Robust Mechanisms for the Financing of Public Goods*

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Abstract

We propose a novel *proportional cost-sharing mechanism* for funding public goods with interdependent values: the agents simultaneously submit bids, which are non-negative numbers; the expenditure on the public good is an increasing and concave function of the sum of the bids; and each agent is responsible for the fraction of the expenditure proportional to their bid. The proportional cost-sharing mechanism provides a non-trivial guarantee for social welfare, regardless of the structure of the agents' information and the equilibrium that is played, as long as the social value for the public good is sufficiently large. Moreover, this guarantee is shown to be unimprovable in environments where the designer knows a lower bound on the social value. The guarantee converges to the entire efficient surplus when the social value grows large. When there are two agents, our model can be reinterpreted as one of bilateral trade, and the proportional cost-sharing is reinterpreted as proportional pricing.

Keywords: Mechanism design, information design, public good, interdependent values, robustness.

JEL Classification: C72, D44, D82, D83.

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1 Introduction

Consider a society that is deciding how much resources to expend on a public good. The good can be produced in continuous quantity, and at linear cost, up to a maximum amount. Valuations are linear and increasing in the expenditure. The agents who make up the society can form a social contract, consisting of a mechanism that will determine how much to spend on the good and each agent's share of the expense. The agents possess private information about their respective valuations of the good, which may differ across agents. The agents must agree to the proposed mechanism after they are endowed with private information. This gives rise to a free-rider problem, wherein the agents may behave as if their value for the good is lower than it truly is, in order to reduce their share of the expense. The question is: What kind of mechanism should the society implement, in order to maximize their joint welfare?

The public expenditure problem just described has been studied in various forms going back to Samuelson (1954). The formulation closest to ours is that of Güth and Hellwig (1986), who assume that the agents' values for the public good are independently distributed, each agent knows their own value and nothing more, and agents must be induced to participate in the mechanism at the interim stage. We describe their contribution in more detail below. In general, the exact solution of the Bayesian mechanism design problem, including in the public goods case, is sensitive to assumptions about the distribution of values and the higher-order beliefs held by the agents, which are typically modeled as an information structure (i.e., a Harsanyi type space). However, it seems hard to say which information structure is empirically relevant, especially if we allow for the possibility that signals may be correlated and values may be interdependent, meaning that one agent's private signal may be informative about another agent's value.

In real-world applications, mechanism designers will not be able to describe fine details of the agents' higher order beliefs. In particular, a mechanism designer may be unwilling or unable to commit to a particular information structure as the correct description of what the agents know. Rather, the designer is more likely to know relatively coarse features of the environment, and especially about payoff-relevant fundamentals, such as the average value of the agents or the range of possible values. Thus, a useful formulation of the mechanism design problem will consider the performance of a mechanism across a range of information structures that are consistent with those coarse features.

This paper contributes such a theory. In our baseline model, the mechanism designer simply knows a lower bound on the social value of the good (i.e., the sum of the agents' values). This lower bound is assumed to be greater than the cost, to avoid the degenerate case where the efficient outcome is to not produce. We later consider a richer version of the model where the designer further specifies an upper bound on the social value and a lower bound on the expected social value, and again this lower bound is at least the cost of production. We first describe the results for the baseline model, and then we will describe the extension.

The *welfare guarantee* of a mechanism defined to be its infimum social welfare across all information structures and equilibria. Importantly, subject to the restrictions in the previous paragraph, we allow for arbitrary (common prior) information, for which there may be correlation in signals and interdependence in values. Loosely speaking, we compute mechanisms that maximize the welfare guarantee. The solutions that we identify are *proportional cost-sharing mechanisms*: Each agent's action is a non-negative number; the total expenditure is an increasing and concave function of the aggregate action; and each agent's share of the cost is proportional to their action. These mechanisms create a natural tradeoff for the agents that mitigates free riding: By reducing their action, agents can reduce their share of the expense, but they also reduce the total expenditure.

To see how one might implement the proportional cost-sharing mechanism in practice, imagine that China and the US have signed a treaty to address the climate change crisis, which entails a protocol for assigning commitments to reduce greenhouse gas emissions. (This problem is not a bad fit for our model, because it involves a public good with uncertain and heterogeneous values, the social value is undoubtedly large, and the ratification process is consistent with our approach to participation constraints.) The way the treaty works is as follows. First, we divide the total amount $\overline{E}$ of emission reductions (in tons of carbon dioxide, say) into K equal-sized tiers. Each tier has a pre-specified number of shares y_k, so a share in tier k has a face value of $\frac{\overline{E}}{K y_k}$ tons. Each nation can demand a number of shares and is committed to reduce emissions according to the face value of its allocated shares. The demands are matched first to the shares from tier 1, and then to tier 2 once tier 1 is exhausted, and so on; within each tier each nation's allocation is proportional to its demand. For example, suppose there are two tiers with $y_1 = 40$ and $y_2 = 60$. If China demands 30 shares and the US demands 40 shares, then their respective shares of aggregate demand are 3/7 and 4/7. The resulting total demand is 70, so that all 40 tier 1 shares will be allocated, and 30 tier 2 shares are allocated as well. China is ultimately responsible for reducing emissions by $\frac{3}{7} \times \overline{E} \times \left(40 \times \frac{0.5}{40} + 30 \times \frac{0.5}{60}\right)$ tons, while the US is responsible for reducing emissions by $\frac{4}{7} \times \overline{E} \times \left(40 \times \frac{0.5}{40} + 30 \times \frac{0.5}{60}\right)$ tons.

To our knowledge, these mechanisms are new to the literature. Their robustness derives from the way they balance social welfare against agents' incentives in a manner that limits the scope for free riding, as we now explain. In the mechanisms we construct, agents can indicate that they are unwilling to pay for the good by taking a low action, so that low action profiles are associated with low expenditure and inefficient production. The concern about free riding is that a strategy profile in which expected expenditure is relatively large might unravel because the agents are tempted to deviate to actions associated with lower expenditure and lower cost, which in our mechanism are lower actions. A natural subset of deviations to consider is those that are marginal, meaning that they correspond to an infinitesimal change in an agent's action. To guard against unraveling, we engineer the mechanism so that marginal effects are high when actions are low, which is also when expenditure is low. Thus, if low expenditure were to occur too often in equilibrium, the agents would tempted to deviate to higher actions. Of course, in equilibrium, the expected marginal effect must be zero (otherwise an agent would want to deviate), and hence we make marginal effects low at high action profiles and when expenditure is relatively high.

This is precisely the structure that we obtain with proportional cost-sharing mechanisms. These mechanisms have the special feature that the average of the agents' marginal effects depends only on the aggregate value and aggregate action. Hence, the relationship between social welfare and average marginal incentives is reduced to the choice of the mapping from the aggregate action to total expenditure. It turns out that if the total expenditure function is linear (such as if there were only a single tier in our example above), then the average marginal effect is independent of the aggregate demand (and hence doesn't depend on actions at all). But as we just observed, this is undesirable from a robustness perspective: We forestall unraveling by positively relating marginal effects to expenditure on the public good. This is achieved with a concave expenditure function, e.g., the two tiers in our example. In fact, under the optimal specification, social welfare and marginal effects are exactly balanced. More precisely, the optimal proportional cost-sharing equalizes the *strategic virtual objective*—the sum of social welfare and the agents' marginal incentives—across all states and action profiles.

In addition to studying guarantee-maximizing mechanisms, we also study the worst-case information structures against which the proportional cost-sharing mechanisms are implicitly guarding. The *welfare potential* of an information structure is its maximum welfare across all mechanisms and equilibria. We construct information structures that have a potential arbitrarily close to the guarantee of the proportional cost-sharing mechanism. In a sense that we will expand on shortly, these information structures minimize the potential. The information structure is of the following form: the social value of the good is (not surprisingly) equal to its lower bound with probability one; the agents' signals are non-negative real numbers; the density only depends on the sum of the signals; and each agent's expected value for the good is proportional to their signal.

In order to describe our results more precisely, we must address an important technical issue that arises in our model. The guarantee-maximizing mechanism has a continuum of actions and the potential-minimizing information has a continuum of signals. An obvious concern with such infinite objects is that the guarantee and potential might be either ill-defined or vacuous, because equilibria fail to exist for some or all information structures or mechanisms, respectively. A similar issue arises in our earlier work (Brooks and Du, 2021, 2023). In the present paper, we address this concern by proving that an equilibrium exists at the saddle point consisting of the guarantee-maximizing mechanism and potential-minimizing information structure.

The solution just described works as long as the minimum social value is sufficiently large. As long as the social value per unit expenditure is greater than one, it is socially efficient to produce the good. However, even when this is the case, it may still be impossible to implement efficient expenditure because of the free-rider problem, so that the maximum guarantee is less than the efficient surplus. In the case where the per capita social value is at least one, the solution is as just described. However, when the per-capital social value is less than one, so that the value of a dollar of public expenditure is worth less than a dollar to the average agent, we construct information structures for which the welfare potential is arbitrarily close to zero. That it should be so challenging to generate non-trivial guarantees is perhaps not surprising given the weakness of our assumptions on the information structure. Nonetheless, it is quite striking that even though there is common knowledge that the efficient outcome is full expenditure, there are information structures for which the free-rider problem is so severe that no mechanism can generate a non-negligible amount of surplus in any equilibrium.

As mentioned above, we also consider the more general model where there is an upper bound on the social value and a lower bound on the expected social value. The value of this extension is that it allows for positive probability that the social value is arbitrarily small.

The main finding is that proportional cost-sharing mechanisms continue to maximize the guarantee. If the expected social value per capita is less than one, the model again collapses, and the max guarantee and min potential are both zero. But as long as the expected social value per capita is greater than one, there are strong maxmin solutions with non-trivial guarantee. Moreover, a proportional cost-sharing mechanism is part of the solution and attains the guarantee. The difference with our baseline model is in the particular form of the concave total expenditure function.

Thus, while the exact optimal total expenditure rule depends on the particular assumptions we impose on fundamentals, the main takeaway from our model is that the class of proportional cost-sharing mechanisms provides unimprovable welfare guarantees, if we allow for relatively large uncertainty about the form of private information and the dispersion in values.

An important consideration is what happens if the assumptions about fundamentals are also misspecified. For example, what happens to the welfare guarantee of a guarantee-maximizing proportional cost-sharing mechanism if the social value ends up being significantly larger than the lower bound? We show that even if these assumptions are slightly misspecified, the mechanism will still provide a welfare guarantee that is close to the optimum. Moreover, Proposition 4 shows that as the social value grows large, the proportional cost-sharing mechanism guarantees a fraction of the efficient surplus that converges to one. Thus, when the social value is large, proportional cost-sharing mechanisms achieve approximately efficient outcomes.

Our work is connected to the literatures on the public expenditure problem and on informationally-robust mechanism design. Most of what is known about the public goods problem concerns information structures with private values. In this case, it is well-known that if there is no budget constraint, then the efficient outcome can be implemented in dominant strategies with the Vickrey-Clarke-Groves mechanism. Moreover, d'Aspremont and Gérard-Varet (1979) showed that it is still possible to achieve efficient outcomes with Bayes Nash implementation and ex post budget balance, as long as participation constraints are ex ante. The closest paper to ours appears to be Güth and Hellwig (1986), who study Bayes Nash implementation, ex ante budget balance, and interim participation constraints. They characterize social-welfare maximizing direct mechanisms, subject to ex ante budget balance. The mechanisms they describe have full expenditure if the sum of the agents' reported virtual values exceeds a cutoff. Otherwise, the expenditure is zero. The agents' interim payments are pinned down from the production rule via the standard envelope argument. Importantly, the virtual values and the transfers depend on the value distribution, so in that sense, the designer needs to know a great deal about the environment in order to calibrate the mechanism.

We are unaware of other work that addresses optimal mechanism design for the public goods problem with interdependent values and correlated types. However, as we will expand on towards the end of the paper, the special case of our model with two agents can be reinterpreted as an instance of the bilateral trade problem, where there is an upper bound on the seller's value and a lower bound on the gains from trade. Our analysis is therefore also connected to the literature on bilateral trade, which has highlighted the inefficiency that results from budget balance and incentive and participation constraints (Myerson and Satterthwaite, 1983), as well as the potential welfare-reducing effects of interdependence in values and correlation in signals (Akerlof, 1970; Carroll, 2016). In the bilateral trade context, the proportional cost-sharing mechanism can be reinterpreted as a *proportional-price trading mechanism*: The buyer and seller submit non-negative numbers, trade occurs with a probability that depends on the aggregate action, and the price is a weighted average of the lowest possible value of the buyer and the highest possible value of the seller. These weights are proportional to the actions of the seller and the buyer, respectively. Thus, by increasing their action, an agent can increase the probability of trade, but at the cost of moving the terms of trade in a direction that is unfavorable. Moreover, the trading probability is more sensitive to the agents' actions when the trading probability is small, which guards against the expected probability of trade falling too low in equilibrium. Theorem 3 shows that proportional-price trading mechanisms provide optimal guarantees for gains from trade.

Within the robust mechanism design literature, our work is most closely related to the recent literature on maxmin mechanism design (Chung and Ely, 2007; Bergemann et al., 2016; Du, 2018; Brooks and Du, 2021, 2023). The proportional cost-sharing mechanisms we derive are reminiscent of the proportional auctions that were found to be robustly optimal in Brooks and Du (2021). We discuss in greater detail the similarities and differences between the two mechanisms after presenting our main results. Most closely related is Brooks and Du (2023), who describe a general framework for informationally robust optimal mechanism design. Setting aside technical differences regarding finite versus infinite mechanisms and information structures, our model is a special case of that of Brooks and Du (2023). That paper introduced the notions of strategic and informational virtual objectives, which are the designer's objective plus an adjustment corresponding to local equilibrium constraints. In the case of the strategic virtual objective, this adjustment is the sum of the agents' gains from deviating to nearby actions, and in the case of the informational virtual objective, the adjustment is the sum of agents' gains from mimicking nearby types. Brooks and Du (2023) argued that the expectation (across payoff-relevant states) of the lowest (across actions) strategic virtual objective is a lower bound on a mechanism's guarantee. Similarly, the expectation (across signals) of the highest (across outcomes) informational virtual objective is an upper bound on the potential. The present paper applies this bounding methodology to the public goods problem: the proportional cost-sharing mechanisms maximize the expected lowest strategic virtual objective, and the worst-case information structures minimize the expected highest informational virtual objective. Some of our results on the baseline model are summarized in Brooks and Du (2023) at a high level as an application of the more general framework.

The rest of this paper is organized as follows. Section 2 describes the model. Section 3 analyzes the baseline model where there is only a lower bound on the social value. Section 4 analyzes the extension where there are lower and upper bounds and a known expected social value. Section 5 contains some additional results, including on robustness to misspecification of fundamentals, the connection to bilateral trade, and comparisons with alternative mechanisms. Section 6 is a conclusion. Omitted proofs are in the Appendix.

2 Model

2.1 Fundamentals, information structures, and mechanisms

Society chooses how much resources to expend on a public good. The expenditure is denoted $E \in [0,1]$. The marginal value of expenditure to agent i is $\theta_i \in \mathbb{R}_+$. All of the expenditures must be raised from the N agents. Let $e_i \in \mathbb{R}$ denote the amount supplied by agent i. (Note that we allow expenditure to be negative, in which case an agent receives a subsidy from the rest of society.) Budget balance requires that $E = \sum_i e_i \equiv \Sigma e$. (In general we write $\Sigma x \equiv \sum_i x_i$ for $x \in \mathbb{R}^k$ for some k.) Agent i's payoff is $u_i = \theta_i E - e_i$. The mechanism designer's objective is to maximize social welfare, which is $\Sigma u = E(\Sigma\theta - 1)$. Full expenditure is socially efficient as long as $\Sigma\theta \geqslant 1$.

The agents' higher-order beliefs about θ are described by an *information structure* $I = (S, \rho, v)$, where S_i is a measurable set of signals for agent i, $S = \prod_{i=1}^N S_i$ is the set of signal profiles, and $\rho \in \Delta(S)$ is the joint distribution of signals, and $v : S \to \mathbb{R}_+^N$ is the expectation of θ conditional on the signal profile s. Due to the linearity of utility in θ, we can write agent i's expected utility of e at s as $v_i(s)E - e_i$, and the designer's expected welfare as $(\Sigma v(s) - 1)E$.

We will consider two sets of assumption about the information structure. In the baseline model, there is a lower bound $\theta_L \geqslant 0$ on the social value, so $\Sigma v(s) \geqslant \theta_L$ for all s. The baseline model is studied in Section 3. In Section 4, we consider a richer model where we further assume that the expected social value is at least $\widehat{\theta}$ and, moreover, there is an upper bound θ_H on the social value.

The agents interact through a *mechanism* $M = (A, e)$, where A_i is a measurable set of actions for agent i, $A = \prod_{i=1}^N A_i$ are the action profiles, and $e : A \to \Omega$ where $\Omega = \{e \in \mathbb{R}^N : \Sigma e \in [0,1]\}$ is the set of outcomes. We write $E(a) = \Sigma e(a)$ for the associated total expenditure.

A mechanism is *participation secure* if for every agent i, there exists an action $0 \in A_i$ such that $\theta_i E(0, a_{-i}) - e_i(0, a_{-i}) \geqslant 0$ for all $a_{-i} \in A_{-i}$ and $\theta \in \Theta$. Since $\theta_i = 0$ is a possibility, this condition is equivalent to $e_i(0, a_{-i}) \leqslant 0$ for all $a_{-i} \in A_{-i}$.

2.2 Solution concepts

A pair (M, I) of a mechanism and an information structure is a simultaneous-move Bayesian game, in which the behavioral strategy of agent i is a mapping $b_i : S_i \to \Delta(A_i)$. We identify a profile b with the function $b : S \to \Delta(A)$ given by $b(da|s) = \prod_i b_i(da_i|s_i)$. Given a strategy profile b, the ex ante expected payoffs are

$$U_i(M, I, b) = \int_{s,a} (v_i(s)E(a) - e_i(a))b(da \mid s)\rho(ds)$$

The profile b is a *(Bayes Nash) equilibrium* if $U_i(b; M, I) \geqslant U_i(b'_i, b_{-i}; M, I)$ for all i and b'_i. The objective of the designer is the expected *social welfare*

$$W(M, I, b) = \sum_i U_i(M, I, b) = \int_{s,a} (\Sigma v(s) - 1)E(a)b(da \mid s)\rho(ds).$$

For a mechanism M, define its *welfare guarantee* as the minimum expected welfare across all information structures I' and equilibria of (M, I').

