i' + y)^{-\alpha}\, dy.
\end{aligned}$$

Thus, for $s_i' \in [s_i, \overline{x}]$,

$$\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
&= \exp(s_i - s_i')(\overline{x} - s_i')^{N-1} (\overline{x} - s_i' + s_i)^{-\frac{(N-1)\theta_L}{\theta_L-1}} \overline{x}^{\frac{N-\theta_L}{\theta_L-1}} \left(1 - \frac{\overline{x}}{\alpha}\right) \int_{z=0}^{\overline{x}} \exp(z) z^{-\alpha}\, dz \\
&\quad + \int_{y=0}^{\overline{x}-s_i'} \left(\frac{s_i}{s_i + y}\theta_L - 1 + \frac{y}{\alpha} \right) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
&\quad - \int_{y=\overline{x}-s_i'}^{\overline{x}-s_i} \frac{y}{(s_i' + y)^2} g(\overline{x}) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy.
\end{aligned} \tag{20}$$

Changing variable to $x = \overline{x} - s_i'$ and applying (19), we get for $s_i' \in [s_i, \overline{x}]$:

$$
\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
&= \exp(x+s_i)x^{N-1}(x+s_i)^{-\alpha-(N-1)}\left(\frac{1}{\overline{x}} - \frac{1}{\alpha}\right)g(\overline{x}) \\
&\quad + \int_{z=s_i}^{x+s_i}\left(\frac{s_i}{z}\theta_L - 1 + \frac{z-s_i}{\alpha}\right)\exp(z)z^{-\alpha-(N-1)}(z-s_i)^{N-2}\,dz \\
&\quad - \int_{y=\overline{x}-s_i'}^{\overline{x}-s_i}\frac{y}{(s_i'+y)^2}g(\overline{x})\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy \\
&= \exp(x+s_i)x^{N-1}(x+s_i)^{-\alpha-(N-1)}\left(\frac{1}{\overline{x}} - \frac{1}{\alpha}\right)g(\overline{x}) \\
&\quad + \frac{1}{\alpha}\left(1 - \frac{s_i}{x+s_i}\right)^{N-1}\exp(x+s_i)(x+s_i)^{-\alpha} \\
&\quad - \int_{y=\overline{x}-s_i'}^{\overline{x}-s_i}\frac{y}{(s_i'+y)^2}g(\overline{x})\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy \\
&= -\int_{y=\overline{x}-s_i'}^{\overline{x}-s_i}\frac{y}{(s_i'+y)^2}g(\overline{x})\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy \leqslant 0
\end{aligned}
$$

where in the last equality we use $\left(\frac{\alpha}{\overline{x}} - 1\right)g(\overline{x}) + 1 = g'(\overline{x}) = 0$. □

B Proof of Theorem 2

Recall the notion of a smooth mechanism on page 8 and the definition of the strategic virtual welfare $\lambda(\theta, a)$ in equation (5). Proposition 1 implies the following lower bound on the welfare guarantee:

$$
W(M, I, b) \geqslant \inf_{\mu \in \Delta_{\widehat{\theta}}} \int \left(\inf_{a \in \mathbb{R}_+^N} \lambda(\theta, a)\right)\mu(d\theta), \tag{21}
$$

where $\Delta_{\widehat{\theta}} = \{\mu \in \Delta(\Theta) : \int \Sigma\theta\mu(d\theta) \geqslant \widehat{\theta}\}$ and $\Theta = \{\theta \in \mathbb{R}_+^N : \theta_L \leqslant \Sigma\theta \leqslant \theta_H\}$.

For a proportional cost-sharing mechanism, the strategic virtual welfare only depends on $\Sigma\theta$ and Σa, and in a slight abuse of notation simplifies to

$$
\lambda(\Sigma\theta, \Sigma a) = (\Sigma\theta - 1)E(\Sigma a) + (\Sigma\theta - 1)E'(\Sigma a) - \frac{(N-1)E(\Sigma a)}{\Sigma a}.
$$

