Online Appendix to "Optimal auction design with common values: An informationally-robust approach"

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B Proofs for Section 5

B.1 Proof of Proposition 5

Let $\Delta = 1/K$, and recall that the message space for $\overline{\mathcal{M}}(\underline{m}, K)$ is

$$M_i = \{\underline{m}, \underline{m} + \Delta, \ldots, \underline{m} + K\}.$$

Note that the highest message $\overline{m} = \underline{m} + K$ is at least Δ^{-1}. We shall extend the domain of the allocation and transfer rules to all of $\mathbb{R}^N_+$ for notational convenience. The discrete aggregate allocation sensitivity is

$$\mu(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \mathbb{I}_{m_i < \overline{m}} (q_i(m_i + \Delta, m_{-i}) - q_i(m)),$$

and the discrete aggregate excess growth is

$$\Xi(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \mathbb{I}_{m_i < \overline{m}} (t_i(m_i + \Delta, m_{-i}) - t_i(m)) - \Sigma t(m).$$

Now, define

$$\lambda(m; v) = v\mu(m) - \Xi(m) - c\overline{Q}(\Sigma m),$$

and let $\lambda(v) = \min_{m \in M} \lambda(m; v)$.

Lemma 1. *For any information structures $\mathcal{S}$ and equilibrium β of $(\mathcal{S}, \overline{\mathcal{M}}(\underline{m}, K))$, expected profit is at least $\int_V \lambda(v) H(dv)$.*

Proof of Lemma 1. The equilibrium hypothesis implies that for all i,

$$\int_S \sum_{m \in M} \big[w(s)(q_i(\min\{m_i + \Delta, \overline{m}\}, m_{-i}) - q_i(m))$$

$$- \left(t_i(\min\{m_i+\Delta,\overline{m}\},m_{-i}) - t_i(m)\right)\Big]\beta(m|s)\pi(ds) \leq 0,$$

which corresponds to the incentive constraint for deviating to $\min\{m_i + \Delta, \overline{m}\}$. Summing across bidders, and dividing by Δ, we conclude that

$$\int_S \sum_{m \in M} \left[w(s)\mu(m) - \Xi(m) - \Sigma t(m)\right] \beta(m|s)\pi(ds) \leq 0.$$

Hence, expected profit is

$$\begin{aligned}
&\int_S \sum_{m \in M} \left[\Sigma t(m) - cQ(\Sigma m)\right] \beta(m|s)\pi(ds) \\
&\geq \int_S \sum_{m \in M} \left[\Sigma t(m) - cQ(\Sigma m) + w(s)\mu(m) - \Xi(m) - \Sigma t(m)\right] \beta(m|s)\pi(ds) \\
&= \int_S \sum_{m \in M} \left[w(s)\mu(m) - \Xi(m) - cQ(\Sigma m)\right] \beta(m|s)\pi(ds) \\
&\geq \int_S \lambda(w(s))\pi(ds) \\
&\geq \int_V \lambda(v)H(dv),
\end{aligned}$$

where the last line follows from the mean-preserving spread condition on $w(s)$ and that λ is concave, being the infimum of linear functions. $\square$

Lemma 2. *For all $m \in M$,*

$$\mu(m) \geq \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{\mu}(\Sigma m + y)dy - \widehat{L}(\underline{m}, \Delta),$$

where

$$\widehat{L}(\underline{m}, \Delta) = N(N+1)\Delta + \frac{N(N-1)}{\Delta}\left(\log(N\underline{m} + \Delta) + \frac{N\underline{m}}{N\underline{m} + \Delta} - \log(N\underline{m}) - 1\right).$$

Moreover, for all $\underline{m} > 0$, $\widehat{L}(\underline{m}, \Delta) \to 0$ as $\Delta \to 0$.