For an information structure I, define its *welfare potential* as the maximum expected welfare across all mechanisms M' and equilibria of (M', I).

As in Brooks and Du (2021), we seek to characterize a *strong maxmin solution* for the mechanism designer. Formally, a tuple (M, I, b) is a *ϵ-strong maxmin solution* if

(i) b is an equilibrium of (M, I)

(ii) The welfare guarantee of M is at least $W(M, I, b) - \epsilon$;

(iii) The welfare potential of I is at most $W(M, I, b) + \epsilon$.

The equilibrium b in Condition (i) is needed so that statements (ii) and (iii) about the guarantee and potential are not vacuous (because of equilibrium nonexistence). See Section 5.1 for a discussion of equilibrium existence.

When $\epsilon = 0$, we call the tuple (M, I, b) a *strong maxmin solution*, M a *guarantee-maximizing mechanism*, I a *potential-minimizing information structure*, and $W(M, I, b)$ the *value of the solution*. We also use the terms "guarantee-maximizing" and "potential-minimizing" when the mechanism and information structure are limits of ϵ-strong maxmin solutions as $\epsilon \to 0$.

2.3 A note on interpretation

One might consider a more general model where there is a lower bound $\underline{\theta}$ on each agent's individual value θ_i, as well as a lower bound $\tilde{\theta}$ on the social value $\Sigma\theta$. Let us also explicitly introduce a cost of production c (in Section 2.1 we normalized c to be 1). In this case, the mechanism can always demand a "baseline" contribution from agent i of $\underline{\theta}E$ without violating participation security. Thus, if $N\underline{\theta} > c$, it is possible to implement an efficient outcome. On the other hand, if $N\underline{\theta} < c$, then efficient production cannot be financed just from such baseline contributions, and the question is whether there is a mechanism that can cover the budget shortfall $c - N\underline{\theta}$. If in addition we have $\tilde{\theta} > c$, then we know that production is efficient, but there is uncertainty about how the uncertain component of the social value $\tilde{\theta} - N\underline{\theta}$ is distributed across agents. Thus, we should interpret θ_L as a lower bound on the normalized uncertain portion of the social value relative to the shortfall, i.e., $\theta_L = (\tilde{\theta} - N\underline{\theta})/(c - N\underline{\theta})$. We will revisit this interpretation in Section 3.4, when we consider the large N limit.

3 Known lower bound on the social value

3.1 Overview of the analysis

For this section, we adopt the baseline assumption on the information structure, that the social value is at least θ_L. We will construct ϵ-strong maxmin solutions, for ϵ arbitrarily small, of the form described in the introduction. In order to make the treatment accessible, our discussion will be mostly informal, with rigorous proofs relegated to the Appendix.

3.2 Guarantee-maximizing mechanisms

We will begin with a heuristic derivation of proportional cost-sharing as a guarantee-maximizing mechanism, based on a logic analogous to that in Brooks and Du (2023). We do so by first deriving a lower bound on welfare across all information structures and equilibria, and then constructing a mechanism which maximizes this lower bound.

3.2.1 Constructing the lower bound on the welfare guarantee

In our construction, we consider a particular kind of mechanism in which actions are linearly ordered and observe that, in any equilibrium and under any information structure, local deviations must not be attractive. We then bound the welfare (across all information structures and equilibria) from below through the welfare from this mechanism, decreased by a term related to its local incentive constraints.

In particular, consider a mechanism for which the action space is $A_i = \mathbb{R}_+$, and the partial derivatives of $e_i(a)$ exist and are bounded. We refer to such a mechanism as *smooth.*

Now, one deviation available to agent i, regardless of the information structure and equilibrium, would be to increase her action by $\epsilon > 0$, i.e., whenever she would have played a_i, play $a_i + \epsilon$ instead. Clearly, such deviations must not be attractive in equilibrium, and hence

$$\int_{s,a} \left[(v_i(s)E(a_i+\epsilon, a_{-i}) - e_i(a_i+\epsilon, a_{-i})) - (v_i(s)E(a) - e_i(a))\right] b(da|s)\rho(ds) \leqslant 0. \tag{1}$$

Dividing by ϵ, and taking the lim sup as $\epsilon \to 0$, we get[1]

$$\int_{s,a} \frac{\partial}{\partial a_i} \left(v_i(s)E(a) - e_i(a)\right) b(da|s)\rho(ds) \leqslant 0. \tag{2}$$

Thus, equilibrium welfare in the mechanism must be at least

$$\int_{s,a} \left[(\Sigma v(s) - 1)E(a) + \sum_i \frac{\partial}{\partial a_i} \left(v_i(s)E(a) - e_i(a)\right)\right] b(da|s)\rho(ds).$$

In fact, we can obtain an even more generous bound, and one that holds for all information structures and equilibria, if we simply take the minimum of the integrand across all values and action profiles. To define this more compactly, let

$$\lambda(\theta, a) \equiv (\Sigma\theta - 1)E(a) + \sum_i \frac{\partial}{\partial a_i} \left(E(a)\theta_i - e_i(a)\right). \tag{3}$$

and $\Theta \equiv \{\theta \in \mathbb{R}_+^N | \Sigma\theta \geqslant \theta_L\}$. The object $\lambda(\theta, a)$ is the *strategic virtual objective*, as described in Brooks and Du (2023). Note that since Θ is convex and $v(s)$ is a conditional expectation of θ, $v(s)$ must be an element of Θ for every s. We therefore have the following result:

[1] Note that (1) divided by ϵ converges to (2) as $\epsilon \to 0$ by the dominated convergence theorem (we have assumed in the definition of a smooth mechanism that E and e_i have bounded derivatives).

Proposition 1. *Suppose the mechanism $M = (\mathbb{R}^N_+, e)$ is smooth. Then for any information structure $I = (S, \rho, v)$ and equilibrium b of (M, I), we have*

$$W(M, I, b) \geqslant \int_s \inf_a \lambda(v(s), a)\rho(ds) \geqslant \inf_{\theta, a} \lambda(\theta, a).$$

This result suggests that one way to engineer a mechanism with a favorable welfare guarantee is to construct $e(a)$ so as to maximize the lowest strategic virtual objective, $\inf_{\theta,a} \lambda(\theta, a)$. In fact, this is precisely what is achieved by the proportional cost-sharing mechanism.

3.2.2 Maximizing the lowest strategic virtual objective

Our argument proceeds as follows: first, we intuit that at the lowest strategic virtual objective, the public good is least valuable, i.e., $\Sigma\theta = \theta_L$; and as long as $\Sigma\theta = \theta_L$, the lowest strategic virtual objective is independent of the individual value θ_i. We then push this intuition one step further and conjecture that the strategic virtual objective does not in fact depend on the individual values θ_i and actions a_i but only on their aggregates $\Sigma\theta$ and Σa.

We therefore fix the total value of the public good at $\Sigma\theta = \theta_L$ and look for $e(a)$ for which the strategic virtual objective $\lambda(\theta_L, a)$[2] does not depend on individual a_i and θ_i. Notice:

$$\lambda(\theta_L, a) = (\theta_L - 1)E(a) + \sum_i \frac{\partial}{\partial a_i}\theta_i E(a) - \sum_i \frac{\partial}{\partial a_i} e_i(a).$$

With respect to the first term in $\lambda(\theta_L, a)$, the individual θ_i obviously do not make a difference. Individual θ_i will generally matter for the second term, that is, unless $\partial E(a)/\partial a_i$ is independent of i. This is the case precisely when the total expenditure only depends on the aggregate action, i.e., $E(a) \equiv \widehat{E}(\Sigma a)$ for some function $\widehat{E}$. Moreover, if total expenditure has this form, then the first two terms of $\lambda(\theta_L, a)$ do not depend on individual values of θ_i and a_i, and only depend on the aggregate action Σa. We therefore restrict attention to such forms for $E(a)$.

Without further functional form restrictions, the remaining term in $\lambda(\theta_L, a)$, to wit the divergence of the expenditure shares $e(a)$, will depend on the whole action profile. But we can derive a functional form such that this divergence also only depends on the aggregate action. To fix ideas, consider the case of $N = 2$, and look at a level curve of action profiles where $a_1 + a_2 = x$. Then, using the fact that $e_2(a) = \widehat{E}(\Sigma a) - e_1(a)$, the divergence is

$$\begin{aligned}
\frac{\partial e_1(a_1, x - a_1)}{\partial a_1} + \frac{\partial e_2(a_1, x - a_1)}{\partial a_2} &= \frac{\partial e_1(a)}{\partial a_1} - \frac{\partial e_1(a)}{\partial a_2} \\
&= \frac{d}{da_1} e_1(a_1, x - a_1).
\end{aligned}$$

Thus, the divergence is constant in a_1 along the level curve if and only if $e_1(a_1, x - a_1)$ is linear in a_1. At the same time, participation security forces $e_1(0, x) = 0$ and $e_1(x, 0) = \widehat{E}(x)$, so it must be that $e_1(a_1, x - a_1) = \widehat{E}(x)(a_1/x)$, i.e., the expenditure shares are *proportional*.

[2]We abuse notation slightly by using the same notation for the strategic virtual objective as a function of the social value.

More generally, a *proportional cost-sharing mechanism* is a smooth mechanism of the form

$$e_i(a) = \begin{cases} \frac{a_i}{\Sigma a}\widehat{E}(\Sigma a) & \text{if } \Sigma a > 0; \\ 0 & \text{if } \Sigma a = 0, \end{cases} \tag{4}$$

where $\widehat{E}(\Sigma a)$ is associated total expenditure function, and is assumed to be differentiable. The resulting strategic virtual objective is

$$(\theta_L - 1)(\widehat{E}(\Sigma a) + \widehat{E}'(\Sigma a)) - \frac{(N-1)\widehat{E}(\Sigma a)}{\Sigma a}.$$

In other words, the strategic virtual objective only depends on the aggregate value $\Sigma\theta = \theta_L$ and the aggregate action Σa; it does not depend on how those objects are distributed across agents.

We may then ask: What is a favorable choice for the total expenditure $\widehat{E}$? The natural guess is to set $\widehat{E}$ so that the strategic virtual objective is independent of Σa. This is equivalent to solving the differential equation

$$(\theta_L - 1)(\widehat{E}(x) + \widehat{E}'(x)) - \frac{(N-1)\widehat{E}(x)}{x} = \widehat{\lambda}, \tag{5}$$

for some constant $\widehat{\lambda}$. Of course, for the mechanism to be participation secure, it has to be that $\widehat{E}(0) = 0$, and the solution of the differential equation (5) subject to the boundary condition is

$$\widehat{E}(x) = \frac{\widehat{\lambda}}{\theta_L - 1} \int_{y=0}^{x} \exp(y - x) \left(\frac{y}{x}\right)^{-(N-1)/(\theta_L - 1)} dy. \tag{6}$$

The integral converges as long as $\theta_L > N$. Otherwise, the integrand blows up too quickly as $y \to 0$. We will shortly revisit what happens in this case, but for now, we maintain the assumption that $\theta_L > N$.

The question then remains, what is the right choice of $\widehat{\lambda}$? One possibility is that the function given by (6) never hits one, in which case we should just make $\widehat{\lambda}$ as large as possible. However, for $\widehat{\lambda}$ sufficiently large, the function $\widehat{E}$ will hit one at a finite value $\widehat{x}(\widehat{\lambda})$, and for $x > \widehat{x}(\widehat{\lambda})$, feasibility forces us to cap the total expenditure at $\widehat{E}(x) = 1$. As such, for $x > \widehat{x}(\widehat{\lambda})$, the strategic virtual objective changes to

$$\theta_L - 1 - \frac{N-1}{x},$$

which is obviously minimized at $x = \widehat{x}(\widehat{\lambda})$. Thus, the lowest strategic virtual objective for this mechanism will be

$$\min\left\{\widehat{\lambda}, \theta_L - 1 - \frac{N-1}{\widehat{x}(\widehat{\lambda})}\right\}.$$

Clearly, to maximize our lower bound on the guarantee, we should choose $\widehat{\lambda}$ to maximize this minimum.

Note that as $\widehat{\lambda}$ increases, the function (6) is scaled up, so that $\widehat{x}(\widehat{\lambda})$ decreases, and therefore the strategic virtual objective at the threshold decreases as well. This suggests that the optimal $\widehat{\lambda}$ solves

$$\widehat{\lambda} = \theta_L - 1 - \frac{N-1}{\widehat{x}(\widehat{\lambda})}. \tag{7}$$

This indeed turns out to be the case. We formally prove the existence of a solution to this equation in Proposition 5 in Appendix A.

We denote the solution to (7) by $\overline{\lambda}$, and $\overline{x} = \widehat{x}(\overline{\lambda})$. The associated optimal proportional cost-sharing rule is denoted by $\overline{e}$, where the total expenditure function $\overline{E}$ is given by (6) with $\widehat{\lambda} = \overline{\lambda}$ for $x < \overline{x}$, and is equal to 1 for $x \geqslant \overline{x}$. It is easy to check that $\overline{E}(x)$ is an increasing and concave function of x. Finally, we denote the entire mechanism just constructed by $\overline{M} = (\mathbb{R}_+^N, \overline{e})$.

For future reference, we observe that because $\overline{E}(\overline{x}) = 1$, we must have

$$\overline{\lambda} = (\theta_L - 1)\frac{\exp(\overline{x})\overline{x}^{-(N-1)/(\theta_L-1)}}{\int_{y=0}^{\overline{x}} \exp(y) y^{-(N-1)/(\theta_L-1)} dy}. \tag{8}$$

This formula will be useful in verifying that we have constructed approximately potential-minimizing information structures, which we turn to next.

3.3 Potential-minimizing information structures

We now provide the corresponding heuristic derivation of the information structure that minimizes potential across all mechanisms and equilibria. We first consider a particular kind of information structure and construct an upper bound on its welfare potential across all mechanisms and equilibria. We then construct an information structure which minimizes this upper bound in expectation. In the case where $\theta_L > N$, this information structure has a potential equal to the guarantee of the mechanism we constructed previously.

3.3.1 Constructing the upper bound on welfare potential

In our construction, we consider certain "smooth" information structures, in which signals are linearly ordered, and observe that in any equilibrium featuring them, local deviations must not be attractive. We use this fact to bound the welfare (accross all mechanisms and equilibria) from above by the welfare generated under these information structures, increased by a term related to local incentive constraints.

We say that an information structure is *smooth* if $S_i = [\underline{s}, \infty)$ [3] and the distribution of signals admits a density $\rho(ds) = \rho(s)ds$.[4] Now, for such an information structure, one deviation available to agent i, regardless of the mechanism and equilibrium, would be to

[3] As it will be clear later, it is convenient to work with an $\underline{s}$ that is not necessarily zero.

[4] We abuse notation slightly by using ρ for both the density and for the measure itself.

mimic an agent with a signal that is lower by $\epsilon > 0$, i.e., whenever their signal is $s_i > \underline{s} + \epsilon$, report $s_i - \epsilon$ instead. Clearly, such deviations must not be attractive in equilibrium, and hence

$$
\begin{aligned}
0 \geqslant \int_{s_i \geqslant \underline{s}+\epsilon, s_{-i}, a} & \left(v_i(s)E(a) - e_i(a)\right) b(da|s_i - \epsilon, s_{-i})\rho(s)ds \\
& - \int_{s_i \geqslant \underline{s}+\epsilon, s_{-i}, a} \left(v_i(s)E(a) - e_i(a)\right) b(da|s)\rho(s)ds.
\end{aligned}
$$

A change of variable in the first term gives

$$
\begin{aligned}
0 \geqslant \int_{s,a} & \left(v_i(s_i+\epsilon, s_{-i})E(a) - e_i(a)\right) b(da|s)\rho(s_i+\epsilon, s_{-i})ds \\
& - \int_{s_i \geqslant \underline{s}+\epsilon, s_{-i}, a} \left(v_i(s)E(a) - e_i(a)\right) b(da|s)\rho(s)ds.
\end{aligned}
$$

Moreover, since the mechanism is participation secure, we know that

$$
\int_{0 \leqslant s_i < \underline{s}+\epsilon, s_{-i}, a} \left(v_i(s)E(a) - e_i(a)\right) b(da|s)\rho(s)ds \geqslant 0. \tag{9}
$$

Subtracting this inequality from the preceding inequality yields

$$
0 \geqslant \int_{s,a} \left[\left(v_i(s_i+\epsilon, s_{-i})E(a) - e_i(a)\right)\rho(s_i+\epsilon, s_{-i}) - \left(v_i(s)E(a) - e_i(a)\right)\rho(s)\right] b(da|s)ds.
$$

Thus, by subtracting this negative term from the expression for expected utility for agent i and summing over agents, we obtain the following upper bound on welfare (for any equilibrium and mechanism):[5]

$$
\begin{aligned}
\int_{s,a} \Big\{ & (\Sigma v(s) - 1)E(a)\rho(s) \\
& - \sum_i \frac{\left(v_i(s_i+\epsilon, s_{-i})E(a) - e_i(a)\right)\rho(s_i+\epsilon, s_{-i}) - \left(v_i(s)E(a) - e_i(a)\right)\rho(s)}{\epsilon} \Big\} b(da|s)ds.
\end{aligned}
$$

In fact, if for each s we take the maximum across all outcomes e, rather than taking the expectation across equilibrium actions and outcomes, then we can obtain an even more generous bound that holds for all mechanisms. To express this new bound more compactly, we define

$$
\gamma(s, e; \epsilon) \equiv \left(\Sigma v(s) - 1\right)\left(\Sigma e\right)\rho(s) - \sum_i \frac{\left(v_i(s_i+\epsilon, s_{-i})\Sigma e - e_i\right)\rho(s_i+\epsilon, s_{-i}) - \left(v_i(s)\Sigma e - e_i\right)\rho(s)}{\epsilon},
$$

where e is an outcome in Ω. The object $\gamma(s, e; \epsilon)$ corresponds to the *informational virtual objective* as described in Brooks and Du (2023). We have then shown the following:

[5] This calculation is very much analogous to the integration by parts step in deriving virtual values and bounding revenue in the analysis of optimal auctions (Myerson, 1981). By working with the discrete downward deviation, we have sidestepped various technical complexities associated with applying the envelop theorem, given the lack of assumptions on v and ρ.