Suppose $E(\Sigma a)$ is non-decreasing in Σa. Since $\lambda(\Sigma\theta, \Sigma a)$ is non-decreasing and linear in $\Sigma\theta$ for every a, $\inf_{a \in \mathbb{R}_+^N} \lambda(\Sigma\theta, \Sigma a)$ is a non-decreasing and concave function of $\Sigma\theta$. Thus, the righthand-side of (21) is minimized when the expected value is exactly $\widehat{\theta}$. If not, then we can reduce the likelihood of some realization for which $\Sigma\theta > \widehat{\theta}$ and increase the likelihood of $\widehat{\theta}$. Moreover, concavity implies that the minimizing μ induces a two-point distribution (supported on $\overline{\theta}_L$ and $\overline{\theta}_H$) for $\Sigma\theta$. Therefore, the welfare guarantee of a proportional cost-sharing mechanism is at least

$$\mu_L \inf_{a \in \mathbb{R}_+^N} \lambda(\theta_L, \Sigma a) + \mu_H \inf_{a \in \mathbb{R}_+^N} \lambda(\theta_H, \Sigma a),$$

where

$$\mu_H = \frac{\widehat{\theta} - \theta_L}{\theta_H - \theta_L}, \mu_L = 1 - \mu_H.$$

B.1 Case 1: $\theta_L \leqslant N$ and $\widehat{\theta} > N$.

Consider the proportional cost-sharing mechanism with the expenditure function

$$\overline{E}(x) = \begin{cases} \frac{\lambda_H}{\theta_H - c} \int_{y=0}^{x} e^{y-x} \left(\frac{y}{x}\right)^{-(N-1)c/(\theta_L - c)} dy & x \leqslant x_H, \\ 1 & x > x_H, \end{cases} \tag{22}$$

where the parameters (λ_H, x_H) satisfy $\overline{E}(x_H) = 1$ and $\overline{E}'(x_H) = 0$ (Proposition 1). By construction we have

$$\lambda(\theta_H, x) = (\theta_H - 1)(\overline{E}(x) + \overline{E}'(x)) - \frac{(N-1)\overline{E}(x)}{x} \begin{cases} = \lambda_H & x \in [0, x_H], \\ > \lambda_H & x \in (x_H, \infty). \end{cases} \tag{23}$$

Substituting in (23), we get

$$\lambda(\theta_L, x) = (\theta_L - 1)(\overline{E}(x) + \overline{E}'(x)) - \frac{(N-1)\overline{E}(x)}{x} \begin{cases} = \lambda_H - (\theta_H - \theta_L)(\overline{E}(x) + \overline{E}'(x)) & x \in [0, x_H], \\ > \lambda_H - (\theta_H - \theta_L)(\overline{E}(x) + \overline{E}'(x)) & x \in (x_H, \infty). \end{cases}$$

By the argument in Lemma 5, $\overline{E}(x) + \overline{E}'(x)$ is decreasing in $x \in [0, \infty)$. Thus, applying Lemma 1, we have

$$\lambda(\theta_L, x) \geqslant \lambda(\theta_L, 0) = \lambda_H - (\theta_H - \theta_L)\frac{\lambda_H}{\theta_H - 1}\frac{1}{1 - \frac{(N-1)}{\theta_H - 1}} = \lambda_H \frac{\theta_L - N}{\theta_H - N}.$$

Therefore, the welfare guarantee of the proportional-cost sharing mechanism is at least

$$\left(\mu_H + \mu_L \frac{\overline{\theta}_L - N}{\theta_H - N}\right) \lambda_H. \tag{24}$$

By Theorem 1, the welfare potential of $I_{\overline{\theta}_H,0}$ is λ_H, while the welfare potential of $I_{N,\underline{x}}$ tends to 0 as $\underline{x} \to 0$. Thus, the welfare potential of the public randomization between $I_{N,\underline{x}}$ and $I_{\theta_H,0}$, with probabilities $\frac{\theta_H - \widehat{\theta}}{\theta_H - N}$ and $\frac{\widehat{\theta} - N}{\theta_H - N}$ respectively, tends to $\lambda_H \frac{\widehat{\theta} - N}{\theta_L - N}$, which is exactly (24), as $\underline{x} \to 0$.

It remains only to show that there exists an equilibrium at $(\overline{M}_{\theta_H}, \tilde{I}_{\underline{x}})$. This can be established by showing separately that there are equilibria of $(\overline{M}_{\theta_H}, I_{N,\underline{x}})$ and $(\overline{M}_{\theta_H}, I_{\overline{\theta}_H,0})$.