Proof of Lemma 2. From Lemma 12, we know that

$$\begin{aligned}
\mu(m) &= \sum_{i=1}^{N} \frac{1}{\Delta}(q_i(m_i + \Delta, m_{-i}) - q_i(m)) - \sum_{i=1}^{N} \mathbb{I}_{m_i = \overline{m}} \frac{1}{\Delta}(q_i(m_i + \Delta, m_{-i}) - q_i(m)) \\
&\geq \sum_{i=1}^{N} \frac{1}{\Delta}(q_i(m_i + \Delta, m_{-i}) - q_i(m)) - N\frac{N+1}{\overline{m}} \\
&\geq \sum_{i=1}^{N} \frac{1}{\Delta}(q_i(m_i + \Delta, m_{-i}) - q_i(m)) - N(N+1)\Delta.
\end{aligned}$$

Recall that

$$\overline{\mu}(x) = \frac{N-1}{x}\overline{Q}(x) + \overline{Q}'(x).$$

Also recall that

$$\frac{\partial q_i(m)}{\partial m_i} = \frac{\Sigma m_{-i}}{(\Sigma m)^2}\overline{Q}(\Sigma m) + \frac{m_i}{\Sigma m}\overline{Q}'(\Sigma m).$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{1}{\Delta}\left(q_i(m_i+\Delta, m_{-i}) - q_i(m)\right) \\
&= \frac{1}{\Delta}\sum_{i=1}^{N}\int_{y=0}^{\Delta} \frac{\partial q_i(m_i+y, m_{-i})}{\partial m_i} dy \\
&= \frac{1}{\Delta}\sum_{i=1}^{N}\int_{y=0}^{\Delta} \left[\frac{\Sigma m_{-i}}{(\Sigma m+y)^2}\overline{Q}(\Sigma m+y) + \frac{m_i+y}{\Sigma m+y}\overline{Q}'(\Sigma m+y)\right] dy \\
&= \frac{1}{\Delta}\int_{y=0}^{\Delta} \left[\frac{(N-1)\Sigma m}{(\Sigma m+y)^2}\overline{Q}(\Sigma m+y) + \frac{\Sigma m+Ny}{\Sigma m+y}\overline{Q}'(\Sigma m+y)\right] dy \\
&= \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{\mu}(\Sigma m+y)dy - \frac{N-1}{\Delta}\int_{y=0}^{\Delta} \frac{y}{\Sigma m+y}\left[\frac{\overline{Q}(\Sigma m+y)}{\Sigma m+y} - \overline{Q}'(\Sigma m+y)\right] dy.
\end{aligned}$$

We need to bound the last integral from above. If x is in a non-graded interval, then $\overline{Q}(x)/x - \overline{Q}'(x)$ is just $1/x$. If x is in a graded interval $[a,b]$, then

$$\frac{\overline{Q}(x)}{x} - \overline{Q}'(x) = \frac{C(a,b)}{N} + \frac{D(a,b)}{x^N} - \frac{C(a,b)}{N} + (N-1)\frac{D(a,b)}{x^N} = \frac{ND(a,b)}{x^N}.$$

From equation (33), $D(a,b) \leq x^{N-1}$, so that the integrand in this case is at most N/x, and

$$\begin{aligned}
\int_{y=0}^{\Delta} \frac{y}{x+y}\left[\frac{\overline{Q}(x+y)}{x+y} - \overline{Q}'(x+y)\right] dy &\leq N\int_{y=0}^{\Delta} \frac{y}{(x+y)^2}dy \\
&= N\int_{y=0}^{\Delta}\left(\frac{1}{x+y} - \frac{x}{(x+y)^2}\right) dy \\
&= N\left(\log(x+\Delta) + \frac{x}{x+\Delta} - \log(x) - 1\right).
\end{aligned}$$

The derivative with respect to x is

$$N\left(\frac{1}{x+\Delta} - \frac{1}{x} + \frac{\Delta}{(x+\Delta)^2}\right) = N\Delta\left(\frac{1}{(x+\Delta)^2} - \frac{1}{x(x+\Delta)}\right)$$

which is clearly negative, so subject to $x \geq N\underline{m}$, the expression is maximized with $x = N\underline{m}$, which gives us the lower bound on μ.