Proposition 2. *Suppose the information structure $I = ([\underline{s}, \infty)^N, \rho, v)$ is smooth. Then for any mechanism M and equilibrium b of (M, I), we have*

$$W(M, I, b) \leqslant \int_{s,a} \gamma(s, e(a); \epsilon) b(da|s) ds \leqslant \int_s \max_{e \in \Omega} \gamma(s, e; \epsilon) ds.$$

Note that if v and ρ are both differentiable at s, then

$$\lim_{\epsilon \to 0} \gamma(s, e; \epsilon) = (\Sigma v(s) - 1)(\Sigma e) \rho(s) - \sum_i \frac{\partial}{\partial s_i} [(v_i(s) \Sigma e - e_i) \rho(s)] \equiv \gamma(s, e).$$

One might have thought that we would strengthen our definition of a smooth information structure so that v and ρ are differentiable everywhere, and use this limit form of the informational virtual objective in our upper bound. It turns out, however, that the potential-minimizing information structures have discontinuities, and understanding the role played by those discontinuities is essential for deriving the optimal form of v and ρ.

3.3.2 Minimizing the expected highest informational virtual objective

We will now engineer an information structure that minimizes the the expected highest $\gamma(s, e; \epsilon)$, in the limit as $\epsilon \to 0$. We first argue that for this structure $\partial \rho(s)/\partial s_i$ has to be the same for all i, for every s. Otherwise, for some s, $\max_e \gamma(s, e)$ would be infinite—the designer could choose an outcome e with $e_i = -K$ and $e_j = K$ for i, j such that $\partial \rho(s)/\partial s_i < \partial \rho(s)/\partial s_j$, and $e_k = 0$ for $k \notin \{i, j\}$; for this choice of e we have $\gamma(s, e) = \left(\frac{\partial \rho(s)}{\partial s_j} - \frac{\partial \rho(s)}{\partial s_i} \right) K$ which tends to infinity as $K \to \infty$. Thus, in the structure of interest, $\partial \rho(s)/\partial s_i$ is independent of i, that is, the density is just a function of the aggregate signal: $\rho(s) = \widehat{\rho}(\Sigma s)$. For such information structures, we have

$$\gamma(s, e) = (\Sigma v(s) - 1)(\Sigma e) \widehat{\rho}(\Sigma s) - (\Sigma e) \widehat{\rho}(\Sigma s) \sum_i \frac{\partial v_i(s)}{\partial s_i} - (\Sigma v(s) - 1)(\Sigma e) \frac{\partial \widehat{\rho}(\Sigma s)}{\partial \Sigma s}$$

We can further intuit that for the upper-bound-minimizing structure, $\Sigma v(s)$ will equal to θ_L—the lower bound on social value.

Now, analogously to the construction of the lower-bound-maximizing mechanism, we would like to select an information structure for which $\gamma(s, e)$ only depends on aggregates Σe and Σs. Notice $\gamma(s, e)$ already does not depend on individual e_i. In addition, the density $\widehat{\rho}$ only depends on the aggregate signal. Then the only remaining term that depends on individual values is the divergence $\sum_i \partial v_i(s)/\partial s_i$. We therefore want to select a functional form for $v(s)$ so that the divergence only depends on Σs. By the argument previously made for the expenditure rule, this will be the case only if $v(s)$ has a *proportional form*, where each agent's share of the social value is proportional to their signal.

Thus, we are led to consider information structures of the form

$$\overline{v}_i(s) \equiv \begin{cases} \frac{s_i}{\Sigma s} \theta_L & \text{if } \Sigma s > 0; \\ \frac{1}{N} \theta_L & \text{if } \Sigma s = 0, \end{cases} \tag{10}$$

with a density of the form $\rho(s) = \widehat{\rho}(\Sigma s)$, where $\widehat{\rho}(x)$ is positive on an interval $[\underline{x}, \widehat{x}]$ for some $\widehat{x}$ and with $\underline{x} = N\underline{s}$. On the low region, we choose $\widehat{\rho}$ to be differentiable so that the limit informational virtual objective $\gamma(s, e)$ reduces to

$$\gamma(s, e) = \left[(\theta_L - 1)(\widehat{\rho}(\Sigma s) - \widehat{\rho}'(\Sigma s)) - \frac{(N-1)\theta_L \widehat{\rho}(\Sigma s)}{\Sigma s}\right] \Sigma e. \tag{11}$$

Thus, γ only depends on Σe and not on the distribution of expenditure across agents.

Indeed, we can go a step further and choose $\widehat{\rho}$ so that the informational virtual objective is exactly zero, regardless of the total expenditure. This is equivalent to solving the first-order ordinary differential equation equating the term in square brackets to zero. The precise solution is

$$\widehat{\rho}(x) \equiv \begin{cases} \frac{\exp(x) x^{-(N-1)\theta_L/(\theta_L - 1)}}{\int_{y=\underline{x}}^{\widehat{x}} \exp(y) y^{-(N-1)\theta_L/(\theta_L-1)} \frac{(y-\underline{x})^{N-1}}{(N-1)!} dy} & x \in [\underline{x}, \widehat{x}], \\ 0 & x > \widehat{x}, \end{cases} \tag{12}$$

where each agent's signal space is $[\underline{x}/N, \infty)$. The constant of integration is pinned down by the requirement that $\widehat{\rho}$ integrate to 1 on the simplex $\{s \in \mathbb{R}^N_+ | \Sigma s \leqslant \widehat{x}\}$.

The information structure constructed above ensures that the informational virtual objective is zero when $\Sigma s < \widehat{x}$. But the density is non-zero at $x = \widehat{x}$, and then discontinuously drops to zero. In fact, for ϵ small, the informational virtual objective $\gamma(s, e; \epsilon)$ at the boundary with full expenditure ($\Sigma s = \widehat{x}$ and $\Sigma e = 1$) is approximately $(\theta_L - 1)\widehat{\rho}(\widehat{x})/\epsilon$, whereas the Lebesgue measure of the boundary is approximately $\epsilon(\widehat{x} - \underline{x})^{N-1}/(N-1)!$. Indeed, in the limit as $\epsilon \to 0$, it is only this boundary that contributes to the expected highest informational virtual objective, which converges to

$$(\theta_L - 1)\widehat{\rho}(\widehat{x}) \frac{(\widehat{x} - \underline{x})^{N-1}}{(N-1)!} = (\theta_L - 1) \frac{\exp(\widehat{x}) \widehat{x}^{-(N-1)\theta_L/(\theta_L-1)} (\widehat{x} - \underline{x})^{N-1}}{\int_{y=\underline{x}}^{\widehat{x}} \exp(y) y^{-(N-1)\theta_L/(\theta_L-1)} (y - \underline{x})^{N-1} dy}. \tag{13}$$

Finally, if we choose $\widehat{x} = \overline{x}$ and send $\underline{x} \to 0$, (where $\overline{x}$ is the threshold from the mechanism $\overline{M}$) then the limiting upper bound on the potential is precisely equal to $\overline{\lambda}$ given by (8).

3.3.3 The ϵ-strong maxmin solution

Thus, we conclude that when $\theta_L > N$, there is a sequence of information structures of the form $\overline{I} = ([\underline{x}/N, \infty)^N, \overline{v}, \overline{\rho})$, where $\overline{\rho}$ is given by (12) with $\widehat{x} = \overline{x}$, whose potentials converge to the guarantee of $\overline{M}$ as $\underline{x} \to 0$. We can appeal to the theorem of Milgrom and Weber (1985) to obtain existence of equilibrium in $(\overline{I}, \overline{M})$. This completes the construction of an ϵ-strong maxmin solution for any $\epsilon > 0$.

3.4 $\theta_L \leqslant N$

We have so far covered the case in which $\theta_L > N$, meaning that the minimum per-capita social value of the good is greater than its cost. That condition was used above in order to ensure that the integrals in the definitions of the total expenditure function and the signal density converged. What about when $\theta_L \leqslant N$? In this case, the upper bound on the potential in (13) is still valid as long as $\underline{x} > 0$. But for any fixed $\widehat{x}$, in the limit as $\underline{x} \to 0$, the integral in the denominator blows up. As a result, the upper bound on the potential goes to zero. Thus, for any $\epsilon > 0$, there exists a $\underline{x}$ such that $\overline{I} = ([\underline{x}/N, \infty)^N, \overline{v}, \overline{\rho})$, where $\overline{\rho}$ is given by (12) for a fixed $\widehat{x}$, has a welfare potential of at most ϵ. It forms an ϵ-strong maxmin solution with a mechanism $\overline{M}$ that always allocates zero expenditure (and hence has a welfare guarantee of zero).

What is it that makes these information structures so challenging for the designer? All of the information structures that we constructed, (i.e., of the proportional form and with densities $\widehat{\rho}$ that satisfy (12)), are engineered so that the designer cannot generate positive surplus from types below the boundary $\Sigma s = \widehat{x}$.[6] This is because any surplus generated by producing the public good when types are below the boundary is offset by inefficient distortions in production that are needed elsewhere to deter agents from misreporting. In contrast, for the types on the boundary, there is no cost of incentives, and the informational virtual objective is just the social surplus generated from production. The only question then is how large is the mass of types on the boundary. When $\theta_L > N$, there is a lower bound to this mass on the boundary, because the mass of types that can be packed in close to zero in $\widehat{\rho}$ is bounded. Hence, under such information structures the surplus generated from the boundary is strictly positive. But when $\theta_L \leqslant N$, we can pack an arbitrarily large mass of types below the boundary and close to zero, so that the mass of types that can be used to generate positive surplus is arbitrarily small.

The condition $\theta_L > N$ seems like it should become more demanding as N grows larger, since we would not expect the social value of most public goods to scale linearly with N. But we should be careful with this interpretation. Recall the more general parametrization introduced at the end of Section 2, where there is a lower bound $\underline{\theta}$ on individual values, as well as a lower bound $\tilde{\theta}$ on the social value and an explicit cost c. Then the condition for public good provision to break down is that

$$N \geqslant \frac{\tilde{\theta} - N\underline{\theta}}{c - N\underline{\theta}}.$$

We might want to think of $\tilde{\theta}$, $\underline{\theta}$, and c as all varying with N, and depending on how these parameters vary, we may or may not get non-trivial guarantees when N is large. The economy collapses only if N grows large, but the minimum value for the good also shrinks to zero faster than N.

At any rate, when the minimum per capita social value is less than the cost of production, then there are extreme information structures where no equilibrium of any mechanism can guarantee a non-negligible amount of surplus. Note that this conclusion only relies on interim participation, which implies the inequality (9) that we used to prove Proposition 2. In particular, it does not rely on participation security or any assumptions about equilibrium selection.

The existence of such information structures sheds new light on the inherent difficulty in overcoming the free rider problem in settings with interdependent values. Indeed, our conclusion is in many ways much stronger than those identified in the prior literature. As we

[6] This feature is reminiscent of the information structure in Roesler and Szentes (2017) for revenue maximization with a single buyer.

discuss below, the special case of our model with two agents can be reinterpreted as a model of bilateral trade, where production of the public good means that the agents trade, and the minimum social value of the good $\theta_L - 1$ is a lower bound on the gains from trade. There is large literature that investigates inefficiencies that can arise in the bilateral trade setting due to private information. For example, Myerson and Satterthwaite (1983) give mild conditions under which there is necessarily *some* inefficiency in bilateral trade, in a setting with private values and independent types. Akerlof (1970) gives examples of an interdependent values "lemons" market where there is no posted price mechanism that induces positive trade (see also Carroll, 2016). In contrast, for the information structures we construct, there is an arbitrarily small amount of trade in *any* equilibrium under *any* mechanism. This is true in spite of the fact that there is common knowledge that the gains from trade are bounded away from zero. The closest result to ours seems to be Mailath and Postlewaite (1990), who study public goods provision with private values and many agents, and show that the welfare potential converges to zero as the number of agents grows large. Our result, in comparison, is valid for any finite number of agents, as long as the social value per capita is less than the cost of the public good.

3.5 Main result

We now summarize our characterization of the baseline model with a formal result:

Theorem 1. *If $\theta_L > N$, then there exist $\overline{\lambda} > 0$ and $\overline{x} > 0$ and associated proportional cost-sharing mechanism $\overline{M}$ with the following property: For any $\epsilon > 0$ there exists a $\underline{x} > 0$ and associated information structure $\overline{I}$ and strategies b such that $(\overline{M}, \overline{I}, b)$ is a ϵ-strong maxmin solution. As $\epsilon \to 0$, the associated guarantee and potential both converge to $\overline{\lambda}$.*

If $\theta_L \leqslant N$, let $\overline{M}$ be the zero-expenditure mechanism. Then for any $\epsilon > 0$ there exists a $\underline{x} > 0$ such that $(\overline{M}, \overline{I}, b)$ is a ϵ-strong maxmin solution for some b. As $\epsilon \to 0$, the associated guarantee and potential both converge to 0.

When $\theta_L > N$, the information structure $\overline{I}$ in the ϵ-strong maxmin solution obviously converges as $\epsilon \to 0$ to an information structure defined by (12) with $\underline{x} = 0$. We conjecture that this limit information structure forms a strong maxmin solution with $\overline{M}$. There is, however, a technical difficulty in establishing that the limit information structure has a potential equal to $\overline{\lambda}$. The reason is that $\overline{\rho}'(x)$ is not integrable against $x^{N-1}dx$ around $x = 0$ (even though $\overline{\rho}(x)$ itself is), which seems to preclude the application of the dominated convergence theorem to pass from the integral of $\gamma(s, e; \epsilon)$ for $\epsilon > 0$ to the integral of $\gamma(s, e)$.

Nonetheless, the limit information structure with $\underline{x} = 0$ sheds light on equilibrium behavior in the limit as $\epsilon \to 0$:

Proposition 3. *Suppose $\theta_L > N$. Let $\overline{I}$ be the information structure from equation* (12) *with $\underline{x} = 0$ and $\widehat{x} = \overline{x}$. Then the truth-telling strategy profile $\overline{b}$ is an equilibrium in $(\overline{M}, \overline{I})$.*

Thus, the truth-telling strategies mediate between the guarantee-maximizing mechanism and the potential-minimizing information structure. We identified a similar structure in our prior analysis of common value auctions (Brooks and Du, 2021) and termed this result the "double revelation principle," that is, the guarantee-maximizing mechanism is a designer-welfare maximizing, incentive-compatible direct mechanism on the potential-minimizing information structure, while the potential-minimizing information is a designer-welfare minimizing Bayes correlated equilibrium on the guarantee-maximizing mechanism. The proof of Proposition 3 involves a fairly long and detailed calculation of interim utilities. The fact that truthful strategies are an equilibrium is both remarkable and somewhat mysterious. We comment further on this issue in Section 5 below.

3.6 Welfare guarantee

In Figure 1 we plot $\overline{\lambda}$ as a fraction of the surplus $(\theta_L - 1)$ when $N = 2$. As θ_L becomes larger than N, we see that the guaranteed fraction of the full surplus quickly approaches one. On the other hand, when $\theta_L \leqslant N$ we have $\overline{\lambda} = 0$.

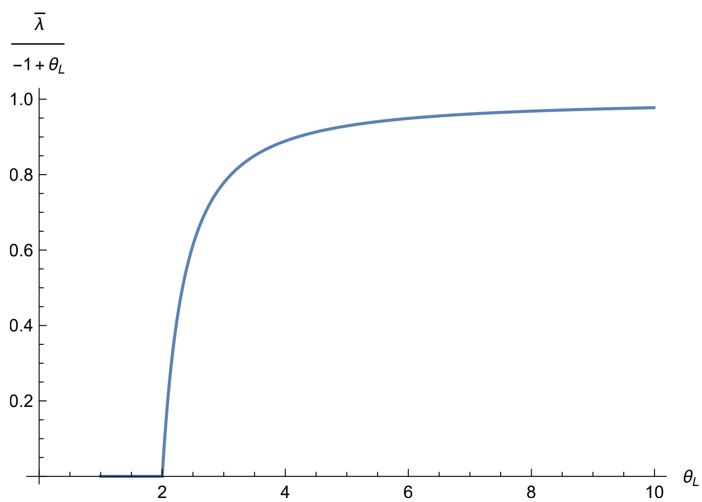

Figure 1: A plot of the optimal welfare guarantee as a fraction of the full surplus when $N = 2$.

Surprisingly, the mechanism $\overline{M}$ guarantees $\overline{\lambda}/(\theta_L - 1)$ fraction of the efficient surplus not just when the efficient surplus is equal to the lower bound $\theta_L - 1$, but also when the efficient surplus exceeds the lower bound. To see why, recall that the strategic virtual objective under the mechanism $\overline{M}$ is

$$\lambda(\theta, a) = (\Sigma\theta - 1)(\overline{E}(\Sigma a) + \overline{E}'(\Sigma a)) - \frac{(N-1)\overline{E}(\Sigma a)}{\Sigma a}.$$

Equivalently:[7]

$$\lambda(\theta, a) = \lambda(\theta_L, a) + (\Sigma\theta - \theta_L)(\overline{E}(\Sigma a) + \overline{E}'(\Sigma a)).$$

Recall $\lambda(\theta_L, a)$ is constant across all action profiles $\Sigma a \leqslant \overline{x}$. Now, $\overline{E}(\Sigma a) + \overline{E}'(\Sigma a)$ is decreasing in Σa (Lemma 5 in the Appendix). Hence, when $\Sigma\theta = \Sigma v(s) > \theta_L$, the second term is minimized by $\Sigma a = \overline{x}$. Therefore, the lowest strategic virtual objective is $(\Sigma v(s) -$

[7]Like before, we abuse notation slightly and denote $\lambda(\theta, a)$ when $\Sigma\theta > \theta_L$ by $\lambda(\theta_L, a)$.