To prove existence in both cases, we will simply verify the hypotheses of Theorem 1 of Milgrom and Weber (1985), namely, that payoffs are equicontinuous and that the distribution of types is absolutely continuous with respect to the product of the marginals. The latter condition follows immediately from the fact that the distribution of types is absolutely continuous with respect to Lebesgue measure. With regard to payoffs, note that a player's payoff is

$$\left(\eta_i(s) - \frac{a_i}{\Sigma a}\right) \overline{E}(\Sigma a).$$

We have already established that the proportional cost-sharing mechanism is smooth, so equicontinuity follows from the fact that $\eta_i(s)$ is bounded above by θ_H. Hence, there exists an equilibrium in distributional strategies, which implies existence of an equilibrium in behavioral strategies.

This proves the first case of Theorem 2.

B.2 Case 2: $\theta_L > N$.

Recall the proportional cost-sharing mechanism $\widehat{M}$ with expenditure $\widehat{E}$ in equation (9), and the information structure $\widehat{I}$ defined by equations (10) and (12). By construction, $\widehat{E}$ is continuous on $[0, x_H]$ and satisfies

$$\lambda_L = \lambda(\theta_L, x) = (\theta_L - 1)(\widehat{E}(x) + \widehat{E}'(x)) - \frac{(N-1)\widehat{E}(x)}{x} \tag{25}$$

for $x \in [0, x_L]$, and

$$\lambda_H = \lambda(\theta_H, x) = (\theta_H - 1)(\widehat{E}(x) + \widehat{E}'(x)) - \frac{(N-1)\widehat{E}(x)}{x} \tag{26}$$

for $x \in [x_L, x_H]$.

We need the following conditions to be satisfied:

$$\widehat{E}(x_H) = \widehat{E}(x_L)\exp(x_L - x)\left(\frac{x_L}{x}\right)^{-(N-1)/(\theta_H-1)} + \frac{\lambda_H}{\theta_H - 1}\int_{y=x_L}^{x} \exp(y-x)\left(\frac{y}{x}\right)^{-(N-1)/(\theta_H-1)} dy = 1, \tag{27}$$

$$\widehat{E}'(x_H) = 0 \iff \theta_H - 1 - \frac{(N-1)}{x_H} = \lambda_H, \tag{28}$$

$$\widehat{E}'(x_L^-) + \widehat{E}(x_L) = \frac{\lambda_H - \lambda_L}{\theta_H - \theta_L} = \widehat{E}'(x_L^+) + \widehat{E}(x_L), \tag{29}$$

and

$$(\theta_H - 1)\widehat{\rho}(x_L^+) = (\theta_L - 1)\widehat{\rho}(x_L^-) \tag{30}$$

Lemma 2. *Condition* (29) *is equivalent to*

$$\frac{\lambda_H + \frac{\widehat{E}(x_L)(N-1)}{x_L}}{\theta_H - 1} = \frac{\lambda_L + \frac{\widehat{E}(x_L)(N-1)}{x_L}}{\theta_L - 1}. \tag{31}$$

Proof. By equations (25) and (26), we have

$$\widehat{E}(x_L) + \widehat{E}'(x_L^-) = \frac{\lambda_L + \frac{\widehat{E}(x_L)(N-1)}{x_L}}{\theta_L - 1},$$
$$\widehat{E}(x_L) + \widehat{E}'(x_L^+) = \frac{\lambda_H + \frac{\widehat{E}(x_L)(N-1)}{x_L}}{\theta_H - 1}.$$

Thus, condition (29) implies condition (31).

Now, suppose condition (31) holds, so $\widehat{E}'(x_L^-) = \widehat{E}'(x_L^+)$. Subtracting (25) from (26) at $x = x_L$ then gives condition (29). $\square$

Lemma 3. *There exist* $(x_L, x_H, \lambda_L, \lambda_H)$ *such that conditions* (27), (28), (29) *and* (30) *are satisfied.*

Proof. Define

$$g(x) = \int_{y=0}^{x} \exp(y-x)\left(\frac{y}{x}\right)^{-(N-1)/(\theta_L-1)} dy.$$

We can rewrite (30) as

$$\int_{x=x_L}^{x_H} \exp(x - x_L)(x/x_L)^{-(N-1)/(\theta_H-1)} dx = g(x_L)\frac{\mu_H(\theta_H-1)}{\mu_L(\theta_L-1)}. \tag{32}$$

Given $x_L > 0$, define $x_H(x_L)$ as the unique $x_H \geqslant x_L$ that satisfies equation (32).