Moreover, as $\Delta \to 0$, $N(N+1)\Delta \to 0$, and by L'Hôpital's rule,

$$\lim_{\Delta\to 0}\left(\frac{\log(N\underline{m}+\Delta)+\frac{N\underline{m}}{N\underline{m}+\Delta}-\log(N\underline{m})-1}{\Delta}\right)=\lim_{\Delta\to 0}\left(\frac{1}{N\underline{m}+\Delta}-\frac{N\underline{m}}{(N\underline{m}+\Delta)^2}\right)=0.$$

□

Now let us write $\Xi^p(m)=\Xi(m)-\underline{v}(\mu(m)-Q(m))$, and recall that $\overline{\Xi}^p(x)=\overline{\Xi}(x)-\underline{v}(\overline{\mu}(x)-\overline{Q}(x))$. These are the excess growths for the "premium" transfers $t_i^p(m)=t_i(m)-\underline{v}q_i(m)$ and $\overline{t}_i^p(m)=\overline{t}_i(m)-\underline{v}\overline{q}_i(m)$, respectively. We similarly denote by $\overline{T}^p(x)=\overline{T}(x)-\underline{v}\overline{Q}(x)$ the aggregate premium transfer, and note that $\overline{T}^p$ satisfies the differential equation

$$\left(\frac{N-1}{x}-1\right)\overline{T}^p(x)+\frac{d}{dx}\overline{T}^p(x)=\overline{\Xi}^p(x),$$

with the boundary condition $\overline{T}^p(0)=0$.

Lemma 3. *Let L_Ξ be an upper bound on $|\overline{\Xi}^p|$ and let L_T be an upper bound on $\overline{T}^p$. Then*

$$\Xi^p(m)\le\frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{\Xi}^p(\Sigma m+y)dy+\tilde{L}(\underline{m})\frac{\Delta}{2}+NL_p\underline{m}$$
$$-\frac{1}{\Delta}\sum_i\mathbb{I}_{m_i=\overline{m}}\left[\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\right],$$

where

$$\tilde{L}(\underline{m})=\left(1+\frac{N-1}{N\underline{m}}\right)L_p+\frac{N-1}{(N\underline{m})^2}L_T.$$

Proof of Lemma 3. Recall that $\overline{T}^p$ is Lipschitz with constant L_p. Furthermore, the function $\overline{T}^p(x)(N-1)/x$ is Lipschitz on $[N\underline{m},\infty)$, and

$$\left|\frac{d}{dx}\left(\frac{N-1}{x}\overline{T}^p(x)\right)\right|=\left|\frac{N-1}{x}\frac{d}{dx}\overline{T}^p(x)-\frac{N-1}{x^2}\overline{T}^p(x)\right|$$
$$\le\frac{N-1}{N\underline{m}}L_p+\frac{N-1}{(N\underline{m})^2}L_T=L_1(\underline{m}).$$

Using the differential equation for $\overline{T}^p$,

$$\frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{\Xi}^p(\Sigma m+y)dy$$
$$=\frac{1}{\Delta}\int_{y=0}^{\Delta}\left[\left(\frac{N-1}{\Sigma m+y}-1\right)\overline{T}^p(\Sigma m+y)+\frac{d}{dx}\overline{T}^p(x)\Big|_{x=\Sigma m+y}\right]dy$$
$$=\frac{1}{\Delta}\left[\int_{y=0}^{\Delta}\left(\frac{N-1}{\Sigma m+y}-1\right)\overline{T}^p(\Sigma m+y)dy+\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right]$$

$$\geq \frac{1}{\Delta}\left[\int_{y=0}^{\Delta}\left(\frac{N-1}{\Sigma m+\Delta}\overline{T}^p(\Sigma m+\Delta)-L_1(\underline{m})(\Delta-y)-\overline{T}^p(\Sigma m)-L_p y\right)dy+\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right]$$
$$=\frac{1}{\Delta}\left[\Delta\frac{N-1}{\Sigma m+\Delta}\overline{T}^p(\Sigma m+\Delta)-\Delta\overline{T}^p(\Sigma m)-(L_1(\underline{m})+L_p)\frac{\Delta^2}{2}+\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right]$$
$$=\frac{1}{\Delta}\left[\frac{\Sigma m+N\Delta}{\Sigma m+\Delta}\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right]-\overline{T}^p(\Sigma m)-\underbrace{(L_1(\underline{m})+L_p)}_{\equiv\tilde{L}(\underline{m})}\frac{\Delta}{2}.$$