$1)-(N-1)/\overline{x}$. By Proposition 1, the ratio between the welfare guarantee and the (expected) efficient surplus is

$$\frac{\int_s \left((\Sigma v(s)-1)-(N-1)/\overline{x}\right)\rho(s)ds}{\int_s \left(\Sigma v(s)-1\right)\rho(s)ds} = 1 - \frac{(N-1)/\overline{x}}{\int_s \left(\Sigma v(s)-1\right)\rho(s)ds}.$$

The above ratio strictly increases with the efficient surplus $\int_s \left(\Sigma v(s)-1\right)\rho(s)ds$ and is at least $\frac{\theta_L-1-(N-1)/\overline{x}}{\theta_L-1} = \frac{\overline{\lambda}}{\theta_L-1}$. As the efficient surplus grows large, this ratio tends to 1. We have therefore proven the following:

Proposition 4. *Let $\overline{M}$ be the guarantee-maximizing proportional cost-sharing mechanism when the lower bound on the social value is $\theta_L > N$. Then the fraction of the efficient welfare guaranteed by $\overline{M}$ is at least $\overline{\lambda}/(\theta_L-1)$. This fraction increases with the efficient surplus and converges to 1 as the efficient surplus tends to ∞.*

This proposition reinforces our main message: As long as the social value is sufficiently large, that society can obtain non-trivial welfare guarantees from proportional cost-sharing mechanisms, even if there is only relatively crude information about informational and fundamentals.

More generally, the proportional cost-sharing mechanism is robust to misspecification of the fundamentals in the following sense. Proposition 1 shows that welfare is bounded below by the expectation (across θ) of the minimum (across a) of $\lambda(\theta,a)$. Since λ is continuous in θ, this lower bound will be weak-* continuous in the distribution of θ. Thus, the lower bound on welfare varies smoothly with fundamentals.[8]

4 Lower bound on the expected social value

We next consider the richer model where we also have an upper bound θ_H on the social value and a lower bound $\widehat{\theta}$ on the expected social value. In other words, the possible information structures are those for which $\theta_L \leqslant \Sigma v(s) \leqslant \theta_H$ for all s and

$$\int_s \Sigma v(s)\rho(ds) \geqslant \widehat{\theta}.$$

The primary purpose of this exercise is to allow for positive probability that the social value is less than N. As we shall see, proportional cost-sharing mechanisms continue to provide unimprovable welfare guarantees.

When $\widehat{\theta} \leqslant N$, the potential-minimizing information structure from Theorem 1 is still feasible and still gives a welfare potential arbitrarily close to zero.

When $\widehat{\theta} > N$, however, we obtain non-trivial welfare guarantee. This is the case even though we may have $\theta_L \leqslant N$ (and even $\theta_L \leqslant 1$).

We will construct new strong maxmin solutions using the mechanisms and information structures from Theorem 1 as building blocks. To that extent, we introduce the mechanism

[8]Similar observations about robustness to misspecification of fundamentals appeared in Du (2018), Brooks and Du (2021), and Brooks and Du (2023).

$\overline{M}_{\tilde{\theta}}$ and the information structure $\overline{I}_{\tilde{\theta}}$. $\overline{M}_{\tilde{\theta}}$ is defined by equations (4) and (6), except with θ_L replaced with $\tilde{\theta}$. Similarly, $\overline{I}_{\tilde{\theta}}$ is defined by equations (10) and (12), except with θ_L replaced with $\tilde{\theta}$. Intuitively, these are the previously considered mechanism/information structure, but the agents are now certain that the social value is $\tilde{\theta}$ instead of θ_L (recall that in the solution from Theorem 1, the social value was equal to its lower bound with probability one).

4.1 $\theta_L \leqslant N$ and $\widehat{\theta} > N$

In this case, the guarantee-maximizing mechanism turns out to be $\overline{M}_{\theta_H}$. In other words, we use the proportional cost-sharing mechanism *as if* there is a common knowledge that the social value is θ_H. The corresponding potential-minimizing information structure is a public randomization between $\overline{I}_N$ and $\overline{I}_{\theta_H}$, where the probability of $\overline{I}_N$ is $\frac{\theta_H-\widehat{\theta}}{\theta_H-N}$. Denote this "public mixture" information structure as

$$\tilde{I} \equiv \frac{\theta_H - \widehat{\theta}}{\theta_H - N} \cdot \overline{I}_N + \frac{\widehat{\theta} - N}{\theta_H - N} \cdot \overline{I}_{\theta_H}$$

That is, under $\tilde{I}$, the agents receive a public signal telling them either the information structure is $\overline{I}_N$ or $\overline{I}_{\theta_H}$, and then they receive private signals according to either $\overline{I}_N$ or $\overline{I}_{\theta_H}$. In $\overline{I}_N$ (respectively, $\overline{I}_{\theta_H}$) there is a common knowledge that $\Sigma\theta = N$ (respectively, $\Sigma\theta = \theta_H$); the probabilities on $\overline{I}_N$ and $\overline{I}_{\theta_H}$ are such that the expected value of $\Sigma\theta$ is $\widehat{\theta}$.

4.2 $\theta_L \leqslant N$ and $\widehat{\theta} > N$

In this case, the guarantee-maximizing mechanism $\widehat{M} = (\mathbb{R}^N_+, \widehat{e})$ is a proportional cost-sharing mechanism with

$$\widehat{E}(x) = \begin{cases} \frac{\lambda_L}{\theta_L - 1} \int_{y=0}^{x} \exp(y-x) \left(\frac{y}{x}\right)^{-(N-1)/(\theta_L-1)} dy & x \in [0, x_L], \\ \widehat{E}(x_L) \exp(x_L - x) \left(\frac{x_L}{x}\right)^{-(N-1)/(\theta_H-1)} & \\ \quad + \frac{\lambda_H}{\theta_H-1} \int_{y=x_L}^{x} \exp(y-x) \left(\frac{y}{x}\right)^{-(N-1)/(\theta_H-1)} dy & x \in (x_L, x_H], \\ 1 & x > x_H. \end{cases} \tag{14}$$

Equation (14) pastes together the expenditure functions from $\overline{M}_{\theta_L}$ and $\overline{M}_{\theta_H}$: below x_L the mechanism is as if there was a commonly knowledge that the social value is θ_L, and above x_L as if a commonly knowledge of θ_H. The equation and the parameters $(\lambda_L, x_L, \lambda_H, x_H)$ ensure smooth pasting at x_L and x_H; that is, $\widehat{E}(x)$ and $\widehat{E}'(x)$ are continuous at $x = x_L$ and at $x = x_H$.

Likewise, the potential-minimizing information structure $\widehat{I} = ([\underline{x}/N, \infty)^N, \widehat{\rho}, \widehat{v})$ pastes together (though not smoothly) $\overline{I}_{\theta_L}$ and $\overline{I}_{\theta_H}$: the density $\widehat{\rho}(s)$ of the signal profile $s \in [\underline{x}/N, \infty)^N$ depends only on Σs and satisfies

$$\widehat{\rho}(x) = \begin{cases} \frac{\mu_L \exp(x) x^{-(N-1)\theta_L/(\theta_L-1)}}{\int_{y=\underline{x}}^{x_L} \exp(y) y^{-(N-1)\theta_L/(\theta_L-1)} (y-\underline{x})^{N-1}/(N-1)! dy} & x \in [\underline{x}, x_L), \\ \frac{\mu_H \exp(x) x^{-(N-1)\theta_H/(\theta_H-1)}}{\int_{y=x_L}^{x_H} \exp(y) y^{-(N-1)\theta_H/(\theta_H-1)} (y-\underline{x})^{N-1}/(N-1)! dy} & x \in [x_L, x_H), \\ 0 & x \geqslant x_H, \end{cases} \tag{15}$$

where

$$\mu_L = \frac{\widehat{\theta} - \theta_L}{\theta_H - \theta_L}, \quad \mu_H = \frac{\theta_H - \widehat{\theta}}{\theta_H - \theta_L}.$$

The interim value function is

$$\widehat{v}_i(s) = \begin{cases} \theta_L \frac{1}{N} & \text{if } \Sigma s = 0, \\ \theta_L \frac{s_i}{\Sigma s} & \text{if } \Sigma s \in (0, x_L), \\ \theta_H \frac{s_i}{\Sigma s} & \text{if } \Sigma s \in [x_L, x_H]. \end{cases} \tag{16}$$

Thus, whether Σs is below or above x_L reveals whether $\Sigma \theta$ is θ_L or θ_H, and conditional on either event $\widehat{I}$ is identical to $\overline{I}_{\theta_L}$ or $\overline{I}_{\theta_H}$.

4.3 Main result

Our result for this section is the following:

Theorem 2. *Suppose $\widehat{\theta} > N$ and $\theta_L \leqslant N$. For every $\epsilon > 0$, there exists a $\underline{x}$ in $\tilde{I}$ so that $(\overline{M}_{\theta_H}, \tilde{I}, b)$ is an ϵ-strong maxmin solution for some strategies b. As $\epsilon \to 0$, the welfare guarantee and potential of the solutions converge to $\frac{\widehat{\theta}-N}{\theta_H - N}\overline{\lambda}_{\theta_H} > 0$, where $\overline{\lambda}_{\theta_H}$ is the optimal welfare guarantee of $\overline{M}_{\theta_H}$ when the social value is commonly known to be θ_H.*

Suppose $\theta_L > N$. Then there exist parameters $(\lambda_L, x_L, \lambda_H, x_H)$ in $\widehat{M}$ with the following property: For every $\epsilon > 0$, there exists a $\underline{x}$ in $\widehat{I}$ such that $(\widehat{M}, \widehat{I}, b)$ is a ϵ-strong maxmin solution for some b. As $\epsilon \to 0$, the welfare guarantee and potential of the solutions converge to $\mu_L \lambda_L + \mu_H \lambda_H > 0$.

The proof of Theorem 2 largely follows that of Theorem 1. We first establish a lower bound on welfare in a smooth mechanism, generalizing Proposition 1. The proof of the first part of the theorem is quite straightforward. The second half takes more effort, and involves a fixed point argument proving that there exist parameters so that the lower bound on the guarantee of $\widehat{M}$ coincides with the upper bound on the potential for $\widehat{I}$. These parameters are engineered to satisfy the complementary slackness property described below in Section 5.

Although we have presented our assumptions about fundamentals in terms of bounds on the social value, the baseline model effectively reduces to one in which the social value is equal to its lower bound. Similarly, the model of this section reduces to one where the social value is either θ_L or θ_H with known probabilities. We suspect that the analysis can be generalized to a model where the entire distribution of the social value is known, and that proportional cost-sharing mechanisms continue to be optimal. However, the optimal total expenditure function will have to satisfy a more complicated differential equation, in which social values are assortatively matched with aggregate actions. We leave this interesting extension to future work.

5 Discussion

5.1 On infinite actions/signals and equilibrium existence

As we mentioned in the introduction, a subtle technical issue is how to handle the possibility of equilibrium non-existence when formulating the joint information/mechanism design problem. As far as we know, the guarantee of the proportional cost-sharing mechanism that we construct is strictly greater than that of any finite mechanism, and similarly, the potential of the information structures that we construct is lower than that of any finite information structure. But if we allow infinite mechanisms and infinite information structures, then there will be certain combinations for which no equilibria exist. This raises the possibility of other pathological maxmin solutions with different values, as we now explain.

Considering the baseline model of Section 3, if a mechanism has an equilibrium on the information structure $\overline{I}$, then its guarantee must exist and be finite and, moreover, the guarantee must be less than that of $\overline{M}$. Thus, $\overline{M}$ maximizes the welfare guarantee in the set of mechanisms that have an equilibrium on $\overline{I}$; (condition (i) of the strong maxmin solution says that $\overline{M}$ is an element of this set). Similarly, if an information structure has an equilibrium on $\overline{M}$, then its potential is finite and is at least that of $\overline{I}$. Since $\overline{M}$ and $\overline{I}$ are well-behaved, requiring that equilibria exist when paired with $\overline{I}$ and $\overline{M}$ seems to be a mild restriction on the sets of mechanisms and information structures, respectively. Indeed, for our main results, we do not explicitly construct an equilibrium, but rather obtain one indirectly via an application of the existence theorem of Milgrom and Weber (1985), using the fact that $\overline{M}$ and $\overline{I}$ are analytically well-behaved. Moreover, we have a unique value within the class of strong maxmin solutions.

Nonetheless, there remains a logical possibility of pathological mechanisms (e.g., an integer game where the agent who says the highest number receives a transfer from the other agents) that have no equilibria on any information structure, and therefore have infinite guarantee, or the somewhat less pathological but still disturbing possibility of mechanisms that do not have equilibria on $\overline{I}$, and therefore could have strictly higher guarantees than that of $\overline{M}$.

In Brooks and Du (2021), we addressed this issue in the context of revenue maximization in common value auctions. We introduced the notion of *finite approximability* of a strong maxmin solution, which is that there exist finite mechanisms and finite information structures whose guarantees and potentials, respectively, are arbitrarily close to the value of the solution. While we have not pursued that exercise in this paper, we have every reason to think that the solution $(\overline{M}, \overline{I}, \overline{b})$ is finitely approximable, using the same techniques as in Brooks and Du (2021). Moreover, as argued in that paper, all finitely approximable strong maxmin solutions must have the same value. This is a straightforward consequence of the existence of a Nash equilibrium when the mechanism and information structure are both finite.

In sum, while some care is needed when describing exact solutions with infinitely many actions and signals, we have every reason to think that the value of the strong maxmin solutions provided in this paper are the natural ones, and do not rely on a controversial use of equilibrium existence.

5.2 Complementary slackness and the double revelation principle

In Proposition 1, we argued that the expected lowest strategic virtual objective is a lower bound on the welfare guarantee of a smooth mechanism. The guarantee-maximizing mechanism maximizes this lower bound. Similarly, in Proposition 2, we argued that an upper bound on the welfare potential is the expected highest informational virtual objective. The potential-minimizing information minimizes this upper bound. These two bounding programs are infinite dimensional linear programming problems.

In Brooks and Du (2023), we studied discrete analogues of these programs, where the number of actions and signals is finite, and we take the limit as that number goes to infinity. In the discrete setting, the bounding programs are "almost" a dual pair, in the following sense: The derivatives in the strategic and informational virtual objective are discrete and local upward. The dual of the program of minimizing the expected highest informational virtual objective has the form of maximizing an expected lowest strategic virtual objective, but where the local derivatives point down instead of up. In the dual pairing, the likelihood of (s, θ) is the Lagrange multiplier on the constraint that the minimum strategic virtual objective is at most that obtained at the value profile θ and a particular action profile equal to s. Similarly, the likelihood of an outcome e given the action a is the Lagrange multiplier on the constraint that the maximum informational virtual objective be at least that attained at the outcome e and signal profile equal to a.

In the continuum limit, we might expect the difference between discrete upwards and discrete downwards derivatives to vanish, so that the two programs become an exact dual pair. We have yet to find a rigorous formulation of this duality directly in the continuum limit. Nonetheless, the strong maxmin solution we described exhibits the structure that one would expect of a saddle point for a linear programming problem. In particular, it satisfies a form of *complementary slackness*, as we now explain.

Under the limiting potential-minimizing information structure, a profile (s, θ) has positive likelihood only if the associated actions $a = s$ in the guarantee-maximizing mechanism minimize the strategic virtual objective at θ. In the baseline model with $\theta_L > N$, all signals with $\Sigma s \leqslant \overline{x}$ have positive likelihood under $\overline{I}$, and the minimizers of the strategic virtual objective in $\overline{M}$ are those for which $\Sigma a \leqslant \overline{x}$. Similarly, in the enriched model when $\theta_L > N$, under $\widehat{I}$, there is positive likelihood of (s, θ) with $\Sigma\theta = \theta_L$ only if $\Sigma s \in [0, x_L]$, and positive likelihood of with $\Sigma\theta = \theta_H$ only if $\Sigma s \in [x_L, x_H]$. This exactly accords with the minimizers of the strategic virtual objective in $\widehat{M}$: If $\Sigma\theta = \theta_L$, then the minimizers are action profiles with $\Sigma a \in [0, x_L]$, and if $\Sigma\theta = \theta_H$, the minimizers satisfy $\Sigma a \in [x_L, x_H]$.

Similarly, under the guarantee-maximizing mechanism, an outcome has positive likelihood under an action profile a only if it maximizes the associated informational virtual objective for the potential-minimizing information structure. Since in the baseline model the informational virtual objective is zero for all outcomes and s with $\Sigma s \in [0, \overline{x})$, we can have any interior expenditure and shares in $\overline{e}(a)$ for action profiles with $\Sigma a < \overline{x}$. For $\Sigma s = \overline{x}$, the informational virtual objective is maximal and positive at full expenditure, but it does not depend on the particular shares. This accords with the fact that $\overline{E}(\overline{x}) = 1$ and $\overline{e}(a)$ can be any shares of the full expenditure when $\Sigma a = \overline{x}$. The enriched model features exactly the same pattern when we replace $\overline{x}$ with x_H.

A similar form of complementary slackness manifested in our solution of the common value optimal auctions problem in Brooks and Du (2021). We suspect that the phenomenon is more general. In fact, we used the ansatz that complentary slackness would hold to engineer the strong maxmin solution for the public goods problem.

Finally, another striking feature of the strong maxmin solution for the baseline model is that the truthful/obedient strategies are an equilibrium of the Bayesian game consisting of the guarantee-maximizing mechanism and the limit of the potential-minimizing information structures. This "double revelation principle" also appeared in Brooks and Du (2021). While we have only proven this for the baseline model, we have every reason to think that it is also true in the enriched model. The detailed calculations needed to verify optimality of truthtelling in the baseline model are quite involved, and extending them to the richer solution described in Section 4 is a non-trivial task. Again, we suspect it is a general phenomenon of continuous strong maxmin solutions, and that it can be proved via a higher level argument that remains to be discovered.

5.3 Connection to bilateral trade

When there are two agents, our model can be reinterpreted as one of bilateral trade, as we now explain. There is a seller $i = 1$ who owns a single unit of a good that can be traded to a buyer $i = 2$. The seller's value for the good is v_1, which is between 0 and $v_H > 0$, and the buyer's value v_2 is known to be at least zero. Moreover, there is common knowledge that $v_2 - v_1 \geqslant g$, meaning that there is a lower bound g on the gains from trade. The outcome is simply a likelihood of trade $q \in [0, 1]$ and a price p at which the buyer and seller trade. We assume $g > 0$, so that the gains from trade are strictly positive, and the efficient outcome is for the agents to always trade. Note that if $g > v_H$, then it is common knowledge that the buyer's value is greater than the seller's, and we can implement efficient trade with, e.g., a posted price of $(v_H + g)/2$. So, to keep things non-trivial, we assume that $g < v_H$.