Substituting the formula for $\widehat{E}(x_L)$, we can rewrite (31) as

$$\theta_H - 1 + \frac{(N-1)(\theta_H - \theta_L)}{x_L(\theta_L - 1)} g(x_L) = \frac{\lambda_H(\theta_L - 1)}{\lambda_L}. \tag{33}$$

Since $\lambda_H = \theta_H - 1 - \frac{(N-1)}{x_H(x_L)}$ (condition (28)), equation (33) gives λ_L as a function of x_L. Thus, equation (27) can be rewritten as

$$\begin{aligned}
&\exp(x_H(x_L) - x_L)\left(\frac{x_H(x_L)}{x_L}\right)^{-(N-1)/(\theta_H-1)} \\
&= \frac{\lambda_L}{\theta_L - 1} g(x_L) + \frac{\lambda_H}{\theta_H - 1}\frac{\mu_H(\theta_H-1)}{\mu_L(\theta_L-1)} g(x_L) \\
&= \left(\theta_H - 1 - \frac{(N-1)}{x_H(x_L)}\right)\left(\frac{1}{\theta_H - 1 + \frac{(N-1)(\theta_H-\theta_L)}{x_L(\theta_L-1)} g(x_L)} + \frac{\mu_H}{\mu_L(\theta_L-1)}\right) g(x_L),
\end{aligned}$$

where we used (32) in the first equality and (33) in the second equality.

Thus, finding a $(x_L, x_H, \lambda_L, \lambda_H)$ to satisfy conditions (27), (28), (29) and (30) is equivalent to finding a x_L to satisfy

$$\begin{aligned} &\exp(x_H(x_L) - x_L)\left(\frac{x_H(x_L)}{x_L}\right)^{-(N-1)/(\theta_H-1)} \\ &= \left(\theta_H - 1 - \frac{(N-1)}{x_H(x_L)}\right)\left(\frac{1}{\theta_H - 1 + \frac{(N-1)(\theta_H-\theta_L)}{x_L(\theta_L-1)}g(x_L)} + \frac{\mu_H}{\mu_L(\theta_L-1)}\right)g(x_L). \end{aligned} \tag{34}$$

Let x_L^* be the critical point of g: $g'(x_L^*) = 0$. We will now show that there exists a $x_L \in (0, x_L^*]$ that satisfies (34), which proves the proposition.

The lefthand side of (34) is clearly positive when x_L is sufficiently small, while the righthand side of (34) is clearly negative when x_L is sufficiently small (since $x_H(x_L) \to 0$ as $x_L \to 0$).

We now show that when $x_L = x_L^*$, the lefthand side of (34) is less than or equal to the righthand side.

We have

$$g(x_L^*)\left(1 - \frac{(N-1)}{x_L^*(\theta_L-1)}\right) = 1.$$

Thus,

$$\theta_H - 1 + \frac{(N-1)(\theta_H-\theta_L)}{x_L^*(\theta_L-1)}g(x_L^*) = \left(\theta_H - 1 - \frac{(N-1)}{x_L^*}\right)g(x_L^*).$$

Thus at $x_L = x_L^*$, the righthand side of (34) can be rewritten as

$$\left(\theta_H - 1 - \frac{(N-1)}{x_H(x_L^*)}\right)\left(\frac{1}{\left(\theta_H - 1 - \frac{(N-1)}{x_L^*}\right)g(x_L^*)} + \frac{\mu_H}{\mu_L(\theta_L-1)}\right)g(x_L^*),$$

where $x_H^* = x_H(x_L^*)$.