Now, let us write $T^p(\Sigma m)$ for the aggregate transfer when the messages are m. Thus,

$$\Xi^p(m)=\frac{1}{\Delta}\sum_{i=1}^{N}\left[t_i^p(m_i+\Delta,m_{-i})-t_i^p(m)\right]-T^p(\Sigma m)-\frac{1}{\Delta}\sum_{i=1}^{N}\mathbb{I}_{m_i=\overline{m}}\left[t_i^p(m_i+\Delta,m_{-i})-t_i^p(m)\right]$$
$$=\frac{1}{\Delta}\sum_{i=1}^{N}\left[\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\right]-T^p(\Sigma m)-\frac{1}{\Delta}\sum_{i=1}^{N}\mathbb{I}_{m_i=\overline{m}}\left[\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\right]$$
$$\leq\frac{1}{\Delta}\left[\frac{\Sigma m+N\Delta}{(\Sigma m+\Delta)}\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right]-T^p(\Sigma m)-\frac{1}{\Delta}\sum_{i}\mathbb{I}_{m_i=\overline{m}}\left[\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\right].$$

The lemma follows from combining these two inequalities, with the observation that $T^p(x)=\overline{T}^p(x)-NL_p\underline{m}$. □

Lemma 4. *For all $\epsilon>0$, there exists a K such that for all m such that $\Sigma m>K$ and for all i,*

$$\frac{1}{\Delta}\left|\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\right|<\epsilon.$$

Proof of Lemma 4. Since $\lim_{x\to\infty}\overline{T}^p(x)=-\overline{\Xi}^p(\infty)$, we can find a K large enough so that for $x>K$, $\left|\overline{T}^p(x)+\overline{\Xi}^p(\infty)\right|<\epsilon/4$ and $L_T/K<\epsilon/4$, and thus $\left|d\overline{T}^p(x)/dx\right|<\epsilon/2$. Thus, when $\Sigma m>K$, then using $\Delta=K^{-1}$,

$$\frac{1}{\Delta}\left[\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\right]=\frac{1}{\Delta}\int_{y=0}^{\Delta}\frac{\partial\overline{t}_i^p(m_i+y,m_{-i})}{\partial m_i}dy$$
$$=\frac{1}{\Delta}\int_{y=0}^{\Delta}\left[\frac{\Sigma m_{-i}}{(\Sigma m+y)^2}\overline{T}^p(\Sigma m+y)+\frac{m_i+y}{\Sigma m+y}\left.\frac{d}{dx}\overline{T}^p(x)\right|_{x=\Sigma m+y}\right]dy$$
$$\leq\frac{L_T}{K}+\frac{\epsilon}{2}$$
$$<\epsilon.$$

□

Proof of Proposition 5. We first argue that there exists $\underline{m}$ and a K such that $\lambda(m;v)\geq\inf_{m'\in\mathbb{R}^N}\overline{\lambda}(m';v)-\epsilon$ for all $m\in M$ and $v\in[\underline{v},\overline{v}]$, where

$$\overline{\lambda}(m;v)=(v-\underline{v})\overline{\mu}(\Sigma m)-\overline{\Xi}^p(\Sigma m)+(\underline{v}-c)\overline{Q}(\Sigma m).$$

From Lemma 12, we know that $|\overline{Q}(x+y)-\overline{Q}(x)| \leq y(N-1)/\underline{m}$. Thus,

$$\begin{aligned}\left|\overline{Q}(x)-\frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{Q}(x+y)dy\right| &\leq \frac{1}{\Delta}\int_{y=0}^{\Delta}\left|\overline{Q}(x+y)-Q(x)\right|dy \\ &\leq \frac{1}{\Delta}\int_{y=0}^{\Delta} y\frac{N-1}{\underline{m}}dy = \Delta\frac{N-1}{2\underline{m}}.\end{aligned}$$