We now map this into the the baseline model of the public goods problem. No expenditure is equivalent to no trade, in which case both the seller's and the buyer's net payoffs are zero. Full expenditure with agent 1 paying all of the expense should be identified with an outcome in the bilateral trade setting where trade occurs with probability one and the buyer's lowest possible payoff is zero, meaning that trade occurs at a low price of $p = g$. In this case, the seller's payoff is $g - v_1$ and the buyer's payoff is $v_2 - g \geqslant v_1 + g - g = v_1 \geqslant 0$. On the other hand, full expenditure with agent 2 paying all of the expense should be identified with an outcome where trade occurs with probability one and the seller's lowest possible payoff is zero, meaning that trade occurs at a high price of $p = v_H$. The seller's net payoff is therefore $v_H - v_1 \geqslant v_H - v_H = 0$.

To find the right conversion of units, note that the highest payoff for the buyer and the seller is v_H (trade at the highest price and lowest value for the seller, trade at the lowest price and highest value for the buyer). Thus, we should identify $v_H = \alpha\theta_L$ for some scaling parameter $\alpha > 0$. Also, the gains from trade are $v_2 - v_1 \geqslant g$, which should be identified with the net social value from public expenditure, so that $g = \alpha(\theta_L - 1)$. Note that this is consistent with the aforementioned parametric restriction that $g < v_H$. We obtain:

$$\alpha = v_H - g;$$
$$\theta_L = \frac{v_H}{v_H - g}.$$

It is clear that the condition $\theta_L > 2$ is equivalent to $g > v_H/2$. In that case, we define the *proportional pricing mechanism* $\tilde{M}$ as follows: The seller and the buyer report non-negative real numbers a_1 and a_2, respectively; trade occurs with probability $\overline{E}(a_1 + a_2)$, where $\overline{E}$ is defined according to (6) with $\theta_L = v_H/(v_H - g)$; and trade occurs at a price equal to

$$\overline{p}(a_1, a_2) \equiv g\frac{a_1}{a_1 + a_2} + v_H\frac{a_2}{a_1 + a_2}.$$

In addition, we define the information structure $\tilde{I}$ in which the buyer and the seller receive non-negative signals whose density is given by $\overline{\rho}$ (again when $\theta_L = v_H/(v_H - g)$), with a lower bound of $\underline{x}$; conditional on the signal profile s, the seller's value is v_H with likelihood $s_1/(s_1 + s_2)$, and is zero otherwise; and the buyer's value is equal to the seller's value plus g.

As an immediate consequence of Theorem 1, we obtain the following result for the bilateral trade problem:

Theorem 3. *If $g > v_H/2$, then there exist $(\overline{\lambda}, \overline{x})$ with the following property: For every $\epsilon > 0$, there is an $\underline{x}$ and strategies b for which $(\tilde{M}, \tilde{I}, b)$ is a ϵ-strong maxmin solution. Moreover, as $\epsilon \to 0$, the guarantee and potential for gains from trade converge to $\alpha\overline{\lambda} > 0$.*

If $g \leqslant v_H/2$, then for every $\epsilon > 0$, there exists an information structure for which the probability of trade is at most ϵ in any mechanism and equilibrium.

Importantly, in the case $g \leqslant v_H/2$, trade is impossible even though it is common knowledge that the gains from trade are strictly positive. This is a dramatic strengthening of the result of Akerlof (1970) and Carroll (2016), who only give conditions under which the market may break down under posted prices. In fact, the condition $g \leqslant v_H/2$ is precisely the condition for which efficient trade is impossible with a posted price in the lemons information structure (where the buyer has no information, and the seller knows the value which is uniformly distributed on $[0, v_H]$). But in the information structures we construct, there is no trade at all, no matter which mechanism is used and which equilibrium is played.

Clearly, it is possible to further weaken the assumptions on fundamentals in the bilateral trade model along the lines of Section 4. A strong assumption that is harder to dispense with is the upper bound on the seller's value. It is our hope that the methodology developed in this paper and in Brooks and Du (2023) can be fruitfully applied to richer versions of the bilateral trade problem. We hope to pursue this topic in future work.

5.4 Alternative mechanisms

In this section we numerically compare the guarantee-maximizing proportional cost-sharing mechanism with some alternative mechanisms.

First, consider the *unilateral mechanism* where each agent is responsible for $1/N$ of the public good. The agents simultaneously choose a_i from $A_i = \{0, 1/m, 2/m, \ldots, 1\}$, where the integer $m \geqslant 1$ is a parameter, and $e_i(a) = a_i/N$.

Second, consider the *linear proportional cost-sharing mechanism*: The agents simultaneously choose from $A_i = \{0, 1/m, 2/m, \ldots, 1\}$, and

$$e_i(a) = \begin{cases} a_i & \Sigma a \leqslant 1, \\ \frac{a_i}{\Sigma a} & \Sigma a > 1. \end{cases}$$

Thus, the total expenditure function is $E(\Sigma a) = \Sigma a$ for $\Sigma a \leqslant 1$ and $E(\Sigma a) = 1$ for $\Sigma a > 1$.

Surprisingly, when $N = 2$, numerical simulations suggest that the welfare guarantees of these two mechanisms are independent of the parameter m in both the baseline and extended models. These welfare guarantees are plotted in Figures 2 and 3, along with that of the guarantee-maximizing proportional cost-sharing mechanisms. We see that the unilateral mechanism generally offers poor welfare guarantees; this is not surprising since in the unilateral mechanism, the expenditure contributed by an individual agent is at most 1/2 even if his value is equal to the social value. The welfare guarantees of the linear proportional cost-sharing mechanism are significantly better and come close to the optimal guarantees in the extended model. In the baseline model with a moderate θ_L, the guarantee-maximizing mechanism still significantly outperforms the linear proportional cost-sharing mechanism.

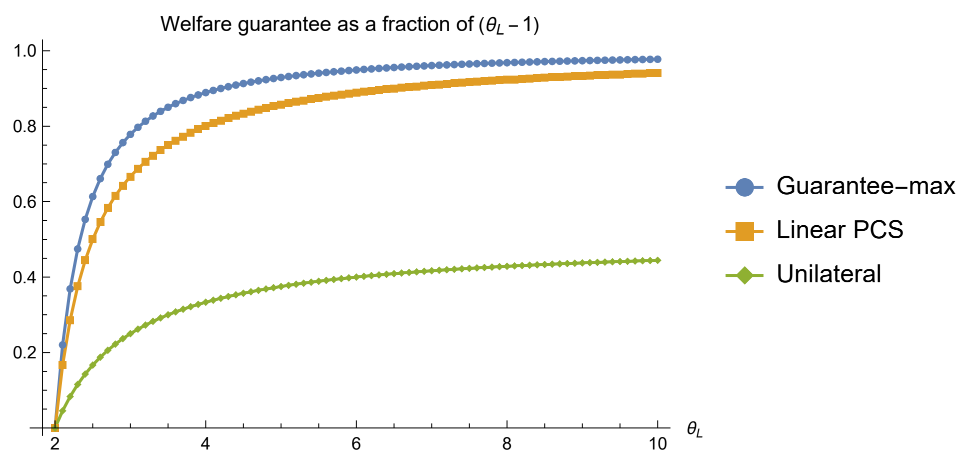

Figure 2: Welfare guarantees of the various mechanisms in the baseline model when $N = 2$.

A takeaway from these numerical simulations is that the linear proportional-cost sharing mechanism offers good welfare guarantees and is a good candidate for practical applications.

5.5 Comparison with proportional auctions

In Brooks and Du (2021), we studied revenue guarantee maximizing mechanisms in the context of a common value private goods allocation problem. In the proportional auction, agents' actions are non-negative real numbers, bidder i is allocated the good with probability

$$q_i(a) = \frac{a_i}{\Sigma a} Q(\Sigma a),$$

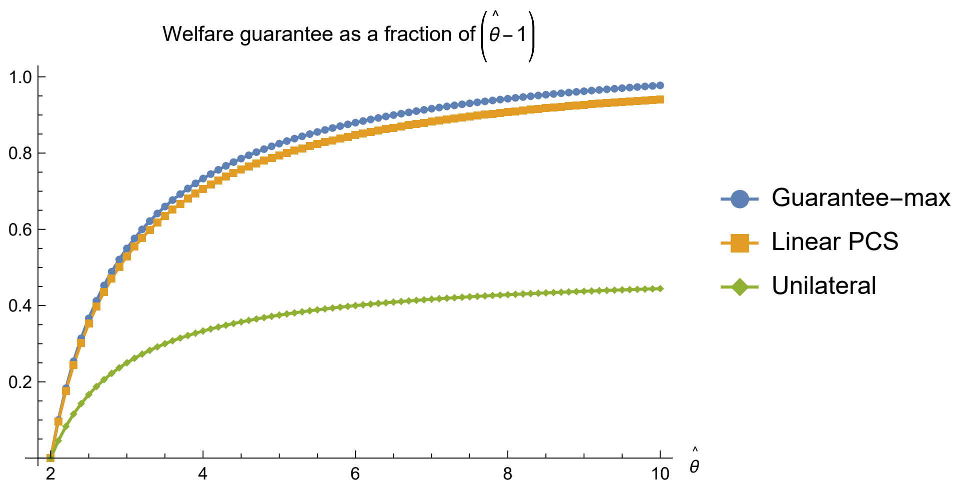

Figure 3: Welfare guarantees of the various mechanisms in the extended model with $\theta_L = 0$, $\theta_H = 10$, and $N = 2$.

and bidder i makes a payment to the seller of

$$t_i(a) = q_i(a)T(\Sigma a).$$

Thus, the auction is parametrized by a pair of functions Q and T, which describe the aggregate allocation and the price per unit, respectively, as a function of the aggregate action.

In proportional auctions, the good is only allocated to one agent, and also the allocation is "decoupled" from the transfer (although the optimal Q and T are related to one another through the strategic virtual objective). In proportional cost-sharing mechanisms, on the other hand, the good is allocated in same quantity to all agents, and budget balance directly links the expenditure on the public good to the total payment of the agents. Nonetheless, the aggregate-proportional functional form appears in both mechanisms (the sum across agents depends only on the aggregate action, and individual shares are proportional to actions). As best we can tell, this is because in both problems, agents' preferences are quasilinear in the utility from the allocation and payments, and the utility from the allocation is multiplicatively separable in the type. As a result, the divergence of the agents' utilities, which appears in the strategic virtual objective, reduces to a weighted sum of the divergences of the payment and the allocation. The aggregate-proportional form gives rise to a divergence for the payment rule that only depends on how the aggregate action is related to the aggregate payment. As a result, incentives are controlled with the choice of the functions Q and T in the case of auctions, and the function E in the case of public goods. In the auction context, the optimal rules induce assortative matching between common values and aggregate actions, in that the higher is the value, the higher are the aggregate actions that minimizes the strategic virtual objective. In the case of public goods, there is also assortative matching, with higher social values being matched to higher aggregate actions, as in the model of Section 4.

To summarize, while there are high level structural similarities between proportional auctions and proportional cost-sharing mechanisms, they are distinct mechanisms tailored to distinct economic environments. The aggregate-proportional form arises in both cases due to the additive separability between payments and utility from consumption of the good. We would not be surprised if similar structures arise from applications of the framework of Brooks and Du (2023) to other settings with quasilinear preferences.

6 Conclusion

This paper has used an informationally-robust welfare criterion to derive new proportional cost-sharing mechanisms for the public goods problem, as well as the proportional-price trading mechanism for the bilateral trade problem. These mechanisms are shown to provide unimprovable guarantees for welfare, even with minimal assumptions about the nature of private information and the social value of the good. These mechanisms seem simple enough that they could actually be implemented in practice.[9] They are parameterized only by the total expenditure rule, and the performance of the mechanism is robust to misspecification of the fundamentals. Moreover, they completely mitigate the free-rider problem in the limit when the social value is large.

An important limitation of our analysis is that the social value has to be relatively large in order for the guarantee to be non-degenerate. In particular, the ratio of per capita social value to each dollar spent must be greater than one. This condition seems more likely to be satisfied when the number of agents is also relatively small. As mentioned in the introduction, a potential application could be to climate change mitigation policies, for instance to the design of a treaty that would assign country-level greenhouse gas emissions reduction targets. Individual countries can refuse to participate in the treaty at the interim stage, but once it is signed, there is an enforcement mechanism that compels countries to meet their assigned targets. As long as the social value of mitigating climate change is relatively large and the number of countries involved is relatively small, our results show that proportional cost-sharing mechanisms provide non-trivial welfare guarantees.

The fact that the guarantee is zero when the social value per capita is small is, in a sense, a consequence of the weakness of our assumptions about the information structure. For the potential-minimizing information structures, no mechanism can achieve positive welfare, no matter what equilibrium is played. Thus, to obtain non-trivial guarantees in such cases, it seems necessary to restrict the degree of heterogeneity in values across agents. Changes to other aspects of the model may be needed as well, as the following example shows. Suppose that the social value is greater than one, so that production is efficient, but the social value per capita is less than one. Further suppose that all agents have the same expected value. Then any participation secure mechanism will have an equilibrium in which all of the agents play their participation secure actions: An agent who deviates from this strategy profile will have to cover the entire cost of the public good, which is less than its private value to the deviator. As a result, under this information structure, any budget balanced and

[9] At no point in our analysis did we explicitly rule out more complicated mechanisms. For example, since the information structure is common knowledge among the agents, it is in principle possible for the designer to ask the agents to report that common knowledge, and then run the social welfare maximizing mechanism for the true information structure. But such a mechanism cannot improve upon proportional cost-sharing mechanisms in the worst case. And focusing on the smooth mechanisms and the strategic virtual objective led us to less elaborate and contrived solutions.

participation secure mechanism will have a welfare guarantee of zero. Thus, in generalizing the theory to the case where the social value is low, it may also be necessary to modify the participation constraint, the budget constraint, the equilibrium selection rule, or all of the above.

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A Omitted proofs for Section 3

Lemma 1. *For $\alpha \in (0,1)$ and $x > 0$, define*

$$g(x) = \exp(-x)x^{\alpha} \int_{y=0}^{x} \exp(y)y^{-\alpha}\, dy. \tag{17}$$

Then we have

$$g'(x) = \alpha g(x)/x - g(x) + 1. \tag{18}$$

Moreover, we have $\lim_{x\to 0} g(x) = 0$, $\lim_{x\to 0} g(x)/x = \lim_{x\to 0} g'(x) = 1/(1-\alpha)$, and $\lim_{x\to\infty} g(x) = 1$. Thus, $g(x)$ and $g'(x)$ are continuous and bounded functions on $[0,\infty)$.

Proof. Equation (18) follows from straightforward calculations.

We have $\lim_{x\to 0} g(x) = 0$ since

$$\int_{y=0}^{x} \exp(y)y^{-\alpha}\, dy \leqslant \int_{y=0}^{x} \exp(x)y^{-\alpha}\, dy = \exp(x)x^{1-\alpha}/(1-\alpha).$$

L'Hôpital's rule implies that

$$\lim_{x\to 0} g(x)/x = \lim_{x\to 0} \frac{\int_{y=0}^{x} \exp(y)y^{-\alpha}\, dy}{x^{1-\alpha}} = \lim_{x\to 0} \frac{\exp(x)x^{-\alpha}}{(1-\alpha)x^{-\alpha}} = \frac{1}{1-\alpha}$$

and

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\exp(x)x^{-\alpha}}{\exp(x)x^{-\alpha} - \alpha\exp(x)x^{-\alpha-1}} = 1.$$

□

Proposition 5. *Suppose $\overline{E}(x) = \frac{\overline{\lambda}}{\theta_L - 1} g(x)$ for $\alpha = \frac{N-1}{\theta_L - 1} \in (0,1)$. There exist $(\overline{\lambda}, \overline{x})$ such that $\overline{E}(\overline{x}) = 1$ and $\overline{E}'(\overline{x}) = 0$.*

Proof. Set

$$\overline{\lambda} = \frac{\theta_L - 1}{g(\overline{x})}.$$

This ensures that $\overline{E}(\overline{x}) = 1$, whatever the value we choose for $\overline{x}$. We now choose a value for $\overline{x}$ such that $\overline{E}'(\overline{x}) = 0$. Note this is is equivalent to $g'(\overline{x}) = 0$. We therefore show that there exists $\overline{x}$ such that $g'(\overline{x}) = 0$.

By Lemma 1, $\lim_{x\to 0} g(x) = 0$, $\lim_{x\to\infty} g(x) = 1$ and $g'(x)$ is continuous for $x > 0$. Suppose there was some x such that $g(x) > 1$. Then there must be some $x' > x$ such that $g'(x') < 0$. Since $\lim_{x\to 0} g(x) = 0$, there must also be some $x'' < x$ for which $g'(x'') > 0$. Hence, by the mean value theorem, $g'(x''') = 0$ for some $x''' \in (x'', x')$. It therefore suffices to show that $g(x) > 1$ for some $x > 0$.

To that end, define

$$\begin{aligned} h(x) &= \frac{g(x) - 1}{x^{\alpha}\exp(x)} \\ &= \int_{y=0}^{x} \exp(y)y^{-\alpha} dy - \exp(x)x^{-\alpha}. \end{aligned}$$

Clearly, $h(x) > 0$ if and only if $g(x) > 1$. Notice

$$h'(x) = \alpha \exp(x) x^{-\alpha-1},$$

which tends to infinity as $x \to \infty$. Hence, we have that $h(x)$ tends to infinity as $x \to \infty$, and so $h(x) > 0$ for some x. But then for this x, we also have $g(x) > 1$, which completes the proof. □

Proof of Theorem 1. For a fixed $\epsilon > 0$, the informational virtual objective of $\overline{I}$ is

$$\gamma(s,e;\epsilon) = \overline{\rho}(\Sigma s)(\theta_L - 1)\Sigma e - \frac{1}{\epsilon}\sum_{i=1}^{N}\left(\overline{\rho}(\Sigma s + \epsilon)\left(\frac{s_i + \epsilon}{\Sigma s + \epsilon}\theta_L \Sigma e - e_i\right) - \left(\overline{\rho}(\Sigma s)\left(\frac{s_i}{\Sigma s}\theta_L \Sigma e - e_i\right)\right)\right).$$

By Proposition 2, the welfare potential of $\overline{I}$ is at most

$$\int_s \max_{e\in\Omega} \gamma(s,e;\epsilon) ds \;=\; \int_{\underline{x} \leqslant \Sigma s < \overline{x} - \epsilon} \max_{e\in\Omega} \gamma(s,e;\epsilon) ds + \int_{\Sigma s \geqslant \overline{x} - \epsilon} \max_{e\in\Omega} \gamma(s,e;\epsilon) ds. \tag{19}$$

When $\Sigma s \in [\underline{x}, \overline{x})$, $\max_{e\in\Omega}\gamma(s,e;\epsilon)$ tends to zero as $\epsilon \to 0$, since by construction the limit satisfies (cf. equation (11))

$$\gamma(s,e) = \left((\overline{\rho}(\Sigma s) - \overline{\rho}'(\Sigma s))(\theta_L - 1) - \frac{(N-1)\theta_L \overline{\rho}(\Sigma s)}{\Sigma s}\right)\Sigma e = 0$$

for every $e \in \Omega$. Since $\overline{\rho}$ and $\overline{\rho}'$ are bounded on $[\underline{x}, \overline{x})$, by the Dominated Convergence Theorem, the first term in (19) tends to zero as $\epsilon \to 0$.