We can rewrite the above as

$$\frac{1 - \frac{\alpha}{x_H^*}}{1 - \frac{\alpha}{x_L^*}} + \int_{x=x_L^*}^{x_H^*} \exp(x - x_L^*)(x/x_L^*)^{-\alpha}dx\left(1 - \frac{\alpha}{x_H^*}\right)$$

where $\alpha = \frac{N-1}{\theta_H-1}$. Thus, we want to show

$$\exp(x_H^* - x_L^*)\left(\frac{x_H^*}{x_L^*}\right)^{-\alpha}\frac{1}{1 - \frac{\alpha}{x_H^*}} \leqslant \frac{1}{1 - \frac{\alpha}{x_L^*}} + \int_{x=x_L^*}^{x_H^*} \exp(x - x_L^*)(x/x_L^*)^{-\alpha}dx. \tag{35}$$

The left-hand side of (35) is clearly equal to the right-hand side when $x_H^* = x_L^*$. The derivative of the left-hand side of (35) with respect to x_H^* can be simplified to be

$$\exp(x_H^* - x_L^*)\left(\frac{x_H^*}{x_L^*}\right)^{-\alpha}\frac{(x_H^*-\alpha)^2 - \alpha}{(x_H^*-\alpha)^2}$$

while the derivative of the right-hand side of (35) with respect to x_H^* is

$$\exp(x_H^* - x_L^*) \left(\frac{x_H^*}{x_L^*}\right)^{-\alpha}.$$

$\square$

Lemma 4. *Suppose conditions* (27), (28), (29) *hold. Then we have* $\widehat{E}(x) \in [0,1]$ *for all* $x \in [0, x_H]$.

Proof. Differentiating (26) with respect to x gives:

$$\widehat{E}''(x) = \left(\frac{(N-1)}{(\theta_H - 1)x} - 1\right)\widehat{E}'(x) - \frac{(N-1)}{(\theta_H - 1)x^2}\widehat{E}(x).$$

Thus, for $x \in [(N-1)/(\theta_H - 1), \infty)$, $\widehat{E}'(x) > 0$ implies $\widehat{E}''(x) < 0$. Thus, by Lemma 2 of Milgrom and Weber (1982), on the interval $[(N-1)/(\theta_H - 1), \infty)$ $\widehat{E}'(x)$ across 0 at most once, and in which case it does so from above. Since condition (28) says that $\widehat{E}'(x_H) = 0$, we conclude that $\widehat{E}(x) \geqslant 0$ for $x \in [(N-1)/(\theta_H - 1), x_H]$. Using equation (26) it is easy to see that $\widehat{E}'(x) > 0$ when $x < (N-1)/(\theta_H - 1)$. Thus, $\widehat{E}'(x) \geqslant 0$ for all $x \in [x_L, x_H]$ for the q defined by (26).

By the smooth pasting condition (29), we have $\widehat{E}'(x_L) \geqslant 0$ for the $\widehat{E}$ defined by (25) as well. By the same argument applied to the differential equation in (26), we conclude that $\widehat{E}'(x) \geqslant 0$ for all $x \in [0, x_L]$ for the q defined by (25).

Finally, condition (27) says that $\widehat{E}(x_H) = 1$. So $\widehat{E}'(x) \geqslant 0$ for all $x \in [0, x_H]$ implies that $\widehat{E}(x) \leqslant 1$ for $x \in [0, x_H]$. $\square$

Lemma 5. *Suppose condition* (29) *holds. Then* $\widehat{E}(x) + \widehat{E}'(x)$ *is decreasing in* $x \in [0, x_H]$.

Proof. By equations (25) and (26), it suffices to show that $f(x) = \widehat{E}(x)/x$ decreases with x. We first focus on the interval $[0, x_L]$. We calculate, using (25),

$$\begin{aligned} f'(x)x^2 &= \widehat{E}'(x)x - \widehat{E}(x) \\ &= \frac{(N-1)}{\theta_L - 1}\widehat{E}(x) - x\widehat{E}(x) + \frac{\lambda_L}{\theta_L - 1}x - \widehat{E}(x) \\ &= x\left(-(1-\alpha)\widehat{E}(x)/x - \widehat{E}(x) + \frac{\lambda_L}{\theta_L - 1}\right), \end{aligned}$$

i.e.,

$$f'(x) = \frac{-(1-\alpha)f(x) - \widehat{E}(x) + \frac{\lambda_L}{\theta_L - 1}}{x},$$

where $\alpha = \frac{N-1}{\theta_L - 1}$.