Combining this inequality with Lemmas 2 and 3, we get that

$$\begin{aligned}\lambda(m;v) &= (v-\underline{v})\mu(m)-\Xi^p(m)+(\underline{v}-c)\overline{Q}(\Sigma m) \\ &\geq \frac{1}{\Delta}\int_{y=0}^{\Delta}\left[(v-\underline{v})\overline{\mu}(\Sigma m+\Delta)-\overline{\Xi}^p(\Sigma m+y)+(\underline{v}-c)\overline{Q}(\Sigma m+y)\right]dy \\ &\quad -(\overline{v}-\underline{v})\widehat{L}(\underline{m},\Delta)-\overline{v}\Delta\frac{N-1}{2\underline{m}}-\frac{\Delta}{2}\tilde{L}(\underline{m})-NL_p\underline{m} \\ &\quad -\frac{1}{\Delta}\sum_i \mathbb{I}_{m_i=\overline{m}}\left|\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\right| \\ &\geq \inf_{\{m'|\Sigma m\leq\Sigma m'\leq\Sigma m+\Delta\}}\overline{\lambda}(m';v) \\ &\quad -(\overline{v}-\underline{v})\widehat{L}(\underline{m},\Delta)-\overline{v}\Delta\frac{N-1}{2\underline{m}}-\frac{\Delta}{2}\tilde{L}(\underline{m})-NL_p\underline{m} \\ &\quad -\frac{1}{\Delta}\sum_i \mathbb{I}_{m_i=\overline{m}}\left|\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\right|.\end{aligned}$$

We can first pick $\underline{m}>0$ so that $NL_p\underline{m}<\epsilon/2$. We can then pick K large enough (and Δ small enough) such that the remaining terms in the last two lines sum to less than $\epsilon/2$ (where for the first term in the middle line and last line, this follows from Lemmas 2 and 4, respectively). We then conclude that

$$\lambda(m;v) \geq \inf_{m'\in\mathbb{R}_N^+}\overline{\lambda}(m';v)-\epsilon \geq \overline{\lambda}(v)-\epsilon.$$

Hence, $\lambda(v) \geq \overline{\lambda}(v)-\epsilon$, and Lemma 1 and Lemma 6 give the result. □

This proof goes through verbatim with the maxmin must-sell mechanism $\widehat{\mathcal{M}}$.

B.2 Proof of Proposition 6

Recall the definition of $\overline{\mathcal{S}}(K)$. Let $\Delta = 1/K$. We subsequently choose K sufficiently large (and equivalently Δ sufficiently small) to attain the desired ϵ. Note that the signal space can be written

$$S_i = \left\{0,\Delta,\ldots,K^2\Delta\right\},$$

and the highest message is simply Δ^{-1}. The probability mass function of s_i is

$$f_i(s_i) = \begin{cases}(1-\exp(-\Delta))\exp(-s_i) & \text{if } s_i<\Delta^{-1}; \\ \exp(-\Delta^{-1}) & \text{if } s_i=\Delta^{-1}.\end{cases}$$

As a result, s_i/Δ is a censored geometric random variable with arrival rate $1 - \exp(-\Delta)$. We write $f(s) = \times_{i=1}^{N} f_i(s_i)$ for the joint probability, and

$$F_i(s_i) = \sum_{s_i' \le s_i} f_i(s_i') = \begin{cases} 1 - \exp(-s_i - \Delta) & \text{if } s_i < \Delta^{-1}; \\ 1 & \text{otherwise,} \end{cases}$$

for the cumulative distribution. The value function is

$$w(s) = \frac{1}{f(s)} \int_{\left\{ s' \in \mathbb{R}_+^N \middle| \tau(s_i') = s_i \forall i \right\}} \overline{w}(\Sigma s') \exp(-\Sigma s') ds',$$

where

$$\tau(x) = \begin{cases} \Delta \lfloor x/\Delta \rfloor & \text{if } x < \Delta^{-1}; \\ \Delta^{-1} & \text{otherwise.} \end{cases}$$

An interpretation is that we draw "true" signals s' for the bidders from $\overline{\mathcal{S}}$ and agent i observes $s_i = \min\{\Delta\lfloor \Delta^{-1} s_i' \rfloor, \Delta^{-1}\}$, i.e., signals above Δ^{-1} are censored and otherwise they are rounded down to the nearest multiple of Δ, and w is the conditional expectation of $\overline{w}$ given the noisy observations s. Thus, the distribution of $\overline{w}$ is a mean-preserving spread of the distribution of w, so that H is a mean-preserving spread of the distribution of w as well.