Now consider the second term in (19). Recall that for $x \in (\overline{x} - \epsilon, \overline{x})$, $\overline{\rho}(x + \epsilon) = 0$. Therefore the second term in is equal to

$$\begin{aligned}
&\int_{\overline{x}-\epsilon \leqslant \Sigma s \leqslant \overline{x}} \max_{e\in\Omega}\left(\overline{\rho}(\Sigma s)(\theta_L - 1)\Sigma e - \sum_{i=1}^{N}\frac{0 - \overline{\rho}(\Sigma s)(\frac{\theta_L s_i}{\Sigma s}\Sigma e - e_i)}{\epsilon}\right) ds \\
=& \int_{x=\overline{x}-\epsilon}^{\overline{x}} \max_{\Sigma e\in\Omega}\left(\overline{\rho}(x)(\theta_L - 1)\Sigma e + \frac{\overline{\rho}(x)(\theta_L - 1)\Sigma e}{\epsilon}\right) ds.
\end{aligned}$$

We now change the variable of integration from s to Σs:

$$\begin{aligned}
=& \int_{x=\overline{x}-\epsilon}^{\overline{x}} \max_{\Sigma e\in[0,1]}\left(\overline{\rho}(x)(\theta_L - 1)\Sigma e + \frac{\overline{\rho}(x)(\theta_L - 1)\Sigma e}{\epsilon}\right)\frac{(x - \underline{x})^{N-1}}{(N-1)!} dx \\
=& \int_{x=\overline{x}-\epsilon}^{\overline{x}} \left(\overline{\rho}(x)(\theta_L - 1) + \frac{\overline{\rho}(x)(\theta_L - 1)}{\epsilon}\right)\frac{(x - \underline{x})^{N-1}}{(N-1)!} dx
\end{aligned}$$

which, as $\epsilon \to 0$, converges to

$$\overline{\rho}(\overline{x})(\theta_L - 1)\frac{(\overline{x} - \underline{x})^{N-1}}{(N-1)!} = \frac{\exp(\overline{x})\overline{x}^{-(N-1)\theta_L/(\theta_L - 1)}(\overline{x} - \underline{x})^{N-1}}{\int_{y=\underline{x}}^{\overline{x}} \exp(y) y^{-(N-1)\theta_L/(\theta_L-1)}(y - \underline{x})^{N-1} dy}. \tag{20}$$

When $\theta_L > N$ (respectively, $\theta_L \leqslant N$), (20) converges to $\overline{\lambda}$ (respectively, 0) as $\underline{x} \to 0$. Thus, the welfare potential of $\overline{I}$ can be made arbitrarily close to $\overline{\lambda}$ (respectively, 0).

To finish the proof of an ϵ-strong maxmin solution, we need to show that an equilibrium exists in $(\overline{M}, \overline{I})$. When $\theta_L > N$ (the other case is trivial), we will verify the hypotheses of Theorem 1 of Milgrom and Weber (1985), namely, that (i) the action spaces are compact, (ii) payoffs are equicontinuous over actions, and (iii) the distribution of signals is absolutely continuous with respect to the product of the marginals. Condition (iii) follows immediately from the fact that the distribution of signals is absolutely continuous with respect to Lebesgue measure. Condition (i) can be satisfied because any action $a_i > \overline{x}$ in $\overline{M}$ is weakly dominated by $a_i = \overline{x}$, since any action beyond $\overline{x}$ does not change the total expenditure but increases one's share of the expenditure; so for equilibrium existence it is without loss to restrict the action space of $\overline{M}$ to $[0, \overline{x}]$. With regard to condition (ii), note that an agent's payoff is

$$\left(\overline{v}_i(s) - \frac{a_i}{\Sigma a}\right) \overline{E}(\Sigma a).$$

Since $\overline{E}$ has bounded derivatives and satisfies $\overline{E}(0) = 0$ by Lemma 1, equicontinuity of the payoff follows from the fact that $\overline{v}_i(s)$ is bounded above by θ_L. Hence, there exists an equilibrium in distributional strategies, which implies existence of an equilibrium in behavioral strategies.

□

Proof of Proposition 3. We will show that under the mechanism $\overline{M}$ and information structure $\overline{I}$, truthtelling, i.e. playing $a_i = s_i$, is an equilibrium. Suppose $a_{-i} = s_{-i}$ and agent i plays s_i' when her signal is s_i. Then i's interim utility can be written as:

$$U_i(s_i, s_i') = \int_{s_{-i}} (\overline{v}_i(s_i, s_{-i})\overline{E}(s_i' + \Sigma s_{-i}) - \overline{e}_i(s_i', s_{-i})) \cdot \overline{\rho}(s_i + \Sigma s_{-i}) ds.$$

We now change the variable of integration from s to Σs_{-i}:

$$\begin{aligned} U_i(s_i, s_i') &= \int_{y=0}^{\infty} \left(\frac{s_i}{s_i + y} \theta_L - \frac{s_i'}{s_i' + y} \right) \overline{E}(s_i' + y) \overline{\rho}(s_i + y) \frac{y^{N-2}}{(N-2)!} \, dy \\ &= C \int_{y=0}^{\overline{x} - s_i} \left(\frac{s_i}{s_i + y} \theta_L - \frac{s_i'}{s_i' + y} \right) g(s_i' + y) \exp(s_i + y)(s_i + y)^{-\alpha - (N-1)} y^{N-2} \, dy, \end{aligned}$$

where $C > 0$ is a constant, $\alpha = \frac{N-1}{\theta_L - 1}$, $g(x)$ is defined by equation (17) when $x \leqslant \underline{x}$, and $g(x) = g(\overline{x})$ when $x \geqslant \overline{x}$. We will show $U_i(s_i, s_i) \geqslant U_i(s_i, s_i')$ for all $s_i \in (0, \overline{x}]$ and $s_i' \geqslant 0$, by showing that $\partial U_i(s_i, s_i')/\partial s_i'$ is non-negative for $s_i' \leqslant s_i$ and non-positive for $s_i' \geqslant s_i$.

We calculate:

$$\begin{aligned} &\frac{1}{C} \frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\ &= \int_{y=0}^{\overline{x} - s_i} \left(\left(\frac{s_i}{s_i + y} \theta_L - \frac{s_i'}{s_i' + y} \right) g'(s_i' + y) - \frac{y}{(s_i' + y)^2} g(s_i' + y) \right) \exp(s_i + y)(s_i + y)^{-\alpha - (N-1)} y^{N-2} \, dy. \end{aligned}$$

Let us simplify the integrand when $s_i' \leqslant s_i$ using the differential equation (18):

$$\begin{aligned}
&\left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y}\right) g'(s_i'+y) - \frac{y}{(s_i'+y)^2} g(s_i'+y) \\
&= \left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha g(s_i'+y)}{s_i'+y} - g(s_i'+y) + 1\right) - \frac{y}{(s_i'+y)^2} g(s_i'+y) \\
&= \frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y} + \left(\left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha}{s_i'+y} - 1\right) - \frac{y}{(s_i'+y)^2}\right) g(s_i'+y).
\end{aligned} \tag{21}$$

Therefore, for $s_i' \leqslant s_i$:

$$\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
&= \int_{y=0}^{\overline{x}-s_i} \left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y}\right) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
&\quad + \underbrace{\int_{y=0}^{\overline{x}-s_i} \left(\left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha}{s_i'+y} - 1\right) - \frac{y}{(s_i'+y)^2}\right) g(s_i'+y) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy}_{A}.
\end{aligned}$$

Note that the term A implicitly involves a double integral, though the function g, which we now aim to simplify. According to Mathematica, we have

$$\begin{aligned}
&\int \left(\left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha}{s_i'+y} - 1\right) - \frac{y}{(s_i'+y)^2}\right) \exp(s_i - s_i')(s_i'+y)^{\alpha}(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
&= \frac{\exp(s_i - s_i')}{N-1} y^{N-1} (s_i+y)^{-\frac{(N-1)\theta_L}{\theta_L - 1}} (s_i'+y)^{\frac{N-\theta_L}{\theta_L-1}} (N - 1 + s_i' + y - (s_i'+y)\theta_L).
\end{aligned} \tag{22}$$

Therefore, integrating by parts, we have

$$\begin{aligned}
A =& \frac{\exp(s_i - s_i')}{N-1} y^{N-1} (s_i+y)^{-\frac{(N-1)\theta_L}{\theta_L - 1}} (s_i'+y)^{\frac{N-\theta_L}{\theta_L-1}} (N - 1 + s_i' + y - (s_i'+y)\theta_L) \int_{z=0}^{y+s_i'} \exp(z) z^{-\alpha}\, dz \Bigg|_{y=0}^{\overline{x}-s_i} \\
&- \int_{y=0}^{\overline{x}-s_i} \frac{\exp(s_i - s_i')}{N-1} y^{N-1} (s_i+y)^{-\frac{(N-1)\theta_L}{\theta_L - 1}} (s_i'+y)^{\frac{N-\theta_L}{\theta_L-1}} (N - 1 + s_i' + y - (s_i'+y)\theta_L) \exp(s_i'+y)(s_i'+y)^{-\alpha}\, dy.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
=& \frac{\exp(s_i - s_i')}{N-1} (\overline{x} - s_i)^{N-1} \overline{x}^{-\frac{(N-1)\theta_L}{\theta_L - 1}} (s_i' + \overline{x} - s_i)^{\frac{N-\theta_L}{\theta_L-1}} (N - 1 - (s_i' + \overline{x} - s_i)(\theta_L - 1)) \int_{z=0}^{\overline{x}-s_i+s_i'} \exp(z) z^{-\alpha}\, dz \\
&+ \int_{y=0}^{\overline{x}-s_i} \left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y} - \frac{y}{s_i'+y}\left(1 - \frac{s_i'+y}{\alpha}\right)\right) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
=& \exp(s_i - s_i')(\overline{x} - s_i)^{N-1} \overline{x}^{-\frac{(N-1)\theta_L}{\theta_L - 1}} (s_i' + \overline{x} - s_i)^{\frac{N-\theta_L}{\theta_L-1}} \left(1 - \frac{s_i' + \overline{x} - s_i}{\alpha}\right) \int_{z=0}^{\overline{x}-s_i+s_i'} \exp(z) z^{-\alpha}\, dz \\
&+ \int_{y=0}^{\overline{x}-s_i} \left(\frac{s_i}{s_i+y}\theta_L - 1 + \frac{y}{\alpha}\right) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy.
\end{aligned} \tag{23}$$

We next argue that (23) is zero when $s_i' = s_i$, i.e., that the first-order condition $\partial U_i(s_i, s_i)/\partial s_i' = 0$ is satisfied. This is equivalent to showing that

$$\begin{aligned}
&-(\overline{x}-s_i)^{N-1}\overline{x}^{-N}\left(1-\frac{\overline{x}}{\alpha}\right)\int_{z=0}^{\overline{x}}\exp(z)z^{-\alpha}\,dz\\
&=\int_{y=0}^{\overline{x}-s_i}\left(\frac{s_i}{s_i+y}\theta_L-1+\frac{y}{\alpha}\right)\exp(s_i+y)(s_i+y)^{-\alpha-(N-1)}y^{N-2}\,dy \qquad (24)\\
&=\int_{z=s_i}^{\overline{x}}\left(\frac{s_i}{z}\theta_L-1+\frac{z-s_i}{\alpha}\right)\exp(z)z^{-\alpha-(N-1)}(z-s_i)^{N-2}\,dz.
\end{aligned}$$

By construction, we have $g'(\overline{x}) = 0$, so by equation (18),

$$\exp(\overline{x})\overline{x}^{-\alpha}=\left(1-\frac{\alpha}{\overline{x}}\right)\int_{z=0}^{\overline{x}}\exp(z)z^{-\alpha}\,dz.$$

Substituting this into (24) gives

$$\begin{aligned}
\left(1-\frac{s_i}{\overline{x}}\right)^{N-1}\exp(\overline{x})\overline{x}^{-\alpha} &=\alpha\int_{z=s_i}^{\overline{x}}\left(\frac{s_i}{z}\theta_L-1+\frac{z-s_i}{\alpha}\right)\exp(z)z^{-\alpha-(N-1)}(z-s_i)^{N-2}\,dz\\
&=\int_{z=s_i}^{\overline{x}}\left(\frac{\frac{s_i}{z}(\alpha+N-1)-\alpha}{z-s_i}+1\right)\exp(z)z^{-\alpha}\left(1-\frac{s_i}{z}\right)^{N-1}dz \qquad (25)\\
&=\int_{z=s_i}^{\overline{x}}\left(-\frac{\alpha}{z}+\frac{s_i(N-1)}{z(z-s_i)}+1\right)\exp(z)z^{-\alpha}\left(1-\frac{s_i}{z}\right)^{N-1}dz,
\end{aligned}$$

using $\alpha\theta_L = \alpha + N - 1$. But the above equation holds since

$$\begin{aligned}
&\int_{z=s_i}^{\overline{x}}\frac{\alpha}{z}\exp(z)z^{-\alpha}\left(1-\frac{s_i}{z}\right)^{N-1}dz\\
&=-z^{-\alpha}\exp(z)\left(1-\frac{s_i}{z}\right)^{N-1}\Bigg|_{z=s_i}^{\overline{x}}+\int_{z=s_i}^{\overline{x}}z^{-\alpha}\frac{d}{dz}\left(\exp(z)\left(1-\frac{s_i}{z}\right)^{N-1}\right)dz\\
&=-\overline{x}^{-\alpha}\exp(\overline{x})\left(1-\frac{s_i}{\overline{x}}\right)^{N-1}+\int_{z=s_i}^{\overline{x}}z^{-\alpha}\exp(z)\left(1-\frac{s_i}{z}\right)^{N-1}\left(1+\frac{(N-1)z}{z-s_i}\frac{s_i}{z^2}\right)dz,
\end{aligned}$$

using integration by parts.

Next, we argue that $\partial U_i(s_i, s_i)/\partial s_i'$ is decreasing in s_i' for $s_i' \leqslant s_i$. Defining $x = \overline{x}-(s_i-s_i')$, the derivative of (23) with respect to s_i' is

$$\begin{aligned}
&\frac{d}{dx}\left(\exp(\overline{x}-x)x^{\frac{N-\theta_L}{\theta_L-1}}\left(1-\frac{x}{\alpha}\right)\int_{z=0}^{x}\exp(z)z^{-\alpha}\,dz\right)\\
&=\exp(\overline{x}-x)x^{\frac{N-\theta_L}{\theta_L-1}}\left(-\left(1-\frac{x}{\alpha}\right)+\frac{N-\theta_L}{\theta_L-1}x^{-1}\left(1-\frac{x}{\alpha}\right)-\frac{1}{\alpha}\right)\int_{z=0}^{x}\exp(z)z^{-\alpha}\,dz\\
&\quad+\exp(\overline{x}-x)x^{\frac{N-\theta_L}{\theta_L-1}}\left(1-\frac{x}{\alpha}\right)\exp(x)x^{-\alpha}\\
&=\exp(\overline{x}-x)x^{\frac{N-\theta_L}{\theta_L-1}}\left(-2+\frac{x}{\alpha}+\frac{N-\theta_L}{(\theta_L-1)x}\right)\int_{z=0}^{x}\exp(z)z^{-\alpha}\,dz+\exp(\overline{x})x^{-1}\left(1-\frac{x}{\alpha}\right)\\
&=\exp(\overline{x})x^{-1}\left(-2+\frac{x}{\alpha}+\frac{\alpha-1}{x}\right)g(x)+\exp(\overline{x})x^{-1}\left(1-\frac{x}{\alpha}\right).
\end{aligned}$$

Now define

$$f(x) = \left(2 - \frac{x}{\alpha} + \frac{1-\alpha}{x}\right) g(x) + \frac{x}{\alpha} - 1$$

for all $x \in [0, \overline{x}]$. We calculate

$$\begin{aligned}
f(0) &= 0, \\
f(\overline{x}) &= \frac{2 - \frac{\overline{x}}{\alpha} + \frac{1-\alpha}{\overline{x}}}{1 - \frac{\alpha}{\overline{x}}} + \frac{\overline{x}}{\alpha} - 1 = \frac{1 - \frac{\overline{x}}{\alpha} + \frac{1}{\overline{x}}}{1 - \frac{\alpha}{\overline{x}}} + \frac{\overline{x}}{\alpha} = \frac{\frac{1}{\overline{x}}}{1 - \frac{\alpha}{\overline{x}}} > 0,
\end{aligned}$$

using the facts that $\lim_{x \to 0} g(x)/x = \frac{1}{1-\alpha}$ and $g'(\overline{x}) = 0$ and hence $g(\overline{x}) = \frac{1}{1-\alpha/\overline{x}} > 0$ from Lemma 1.

It therefore remains to show that $f(x) \geqslant 0$ for all $x \in (0, \overline{x})$. At any point where $f(x) = 0$, we must have

$$\begin{aligned}
g(x) &= \frac{1 - x/\alpha}{2 - x/\alpha + (1-\alpha)/x} \\
&= \frac{x(x-\alpha)}{x^2 - 2\alpha x - \alpha(1-\alpha)} \\
&= \frac{x(x-\alpha)}{(x - \alpha - \sqrt{\alpha})(x - \alpha + \sqrt{\alpha})} \equiv h(x).
\end{aligned}$$

So, it suffices to show that $x = 0$ is the unique point where the functions g and h intersect. Note that because $\alpha \in (0,1)$, only one of the roots of the denominator of h is positive. We have to consider separately what happens for on either side of the positive root $\alpha + \sqrt{\alpha}$.