Lemma 1 implies that $\lim_{x\to 0} f(x) = \frac{\lambda_L}{(\theta_L - 1)(1-\alpha)}$ and $\lim_{x\to 0}\widehat{E}(x) = 0$, so by L'Hôpital's rule we have

$$\lim_{x\to 0} f'(x) = -(1-\alpha)\lim_{x\to 0} f'(x) - \lim_{x\to 0}\widehat{E}'(x),$$

i.e.,

$$\lim_{x\to 0} f'(x) = -\frac{\lambda_L}{(\theta_L - 1)(2-\alpha)(1-\alpha)} < 0,$$

since Lemma 1 implies that $\lim_{x\to 0} \widehat{E}'(x) = \frac{\lambda_L}{(\theta_L-1)(1-\alpha)}$.

For the sake of contradiction, suppose $F = \{x \in \mathbb{R}_+ : f'(x) > 0\} \neq \varnothing$.

Since $\lim_{x\to 0} f'(x) < 0$, there exist $0 < x' < x''$ such that $f'(x') = 0$ and $(x', x''] \subseteq F$. This implies that $f(x'') > f(x')$ and $\widehat{E}(x'') > \widehat{E}(x')$ (since $f'(x) > 0$ implies $\widehat{E}'(x) > 0$), which is a contradiction since

$$f'(x'')x'' = -(1-\alpha)f(x'') - \widehat{E}(x'') + \frac{\lambda_L}{\theta_L - 1} < -(1-\alpha)f(x') - \widehat{E}(x') + \frac{\lambda_L}{\theta_L - 1} = f'(x')x' = 0.$$

Thus, we conclude that $f'(x) \leqslant 0$ for $x \in [0, x_L]$. By (29), this implies that $f'(x_L^+) \leqslant 0$. On the interval $[x_L, x_H]$, because of equation (26) we have

$$f'(x) = \frac{-(1-\alpha)f(x) - \widehat{E}(x) + \frac{\lambda_H}{\theta_H - 1}}{x},$$

for $\alpha = \frac{N-1}{\theta_H - 1}$. By exactly the same argument as in the previous paragraph, we conclude that $f'(x) \leqslant 0$ for $x \in [x_L, x_H]$.

□

Lemma 6. *Suppose conditions* (27), (28), (29) *hold. Then we have*

$$\inf_{x\in\mathbb{R}_+} \lambda(\theta_L, x) = \lambda_L,$$

and

$$\inf_{x\in\mathbb{R}_+} \lambda(\theta_H, x) = \lambda_H.$$

Thus, the welfare guarantee of the proportional cost-sharing mechanism is at least $\mu_L\lambda_L + \mu_H\lambda_H$.

Proof. Given conditions (25) and (26), we need to show that

$$\lambda(\theta_L, x) = \lambda_H + (\widehat{E}(x) + \widehat{E}'(x))(\theta_L - \theta_H) \geqslant \lambda_L, \tag{36}$$

for $x \in [x_L, x_H]$, and

$$\lambda(\theta_H, x) = \lambda_L + (\widehat{E}(x) + \widehat{E}'(x))(\theta_H - \theta_L) \geqslant \lambda_H, \tag{37}$$

for $x \in [0, x_L]$. Conditions (36) and (37) follow from Lemma 5 and equation (29).

Conditions (27) and (28) then imply that

$$\lambda_H = \lambda(\theta_H, x_H) = (\theta_H - 1) - \frac{(N-1)}{x_H} < \lambda(\theta_H, x) = (\theta_H - 1) - \frac{(N-1)}{x}$$

and

$$\lambda(\theta_L, x) = \lambda(\theta_H, x) + (\theta_L - \theta_H) > \lambda_H + (\theta_L - \theta_H) \geqslant \lambda_L,$$

for $x \in (x_H, \infty)$, where the last inequality follows from (36) when $x = x_H$. □

Lemma 7. *Suppose condition* (30) *holds, then the welfare potential of* $\widehat{I}$ *is at most*

$$\frac{(\theta_H - 1)\widehat{\rho}(x_H)(x_H)^{N-1}}{(N-1)!}. \tag{38}$$

Proof. The informational virtual welfare is

$$\gamma(s,e) = \left((\widehat{\rho}(\Sigma s) - \widehat{\rho}'(\Sigma s))(\Sigma\widehat{\eta}(\Sigma s) - 1) - \frac{(N-1)\Sigma\widehat{\eta}(\Sigma s)\widehat{\rho}(\Sigma s)}{\Sigma s}\right) q,$$

where $\Sigma\widehat{\eta}(\Sigma s) = \theta_L$ when $\Sigma s \in [0, x_L)$ and $\Sigma\widehat{\eta}(\Sigma s) = \theta_H$ when $\Sigma s \in [x_L, x_H)$. By construction, we have $\gamma(s,e) = 0$ whenever $\Sigma s \in [0, x_L)$ and $\Sigma s \in (x_L, x_H)$.