Lemma 5. *If $s_i < \Delta^{-1}$ for all i, then $w(s)$ only depends on the sum of the signals $l = \Sigma s$ and*

$$w(s) = \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \rho(x - l) \exp(-x) dx,$$

where $\rho(y)$ is the $N-1$-dimensional volume of the set $\{s \in [0, \Delta]^N | \Sigma s = y\}$.

Proof of Lemma 5. First observe that

$$f(s) = (1 - \exp(-\Delta))^N \exp(-\Sigma s) = (1 - \exp(-\Delta))^N \exp(-l).$$

Thus,

$$\begin{aligned} w(s) &= \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{\left\{ s' \in \mathbb{R}_+^N | \tau_i(s') = s_i \ \forall i \right\}} \overline{w}(\Sigma s') \exp(-\Sigma s') ds' \\ &= \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \int_{\left\{ s' \in \mathbb{R}_+^N | \tau_i(s') = s_i \ \forall i, \Sigma s' = x \right\}} \overline{w}(\Sigma s') \exp(-\Sigma s') ds' dx \\ &= \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \exp(-x) \int_{\left\{ s' \in \mathbb{R}_+^N \middle| \tau_i(s_i' - s_i) = 0 \ \forall i, \Sigma s' = x \right\}} ds' dx \\ &= \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \exp(-x) \int_{\left\{ s' \in \mathbb{R}_+^N | \tau_i(s') = 0 \ \forall i, \Sigma s' = x - l \right\}} ds' dx, \end{aligned}$$

where the inner integral is just $\rho(x - l)$. □

We now abuse notation slightly by writing $w(l)$ for the value when $l = \Sigma s$, and let $\gamma(l) = w(l) - c$.

Lemma 6. *If $l > \Delta$, then $\gamma(l) \leq \exp(\Delta)\gamma(l - \Delta)$.*

Proof of Lemma 6. From Lemma 5, we know that

$$\begin{aligned}
\gamma(l) &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{\gamma}(x)\exp(-x)\rho(x-l)dx \\
&= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l-\Delta}^{l+(N-1)\Delta} \overline{\gamma}(x+\Delta)\exp(-x-\Delta)\rho(x-l+\Delta)dx \\
&\leq \frac{\exp(l-\Delta)}{(1-\exp(-\Delta))^N} \int_{x=l-\Delta}^{l+(N-1)\Delta} \overline{\gamma}(x)\exp(\Delta)\exp(-x)\rho(x-l+\Delta)dx \\
&= \exp(\Delta)\gamma(l-\Delta),
\end{aligned}$$

where the inequality follows from Lemma 2. □

Lemma 7. *If the direct allocation $q_i(s)$ is Nash implemented by a participation secure mechanism, profit is at most*

$$\sum_{s\in S} f(s) \sum_{i=1}^{N} q_i(s) \left[\gamma(\Sigma s) - \frac{1-F_i(s_i)}{f_i(s_i)}(\gamma(\Sigma s+\Delta) - \gamma(\Sigma s))\right]. \tag{1}$$

Proof of Lemma 7. This follows from standard revenue equivalence arguments: If we write $U_i(s_i, s_i')$ for the utility of a signal s_i that reports s_i', with $U_i(s_i) = U_i(s_i, s_i)$, then

$$U_i(s_i) \geq U_i(s_i, s_i') = U_i(s_i') + \sum_{s_{-i}\in S_{-i}} f_{-i}(s_{-i})q_i(s_i', s_{-i})\left(\gamma(s_i + \Sigma s_{-i}) - \gamma(s_i' + \Sigma s_{-i})\right).$$

Thus, for $s_i \geq \Delta$,

$$U_i(s_i) \geq U_i(0) + \sum_{k=0}^{s_i/\Delta - 1} \sum_{s_{-i}\in S_{-i}} f_{-i}(s_{-i})q_i(k\Delta, s_{-i})\left(\gamma((k+1)\Delta + \Sigma s_{-i}) - \gamma(k\Delta + \Sigma s_{-i})\right).$$