If $x < \alpha + \sqrt{\alpha}$, then the denominator of $h(x)$ is positive, and we clearly have that

$$h(x) = \frac{1 - x/\alpha}{2 - x/\alpha + (1-\alpha)/x} \leqslant \frac{1}{2 + (1-\alpha)/x} \leqslant \frac{1}{1 + (1-\alpha)/x} = \frac{x}{x + 1 - \alpha} \equiv \widehat{h}(x),$$

so it suffices to show that $g \geqslant \widehat{h}$, and they are equal only at zero. Note that

$$\widehat{h}'(x) = \frac{1-\alpha}{(x+1-\alpha)^2}.$$

Now, suppose that at $x > 0$ we have $g(x) = \widehat{h}(x)$. Then

$$\begin{aligned}
g'(x) &= \left(\frac{\alpha}{x} - 1\right) \frac{x}{x+1-\alpha} + 1 \\
&= \frac{\alpha - x}{x + 1 - \alpha} + 1 \\
&= \frac{1}{x + 1 - \alpha} \\
&= \frac{x + 1 - \alpha}{1 - \alpha} \widehat{h}'(x),
\end{aligned}$$

so that $g'(x) \geqslant \widehat{h}'(x)$, and the inequality is strict of $x > 0$. Thus, by Lemma 2 of Milgrom and Weber (1982), $g(x) \geqslant \widehat{h}(x)$ for all x, and the inequality is strict if $x > 0$.

Now consider $x > \alpha + \sqrt{\alpha}$. Note that

$$\begin{aligned}
h(\overline{x}) &= \frac{\overline{x}(\overline{x} - \alpha)}{\overline{x}^2 - 2\alpha\overline{x} - \alpha(1-\alpha)} \\
&= \frac{(\overline{x} - \alpha)^2}{\overline{x}^2 - 2\alpha\overline{x} - \alpha(1-\alpha)} \frac{\overline{x}}{\overline{x} - \alpha} \\
&= \frac{\overline{x}^2 - 2\alpha\overline{x} + \alpha^2}{\overline{x}^2 - 2\alpha\overline{x} + \alpha^2 - \alpha} \frac{\overline{x}}{\overline{x} - \alpha} \\
&> \frac{\overline{x}}{\overline{x} - \alpha} = g(\overline{x}).
\end{aligned}$$

Moreover, we claim that $h'(x) < 0$ for $x > \alpha + \sqrt{\alpha}$, and since $g' > 0$, we cannot have $h(x) = g(x)$ in this range:

$$\begin{aligned}
h'(x) &= \frac{(2x-\alpha)(x^2 - 2\alpha x - \alpha(1-\alpha)) - (x^2 - \alpha x)(2x - 2\alpha)}{(x^2 - 2\alpha x - \alpha(1-\alpha))^2} \\
&= \frac{2x^3 - 4\alpha x^2 - 2\alpha(1-\alpha)x - \alpha x^2 + 2\alpha^2 x + \alpha^2(1-\alpha) - 2x^3 + 2\alpha x^2 + 2\alpha x^2 - 2\alpha^2 x}{(x^2 - 2\alpha x - \alpha(1-\alpha))^2} \\
&= \frac{-2\alpha x - \alpha x^2 + 2\alpha^2 x + \alpha^2(1-\alpha)}{(x^2 - 2\alpha x - \alpha(1-\alpha))^2} \\
&= \frac{-\alpha\left[x^2 + 2(1-\alpha)x - \alpha(1-\alpha)\right]}{(x^2 - 2\alpha x - \alpha(1-\alpha))^2}.
\end{aligned}$$

Thus, h' has the opposite sign as the term in brackets, which is clearly increasing in x, and is therefore minimized at $x = \alpha + \sqrt{\alpha}$. Plugging in, we get

$$\begin{aligned}
&(\alpha + \sqrt{\alpha})^2 + 2(1-\alpha)(\alpha + \sqrt{\alpha}) - \alpha(1-\alpha) \\
&= \alpha^2 + 2\alpha\sqrt{\alpha} + \alpha + 2\alpha + 2\sqrt{\alpha} - 2\alpha^2 - 2\alpha\sqrt{\alpha} - \alpha + \alpha^2 \\
&= 2\alpha\sqrt{\alpha} + 2\alpha > 0.
\end{aligned}$$

Thus, $h' < 0$, and we are done. This completes the proof that $f(0) = 0$, and $f(x) > 0$ for all $x \in [0, \overline{x}]$. This in turn completes the proof that $\partial U_i(s_i, s_i')/\partial s_i' \geqslant 0$ for $s_i' \leqslant s_i$.

Finally, we consider the case where $s_i' \geqslant s_i$, in which we have:

$$\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
&= \int_{y=0}^{\max(\overline{x}-s_i',0)} \left(\left(\frac{s_i}{s_i + y}\theta_L - \frac{s_i'}{s_i' + y} \right) g'(s_i' + y) - \frac{y}{(s_i' + y)^2} g(s_i' + y) \right) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
&\quad - \int_{y=\max(\overline{x}-s_i',0)}^{\overline{x}-s_i} \frac{y}{(s_i' + y)^2} g(\overline{x}) \exp(s_i + y)(s_i + y)^{-\alpha-(N-1)} y^{N-2}\, dy,
\end{aligned}$$

since $g'(x) = 0$ and $g(x) = g(\overline{x})$ for $x > \overline{x}$. Clearly, $\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \leqslant 0$ when $s_i' \geqslant \overline{x}$. So to show $\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \leqslant 0$ it suffices to consider only $s_i' \in [s_i, \overline{x}]$.

Applying equation (21), we get for $s_i' \in [s_i, \overline{x}]$:

$$
\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
=&\int_{y=0}^{\overline{x}-s_i'} \left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y}\right) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
&+ \underbrace{\int_{y=0}^{\overline{x}-s_i'} \left(\left(\frac{s_i}{s_i+y}\theta_L - \frac{s_i'}{s_i'+y}\right)\left(\frac{\alpha}{s_i'+y}-1\right) - \frac{y}{(s_i'+y)^2}\right) g(s_i'+y) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy}_{B} \\
&- \int_{y=\overline{x}-s_i'}^{\overline{x}-s_i} \frac{y}{(s_i'+y)^2} g(\overline{x}) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy,
\end{aligned}
$$

where, by (22) we have

$$
\begin{aligned}
B =& \frac{\exp(s_i - s_i')}{N-1} y^{N-1} (s_i+y)^{-\frac{(N-1)\theta_L}{\theta_L-1}} (s_i'+y)^{\frac{N-\theta_L}{\theta_L-1}} (N-1+s_i'+y-(s_i'+y)\theta_L) \int_{z=0}^{y+s_i'} \exp(z) z^{-\alpha}\, dz \Bigg|_{y=0}^{\overline{x}-s_i'} \\
&- \int_{y=0}^{\overline{x}-s_i'} \frac{\exp(s_i - s_i')}{N-1} y^{N-1} (s_i+y)^{-\frac{(N-1)\theta_L}{\theta_L-1}} (s_i'+y)^{\frac{N-\theta_L}{\theta_L-1}} (N-1+s_i'+y-(s_i'+y)\theta_L) \exp(s_i'+y)(s_i'+y)^{-\alpha}\, dy.
\end{aligned}
$$

Thus, for $s_i' \in [s_i, \overline{x}]$,

$$
\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
&= \exp(s_i - s_i')(\overline{x}-s_i')^{N-1}(\overline{x}-s_i'+s_i)^{-\frac{(N-1)\theta_L}{\theta_L-1}} \overline{x}^{\frac{N-\theta_L}{\theta_L-1}} \left(1-\frac{\overline{x}}{\alpha}\right) \int_{z=0}^{\overline{x}} \exp(z) z^{-\alpha}\, dz \\
&\quad + \int_{y=0}^{\overline{x}-s_i'} \left(\frac{s_i}{s_i+y}\theta_L - 1 + \frac{y}{\alpha}\right) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy \\
&\quad - \int_{y=\overline{x}-s_i'}^{\overline{x}-s_i} \frac{y}{(s_i'+y)^2} g(\overline{x}) \exp(s_i+y)(s_i+y)^{-\alpha-(N-1)} y^{N-2}\, dy.
\end{aligned}
$$

Changing variable to $x = \overline{x} - s_i'$ and applying (25), we get for $s_i' \in [s_i, \overline{x}]$:

$$\begin{aligned}
&\frac{1}{C}\frac{\partial U_i}{\partial s_i'}(s_i, s_i') \\
&= \exp(x + s_i)x^{N-1}(x + s_i)^{-\alpha-(N-1)}\left(\frac{1}{\overline{x}} - \frac{1}{\alpha}\right)g(\overline{x}) \\
&\quad + \int_{z=s_i}^{x+s_i}\left(\frac{s_i}{z}\theta_L - 1 + \frac{z - s_i}{\alpha}\right)\exp(z)z^{-\alpha-(N-1)}(z - s_i)^{N-2}\,dz \\
&\quad - \int_{y=\overline{x}-s_i'}^{\overline{x}-s_i}\frac{y}{(s_i' + y)^2}g(\overline{x})\exp(s_i + y)(s_i + y)^{-\alpha-(N-1)}y^{N-2}\,dy \\
&= \exp(x + s_i)x^{N-1}(x + s_i)^{-\alpha-(N-1)}\left(\frac{1}{\overline{x}} - \frac{1}{\alpha}\right)g(\overline{x}) \\
&\quad + \frac{1}{\alpha}\left(1 - \frac{s_i}{x + s_i}\right)^{N-1}\exp(x + s_i)(x + s_i)^{-\alpha} \\
&\quad - \int_{y=\overline{x}-s_i'}^{\overline{x}-s_i}\frac{y}{(s_i' + y)^2}g(\overline{x})\exp(s_i + y)(s_i + y)^{-\alpha-(N-1)}y^{N-2}\,dy \\
&= -\int_{y=\overline{x}-s_i'}^{\overline{x}-s_i}\frac{y}{(s_i' + y)^2}g(\overline{x})\exp(s_i + y)(s_i + y)^{-\alpha-(N-1)}y^{N-2}\,dy \leqslant 0
\end{aligned}$$

where in the last equality we use $\left(\frac{\alpha}{\overline{x}} - 1\right)g(\overline{x}) + 1 = g'(\overline{x}) = 0$. □

B Proof of Theorem 2

B.1 Bound on the welfare guarantee of proportional cost-sharing mechanisms

First, we derive a modified bound for the guarantee of proportional cost-sharing mechanisms. Recall the notion of a smooth mechanism on page 9 and the definition of the strategic virtual objective $\lambda(\theta, a)$ in equation (3). Proposition 1 implies the following lower bound on the welfare guarantee:

$$W(M, I, b) \geqslant \inf_{\mu \in \Delta_{\widehat{\theta}}}\int_\theta\left(\inf_a \lambda(\theta, a)\right)\mu(d\theta), \tag{26}$$

where $\Delta_{\widehat{\theta}} = \{\mu \in \Delta(\Theta) : \int_\theta \Sigma\theta\mu(d\theta) \geqslant \widehat{\theta}\}$ and $\Theta = \{\theta \in \mathbb{R}_+^N : \theta_L \leqslant \Sigma\theta \leqslant \theta_H\}$.

For a proportional cost-sharing mechanism, the strategic virtual objective only depends on $\Sigma\theta$ and Σa, and in a slight abuse of notation simplifies to

$$\lambda(\Sigma\theta, \Sigma a) = (\Sigma\theta - 1)E(\Sigma a) + (\Sigma\theta - 1)E'(\Sigma a) - \frac{(N-1)E(\Sigma a)}{\Sigma a}.$$

Suppose $E(\Sigma a)$ is non-decreasing in Σa, as will be the case for the mechanisms to which we apply what follows. Since $\lambda(\Sigma\theta, \Sigma a)$ is non-decreasing and linear in $\Sigma\theta$ for every a,

$\inf_a \lambda(\Sigma\theta, \Sigma a)$ is a non-decreasing and concave function of $\Sigma\theta$. Thus, the righthand-side of (26) is minimized when the expected value is exactly $\widehat{\theta}$. If not, then we can reduce the likelihood of some realization for which $\Sigma\theta > \widehat{\theta}$ and increase the likelihood of $\widehat{\theta}$. Moreover, concavity implies that the minimizing μ induces a two-point distribution (supported on θ_L and θ_H) for $\Sigma\theta$. Therefore, the welfare guarantee of a proportional cost-sharing mechanism for which E is increasing is at least

$$\mu_L \inf_a \lambda(\theta_L, \Sigma a) + \mu_H \inf_a \lambda(\theta_H, \Sigma a),$$

where

$$\mu_H = \frac{\widehat{\theta} - \theta_L}{\theta_H - \theta_L}, \mu_L = 1 - \mu_H.$$

B.2 Case 1: $\theta_L \leqslant N$ and $\widehat{\theta} > N$.

For Case 1 let us denote $\lambda_H \equiv \overline{\lambda}_{\theta_H}$. Consider the proportional cost-sharing mechanism with the expenditure function

$$\overline{E}(x) = \begin{cases} \frac{\lambda_H}{\theta_H - c} \int_{y=0}^{x} e^{y-x} \left(\frac{y}{x}\right)^{-(N-1)/(\theta_H-1)} dy & x \leqslant x_H, \\ 1 & x > x_H, \end{cases}$$

where the parameters (λ_H, x_H) satisfy $\overline{E}(x_H) = 1$ and $\overline{E}'(x_H) = 0$ (Proposition 5). By construction we have

$$\lambda(\theta_H, x) = (\theta_H - 1)(\overline{E}(x) + \overline{E}'(x)) - \frac{(N-1)\overline{E}(x)}{x} \begin{cases} = \lambda_H & x \in [0, x_H], \\ > \lambda_H & x \in (x_H, \infty). \end{cases} \tag{27}$$

This gives:

$$\lambda(\theta_L, x) = (\theta_L - 1)(\overline{E}(x) + \overline{E}'(x)) - \frac{(N-1)\overline{E}(x)}{x} \begin{cases} = \lambda_H - (\theta_H - \theta_L)(\overline{E}(x) + \overline{E}'(x)) & x \in [0, x_H], \\ > \lambda_H - (\theta_H - \theta_L)(\overline{E}(x) + \overline{E}'(x)) & x \in (x_H, \infty). \end{cases}$$

By the argument in Lemma 5, $\overline{E}(x) + \overline{E}'(x)$ is decreasing in $x \in [0, \infty)$. Lemma 1 implies $\overline{E}(0) = 0$ and $\overline{E}'(0) = \frac{1}{1-\alpha}$ for $\alpha = \frac{N-1}{\theta_H - 1}$. Thus we have

$$\lambda(\theta_L, x) \geqslant \lambda(\theta_L, 0) = \lambda_H - (\theta_H - \theta_L)\frac{\lambda_H}{\theta_H - 1}\frac{1}{1 - \frac{N-1}{\theta_H - 1}} = \lambda_H \frac{\theta_L - N}{\theta_H - N}.$$

Therefore, applying the bound we derived at the beginning of the proof gives that the welfare guarantee of the proportional-cost sharing mechanism is at least

$$\left(\mu_H + \mu_L \frac{\theta_L - N}{\theta_H - N}\right) \lambda_H. \tag{28}$$

By Theorem 1, as $\underline{x} \to 0$ the welfare potential of $\overline{I}_{\theta_H}$ tends to λ_H, while the welfare potential of $\overline{I}_N$ tends to 0. Thus, the welfare potential of $\overline{I}$, which is a public randomization between $\overline{I}_N$ and $\overline{I}_{\theta_H}$ with probabilities $\frac{\theta_H-\hat{\theta}}{\theta_H-N}$ and $\frac{\hat{\theta}-N}{\theta_H-N}$ respectively, tends to $\lambda_H\frac{\hat{\theta}-N}{\theta_L-N}$, which is exactly (28).

It remains only to show that there exists an equilibrium at $(\overline{M}_{\theta_H}, \tilde{I})$. This can be established by showing separately that there are equilibria of $(\overline{M}_{\theta_H}, \overline{I}_N)$ and $(\overline{M}_{\theta_H}, \overline{I}_{\theta_H})$, which follows from the same argument in the proof of Theorem 1.

This proves the first case of Theorem 2.

B.3 Case 2: $\theta_L > N$.

Recall the proportional cost-sharing mechanism $\widehat{M}$ with expenditure $\widehat{E}$ in equation (14), and the information structure $\widehat{I}$ defined by equations (15) and (16). By construction, $\widehat{E}$ is continuous on $[0, x_H]$ and satisfies

$$\lambda_L = \lambda(\theta_L, x) = (\theta_L - 1)(\widehat{E}(x) + \widehat{E}'(x)) - \frac{(N-1)\widehat{E}(x)}{x} \tag{29}$$

for $x \in [0, x_L]$, and

$$\lambda_H = \lambda(\theta_H, x) = (\theta_H - 1)(\widehat{E}(x) + \widehat{E}'(x)) - \frac{(N-1)\widehat{E}(x)}{x} \tag{30}$$

for $x \in [x_L, x_H]$.

We need the following conditions to be satisfied:

$$\widehat{E}(x_H) = \widehat{E}(x_L)\exp(x_L - x)\left(\frac{x_L}{x}\right)^{-(N-1)/(\theta_H-1)} + \frac{\lambda_H}{\theta_H - 1}\int_{y=x_L}^{x} \exp(y-x)\left(\frac{y}{x}\right)^{-(N-1)/(\theta_H-1)} dy = 1, \tag{31}$$

$$\widehat{E}'(x_H) = 0 \iff \theta_H - 1 - \frac{(N-1)}{x_H} = \lambda_H, \tag{32}$$

$$\widehat{E}'(x_L^-) + \widehat{E}(x_L) = \frac{\lambda_H - \lambda_L}{\theta_H - \theta_L} = \widehat{E}'(x_L^+) + \widehat{E}(x_L), \tag{33}$$

and

$$(\theta_H - 1)\widehat{\rho}(x_L^+) = (\theta_L - 1)\widehat{\rho}(x_L^-), \tag{34}$$

when $\underline{x} = 0$.