For a fixed $\epsilon > 0$, the upper bound from Proposition 2 can be written as

$$\int_{\Sigma s \in [0, x_L - \epsilon) \cup [x_L, x_H - \epsilon)} \max_{e \in \Omega} \gamma(s,e;\epsilon) ds + \int_{\Sigma s \in [x_L - \epsilon, x_L]} \max_{e \in \Omega} \gamma(s,e;\epsilon) ds + \int_{\Sigma s \in [x_H - \epsilon, x_H]} \max_{e \in \Omega} \gamma(s,e;\epsilon) ds.$$

By the argument in page 11, as $\epsilon \to 0$ the first term tends to 0 and the third term tends to (38). We can rewrite the second term as

$$\int_{\Sigma s \in [x_L - \epsilon, x_L]} \max_{q \in [0,1]} \left(\widehat{\rho}(\Sigma s)(\widehat{\theta} - 1)q - \frac{1}{\epsilon}\sum_{i=1}^{N}\left(\widehat{\rho}(\Sigma s + \epsilon)\theta_H \frac{s_i + \epsilon}{\Sigma s + \epsilon} - \widehat{\rho}(\Sigma s)\theta_L \frac{s_i}{\Sigma s}\right) q \right.$$
$$\left. + \frac{\widehat{\rho}(\Sigma s + \epsilon) - \widehat{\rho}(\Sigma s)}{\epsilon} cq \right) ds$$

which, as $\epsilon \to 0$, converges to 0 by condition (30). □

Proposition 7. *Suppose conditions* (27), (28), (29) *and* (30) *hold. Then the welfare guarantee of* $\widehat{M}$ *is equal to the welfare potential of* $\widehat{I}$.

Proof. We can rewrite equation (30) as

$$\frac{\mu_L \lambda_L \exp(x_L)(x_L)^{-(N-1)/(\theta_L - 1)}}{\frac{\lambda_L}{\theta_L - 1}\int_{x=0}^{x_L} \exp(x) x^{-(N-1)/(\theta_L - 1)} dx} = \frac{\mu_H \lambda_H \exp(x_L)(x_L)^{-(N-1)/(\theta_H - 1)}}{\frac{\lambda_H}{\theta_H - 1}\int_{x=x_L}^{x_H} \exp(x) x^{-(N-1)/(\theta_H - 1)} dx},$$

or

$$\frac{\mu_L \lambda_L}{\widehat{E}(x_L)} = \frac{\mu_H \lambda_H \exp(x_L - x_H)(x_L/x_H)^{-(N-1)/(\theta_H - 1)}}{1 - \widehat{E}(x_L)\exp(x_L - x_H)(x_L/x_H)^{-(N-1)/(\theta_H - 1)}}.$$

Therefore, the welfare guarantee from Lemma 6 is

$$\begin{aligned}
&\mu_L \lambda_L + \mu_H \lambda_H \\
&= \mu_H \lambda_H \frac{1}{1 - \widehat{E}(x_L)\exp(x_L - x_H)(x_L/x_H)^{-(N-1)/(\theta_H - 1)}} \\
&= \frac{\mu_H \lambda_H}{\frac{\lambda_H}{\theta_H - 1}\int_{y=x_L}^{x_H} \exp(y - x_H)\left(\frac{y}{x_H}\right)^{-(N-1)/(\theta_H - 1)} dy} \\
&= \frac{(\theta_H - 1)\widehat{\rho}(x_H)(x_H)^{N-1}}{(N-1)!}
\end{aligned}$$

which is equal to the welfare potential from Lemma 7. □

Finally, we note that an equilibrium exists for $(\widehat{M}, \widehat{I})$ by exactly the same argument given in Case 1.

The case of $\theta_L > N$ in Theorem 2 then follows from Proposition 7.