Lemma 2. *Condition* (33) *is equivalent to*

$$\frac{\lambda_H + \frac{\widehat{E}(x_L)(N-1)}{x_L}}{\theta_H - 1} = \frac{\lambda_L + \frac{\widehat{E}(x_L)(N-1)}{x_L}}{\theta_L - 1}. \tag{35}$$

Proof. By equations (29) and (30), we have

$$\widehat{E}(x_L) + \widehat{E}'(x_L^-) = \frac{\lambda_L + \frac{\widehat{E}(x_L)(N-1)}{x_L}}{\theta_L - 1},$$
$$\widehat{E}(x_L) + \widehat{E}'(x_L^+) = \frac{\lambda_H + \frac{\widehat{E}(x_L)(N-1)}{x_L}}{\theta_H - 1}.$$

Thus, condition (33) implies condition (35).

Now, suppose condition (35) holds, so $\widehat{E}'(x_L^-) = \widehat{E}'(x_L^+)$. Subtracting (29) from (30) at $x = x_L$ then gives condition (33). □

Lemma 3. *There exist* $(x_L, x_H, \lambda_L, \lambda_H)$ *such that conditions* (31), (32), (33) *and* (34) *are satisfied.*

Proof. Define

$$g(x) = \int_{y=0}^{x} \exp(y-x)\left(\frac{y}{x}\right)^{-(N-1)/(\theta_L-1)} dy.$$

We can rewrite (34) as

$$\int_{x=x_L}^{x_H} \exp(x - x_L)(x/x_L)^{-(N-1)/(\theta_H-1)} dx = g(x_L)\frac{\mu_H(\theta_H-1)}{\mu_L(\theta_L-1)}. \tag{36}$$

Given $x_L > 0$, define $x_H(x_L)$ as the unique $x_H \geqslant x_L$ that satisfies equation (36).

Substituting the formula for $\widehat{E}(x_L)$, we can rewrite (35) as

$$\theta_H - 1 + \frac{(N-1)(\theta_H - \theta_L)}{x_L(\theta_L-1)} g(x_L) = \frac{\lambda_H(\theta_L-1)}{\lambda_L}. \tag{37}$$

Since $\lambda_H = \theta_H - 1 - \frac{(N-1)}{x_H(x_L)}$ (condition (32)), equation (37) gives λ_L as a function of x_L. Thus, equation (31) can be rewritten as

$$\begin{aligned}
&\exp(x_H(x_L) - x_L)\left(\frac{x_H(x_L)}{x_L}\right)^{-(N-1)/(\theta_H-1)} \\
&= \frac{\lambda_L}{\theta_L - 1} g(x_L) + \frac{\lambda_H}{\theta_H - 1}\frac{\mu_H(\theta_H-1)}{\mu_L(\theta_L-1)} g(x_L) \\
&= \left(\theta_H - 1 - \frac{(N-1)}{x_H(x_L)}\right)\left(\frac{1}{\theta_H - 1 + \frac{(N-1)(\theta_H-\theta_L)}{x_L(\theta_L-1)} g(x_L)} + \frac{\mu_H}{\mu_L(\theta_L-1)}\right) g(x_L),
\end{aligned}$$

where we used (36) in the first equality and (37) in the second equality.

Thus, finding a $(x_L, x_H, \lambda_L, \lambda_H)$ to satisfy conditions (31), (32), (33) and (34) is equivalent to finding a x_L to satisfy

$$\begin{aligned}
&\exp(x_H(x_L) - x_L)\left(\frac{x_H(x_L)}{x_L}\right)^{-(N-1)/(\theta_H-1)} \\
&= \left(\theta_H - 1 - \frac{(N-1)}{x_H(x_L)}\right)\left(\frac{1}{\theta_H - 1 + \frac{(N-1)(\theta_H-\theta_L)}{x_L(\theta_L-1)} g(x_L)} + \frac{\mu_H}{\mu_L(\theta_L-1)}\right) g(x_L).
\end{aligned} \tag{38}$$

Let x_L^* be the critical point of g: $g'(x_L^*) = 0$. We will now show that there exists a $x_L \in (0, x_L^*]$ that satisfies (38), which proves the proposition.

The lefthand side of (38) is clearly positive when x_L is sufficiently small, while the righthand side of (38) is clearly negative when x_L is sufficiently small (since $x_H(x_L) \to 0$ as $x_L \to 0$).

We now show that when $x_L = x_L^*$, the lefthand side of (38) is less than or equal to the righthand side.

We have

$$g(x_L^*)\left(1 - \frac{(N-1)}{x_L^*(\theta_L - 1)}\right) = 1.$$

Thus,

$$\theta_H - 1 + \frac{(N-1)(\theta_H - \theta_L)}{x_L^*(\theta_L - 1)} g(x_L^*) = \left(\theta_H - 1 - \frac{(N-1)}{x_L^*}\right) g(x_L^*).$$

Thus at $x_L = x_L^*$, the righthand side of (38) can be rewritten as

$$\left(\theta_H - 1 - \frac{(N-1)}{x_H(x_L^*)}\right)\left(\frac{1}{\left(\theta_H - 1 - \frac{(N-1)}{x_L^*}\right) g(x_L^*)} + \frac{\mu_H}{\mu_L(\theta_L - 1)}\right) g(x_L^*),$$

where $x_H^* = x_H(x_L^*)$.

We can rewrite the above as

$$\frac{1 - \frac{\alpha}{x_H^*}}{1 - \frac{\alpha}{x_L^*}} + \int_{x=x_L^*}^{x_H^*} \exp(x - x_L^*)(x/x_L^*)^{-\alpha} dx \left(1 - \frac{\alpha}{x_H^*}\right)$$

where $\alpha = \frac{N-1}{\theta_H - 1}$. Thus, we want to show

$$\exp(x_H^* - x_L^*)\left(\frac{x_H^*}{x_L^*}\right)^{-\alpha} \frac{1}{1 - \frac{\alpha}{x_H^*}} \leqslant \frac{1}{1 - \frac{\alpha}{x_L^*}} + \int_{x=x_L^*}^{x_H^*} \exp(x - x_L^*)(x/x_L^*)^{-\alpha} dx. \tag{39}$$

The left-hand side of (39) is clearly equal to the right-hand side when $x_H^* = x_L^*$. The derivative of the left-hand side of (39) with respect to x_H^* can be simplified to be

$$\exp(x_H^* - x_L^*)\left(\frac{x_H^*}{x_L^*}\right)^{-\alpha} \frac{(x_H^* - \alpha)^2 - \alpha}{(x_H^* - \alpha)^2}$$

while the derivative of the right-hand side of (39) with respect to x_H^* is

$$\exp(x_H^* - x_L^*)\left(\frac{x_H^*}{x_L^*}\right)^{-\alpha}.$$

□

Lemma 4. *Suppose conditions* (31), (32), (33) *hold. Then we have* $\widehat{E}(x) \in [0, 1]$ *for all* $x \in [0, x_H]$.

Proof. Differentiating (30) with respect to x gives:

$$\widehat{E}''(x) = \left(\frac{(N-1)}{(\theta_H - 1)x} - 1\right)\widehat{E}'(x) - \frac{(N-1)}{(\theta_H - 1)x^2}\widehat{E}(x).$$

Thus, for $x \in [(N-1)/(\theta_H - 1), \infty)$, $\widehat{E}'(x) > 0$ implies $\widehat{E}''(x) < 0$. Thus, by Lemma 2 of Milgrom and Weber (1982), on the interval $[(N-1)/(\theta_H - 1), \infty)$ $\widehat{E}'(x)$ across 0 at most once, and in which case it does so from above. Since condition (32) says that $\widehat{E}'(x_H) = 0$, we conclude that $\widehat{E}(x) \geqslant 0$ for $x \in [(N-1)/(\theta_H - 1), x_H]$. Using equation (30) it is easy to see that $\widehat{E}'(x) > 0$ when $x < (N-1)/(\theta_H - 1)$. Thus, $\widehat{E}'(x) \geqslant 0$ for all $x \in [x_L, x_H]$ for the q defined by (30).

By the smooth pasting condition (33), we have $\widehat{E}'(x_L) \geqslant 0$ for the $\widehat{E}$ defined by (29) as well. By the same argument applied to the differential equation in (30), we conclude that $\widehat{E}'(x) \geqslant 0$ for all $x \in [0, x_L]$ for the q defined by (29).

Finally, condition (31) says that $\widehat{E}(x_H) = 1$. So $\widehat{E}'(x) \geqslant 0$ for all $x \in [0, x_H]$ implies that $\widehat{E}(x) \leqslant 1$ for $x \in [0, x_H]$. □

Lemma 5. *Suppose condition* (33) *holds. Then* $\widehat{E}(x) + \widehat{E}'(x)$ *is decreasing in* $x \in [0, x_H]$.

Proof. By equations (29) and (30), it suffices to show that $f(x) = \widehat{E}(x)/x$ decreases with x. We first focus on the interval $[0, x_L]$. We calculate, using (29),

$$\begin{aligned} f'(x)x^2 &= \widehat{E}'(x)x - \widehat{E}(x) \\ &= \frac{(N-1)}{\theta_L - 1}\widehat{E}(x) - x\widehat{E}(x) + \frac{\lambda_L}{\theta_L - 1}x - \widehat{E}(x) \\ &= x\left(-(1-\alpha)\widehat{E}(x)/x - \widehat{E}(x) + \frac{\lambda_L}{\theta_L - 1}\right), \end{aligned}$$

i.e.,

$$f'(x) = \frac{-(1-\alpha)f(x) - \widehat{E}(x) + \frac{\lambda_L}{\theta_L - 1}}{x},$$

where $\alpha = \frac{N-1}{\theta_L - 1}$.

Lemma 1 implies that $\lim_{x\to 0} f(x) = \frac{\lambda_L}{(\theta_L - 1)(1-\alpha)}$ and $\lim_{x\to 0}\widehat{E}(x) = 0$, so by L'Hôpital's rule we have

$$\lim_{x\to 0} f'(x) = -(1-\alpha)\lim_{x\to 0} f'(x) - \lim_{x\to 0}\widehat{E}'(x),$$

i.e.,

$$\lim_{x\to 0} f'(x) = -\frac{\lambda_L}{(\theta_L - 1)(2-\alpha)(1-\alpha)} < 0,$$

since Lemma 1 implies that $\lim_{x\to 0}\widehat{E}'(x) = \frac{\lambda_L}{(\theta_L - 1)(1-\alpha)}$.

For the sake of contradiction, suppose $F = \{x \in \mathbb{R}_+ : f'(x) > 0\} \neq \varnothing$.

Since $\lim_{x\to 0} f'(x) < 0$, there exist $0 < x' < x''$ such that $f'(x') = 0$ and $(x', x''] \subseteq F$. This implies that $f(x'') > f(x')$ and $\widehat{E}(x'') > \widehat{E}(x')$ (since $f'(x) > 0$ implies $\widehat{E}'(x) > 0$), which is a contradiction since

$$f'(x'')x'' = -(1-\alpha)f(x'') - \widehat{E}(x'') + \frac{\lambda_L}{\theta_L - 1} < -(1-\alpha)f(x') - \widehat{E}(x') + \frac{\lambda_L}{\theta_L - 1} = f'(x')x' = 0.$$

Thus, we conclude that $f'(x) \leqslant 0$ for $x \in [0, x_L]$. By (33), this implies that $f'(x_L^+) \leqslant 0$. On the interval $[x_L, x_H]$, because of equation (30) we have

$$f'(x) = \frac{-(1-\alpha)f(x) - \widehat{E}(x) + \frac{\lambda_H}{\theta_H - 1}}{x},$$

for $\alpha = \frac{N-1}{\theta_H - 1}$. By exactly the same argument as in the previous paragraph, we conclude that $f'(x) \leqslant 0$ for $x \in [x_L, x_H]$.

□

Lemma 6. *Suppose conditions* (31), (32), (33) *hold. Then we have*

$$\inf_x \lambda(\theta_L, x) = \lambda_L,$$

and

$$\inf_x \lambda(\theta_H, x) = \lambda_H.$$

Thus, the welfare guarantee of the proportional cost-sharing mechanism is at least $\mu_L \lambda_L + \mu_H \lambda_H$.

Proof. Given conditions (29) and (30), we need to show that

$$\lambda(\theta_L, x) = \lambda_H + (\widehat{E}(x) + \widehat{E}'(x))(\theta_L - \theta_H) \geqslant \lambda_L, \tag{40}$$

for $x \in [x_L, x_H]$, and

$$\lambda(\theta_H, x) = \lambda_L + (\widehat{E}(x) + \widehat{E}'(x))(\theta_H - \theta_L) \geqslant \lambda_H, \tag{41}$$

for $x \in [0, x_L]$. Conditions (40) and (41) follow from Lemma 5 and equation (33).

Conditions (31) and (32) then imply that

$$\lambda_H = \lambda(\theta_H, x_H) = (\theta_H - 1) - \frac{(N-1)}{x_H} < \lambda(\theta_H, x) = (\theta_H - 1) - \frac{(N-1)}{x}$$

and

$$\lambda(\theta_L, x) = \lambda(\theta_H, x) + (\theta_L - \theta_H) > \lambda_H + (\theta_L - \theta_H) \geqslant \lambda_L,$$

for $x \in (x_H, \infty)$, where the last inequality follows from (40) when $x = x_H$. □

Lemma 7. *Suppose condition* (34) *holds, then the welfare potential of $\widehat{I}$ converges to*

$$\frac{(\theta_H - 1)\widehat{\rho}(x_H)(x_H)^{N-1}}{(N-1)!} \tag{42}$$

as $\underline{x} \to 0$.

Proof. The limit informational virtual objective is

$$\gamma(s, e) = \left((\widehat{\rho}(\Sigma s) - \widehat{\rho}'(\Sigma s))(\Sigma \widehat{v}(\Sigma s) - 1) - \frac{(N-1)\Sigma \widehat{v}(\Sigma s)\widehat{\rho}(\Sigma s)}{\Sigma s} \right) \Sigma e,$$

where $\Sigma\widehat{v}(\Sigma s) = \theta_L$ when $\Sigma s \in [\underline{x}, x_L)$ and $\Sigma\widehat{v}(\Sigma s) = \theta_H$ when $\Sigma s \in [x_L, x_H)$. By construction, we have $\gamma(s, e) = 0$ whenever $\Sigma s \in [\underline{x}, x_L)$ and $\Sigma s \in (x_L, x_H)$.

For a fixed $\epsilon > 0$, the upper bound from Proposition 2 can be written as

$$\int_{\Sigma s \in [\underline{x}, x_L - \epsilon) \cup [x_L, x_H - \epsilon)} \max_{e \in \Omega} \gamma(s, e; \epsilon) ds + \int_{\Sigma s \in [x_L - \epsilon, x_L]} \max_{e \in \Omega} \gamma(s, e; \epsilon) ds + \int_{\Sigma s \in [x_H - \epsilon, x_H]} \max_{e \in \Omega} \gamma(s, e; \epsilon) ds.$$

By the argument in the proof of Theorem 1, sending first $\epsilon \to 0$ and then $\underline{x} \to 0$, the first term tends to 0 and the third term tends to (42). We can rewrite the second term as

$$\int_{\Sigma s \in [x_L - \epsilon, x_L]} \max_{\Sigma e \in [0,1]} \left(\widehat{\rho}(\Sigma s)(\widehat{\theta} - 1)\Sigma e - \frac{1}{\epsilon} \sum_{i=1}^{N} \left(\widehat{\rho}(\Sigma s + \epsilon)\theta_H \frac{s_i + \epsilon}{\Sigma s + \epsilon} - \widehat{\rho}(\Sigma s)\theta_L \frac{s_i}{\Sigma s} \right) \Sigma e \right.$$
$$\left. + \frac{\widehat{\rho}(\Sigma s + \epsilon) - \widehat{\rho}(\Sigma s)}{\epsilon} \Sigma e \right) ds$$

which, as $\epsilon \to 0$ and $\underline{x} \to 0$, converges to 0 by condition (34). □

Proposition 6. *Suppose conditions* (31), (32), (33) *and* (34) *hold. Then the welfare guarantee of $\widehat{M}$ is equal to the welfare potential of $\widehat{I}$ as $\underline{x} \to 0$.*

Proof. We can rewrite equation (34) as

$$\frac{\mu_L \lambda_L \exp(x_L)(x_L)^{-(N-1)/(\theta_L - 1)}}{\frac{\lambda_L}{\theta_L - 1} \int_{x=0}^{x_L} \exp(x) x^{-(N-1)/(\theta_L - 1)} dx} = \frac{\mu_H \lambda_H \exp(x_L)(x_L)^{-(N-1)/(\theta_H - 1)}}{\frac{\lambda_H}{\theta_H - 1} \int_{x=x_L}^{x_H} \exp(x) x^{-(N-1)/(\theta_H - 1)} dx},$$

or

$$\frac{\mu_L \lambda_L}{\widehat{E}(x_L)} = \frac{\mu_H \lambda_H \exp(x_L - x_H)(x_L/x_H)^{-(N-1)/(\theta_H - 1)}}{1 - \widehat{E}(x_L) \exp(x_L - x_H)(x_L/x_H)^{-(N-1)/(\theta_H - 1)}}.$$

Therefore, the welfare guarantee from Lemma 6 is

$$\begin{aligned}
&\mu_L \lambda_L + \mu_H \lambda_H \\
&= \mu_H \lambda_H \frac{1}{1 - \widehat{E}(x_L) \exp(x_L - x_H)(x_L/x_H)^{-(N-1)/(\theta_H - 1)}} \\
&= \frac{\mu_H \lambda_H}{\frac{\lambda_H}{\theta_H - 1} \int_{y=x_L}^{x_H} \exp(y - x_H) \left(\frac{y}{x_H} \right)^{-(N-1)/(\theta_H - 1)} dy} \\
&= \frac{(\theta_H - 1)\widehat{\rho}(x_H)(x_H)^{N-1}}{(N-1)!}
\end{aligned}$$

which is equal to the welfare potential from Lemma 7. □

Finally, we note that an equilibrium exists for $(\widehat{M}, \widehat{I})$ by exactly the same argument given in the proof of Theorem 1.

The case of $\theta_L > N$ in Theorem 2 then follows from Proposition 6.