The expectation of $U_i(s_i)$ across s_i is therefore bounded below by

$$\begin{aligned}
&\sum_{s\in S} f(s) \sum_{k=0}^{s_i/\Delta - 1} q_i(k\Delta, s_{-i})\left(\gamma((k+1)\Delta + \Sigma s_{-i}) - \gamma(k\Delta + \Sigma s_{-i})\right) \\
&= \sum_{s\in S} f(s)q_i(s)(\gamma(\Sigma s + \Delta) - \gamma(\Sigma s))\frac{1-F_i(s_i)}{f_i(s_i)}.
\end{aligned}$$

The formula then follows from subtracting the bound on bidder surplus from total surplus. □

Let Π denote the maximum of the profit bound (1) across all q. Let $\tilde{\Pi}$ denote the profit bound when we set $q_1(s) = 1$ and $q_j(s) = 0$ for all $j \neq 1$.

Lemma 8. $\Pi \leq \tilde{\Pi} + (1 - (1 - \exp(-\Delta^{-1}))^N)\overline{v}$.

Proof of Lemma 8. When signals are all less than Δ^{-1}, the bidder-independent virtual value is

$$\begin{aligned}
&\gamma(l) - \frac{1}{\exp(\Delta) - 1}\left(\gamma(l+\Delta) - \gamma(l)\right) \\
&\geq \gamma(l) - \frac{\exp(-\Delta)}{1-\exp(-\Delta)}(\gamma(l)\exp(\Delta) - \gamma(l)) = 0,
\end{aligned}$$

where the inequality follows from Lemma 6. Thus, the virtual value is maximized pointwise by allocating with probability one to, say, bidder 1. With probability $1-(1-\exp(-\Delta^{-1}))^N$, one of the signals is above Δ^{-1}, in which case $\overline{v}$ is an upper bound on the virtual value. □

Lemma 9. $\lim_{\Delta\to 0} \tilde{\Pi} \leq \overline{\Pi}$.

Proof of Lemma 9. Plugging in $q_1 = 1$, we find that

$$\begin{aligned}
\tilde{\Pi} &= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1 \in S_1} \left[f_1(s_1)\gamma(\Sigma s) - \sum_{s_1' > s_1} f_1(s_1')(\gamma(\Sigma s + \Delta) - \gamma(\Sigma s)) \right] \\
&= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1 \in S_1} \left[f_1(s_1) \left[\gamma(\Sigma s) + \sum_{s_1' < s_1} (\gamma(s_1' + \Sigma s_{-1}) - \gamma(s_1' + \Sigma s_{-1} + \Delta)) \right]\right] \\
&= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1})\gamma(\Sigma s_{-1}).
\end{aligned}$$

Using the definition of γ, this is

$$\begin{aligned}
\tilde{\Pi} &= \frac{1}{1-\exp(-\Delta)} \int_{y=0}^{\Delta} \int_{x=0}^{\infty} \overline{\gamma}(x+y) g_{N-1}(x)\exp(-y)dxdy \\
&= \frac{1}{1-\exp(-\Delta)} \int_{x=0}^{\infty} \overline{\gamma}(x) \int_{y=0}^{\min\{x,\Delta\}} g_{N-1}(x-y)\exp(-y)dydx \\
&\leq \frac{1}{1-\exp(-\Delta)} \left[\int_{x=\Delta}^{\infty} \overline{\gamma}(x) \int_{y=0}^{\Delta} g_{N-1}(x-y)\exp(-y)dydx + G_N(\Delta)\overline{v} \right].
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\int_{y=0}^{\Delta} g_{N-1}(x-y)\exp(-y)dy &= \frac{x^{N-1} - (x-\Delta)^{N-1}}{(N-1)!}\exp(-x) \\
&\leq \frac{\Delta(N-1)x^{N-2}}{(N-1)!}\exp(-x) = \Delta g_{N-1}(x),
\end{aligned}$$

where we have used convexity of x^{N-1}. Thus,

$$\tilde{\Pi} \leq \frac{\Delta}{1-\exp(-\Delta)} \int_{x=0}^{\infty} \overline{\gamma}(x) g_{N-1}(x)dx + \frac{G_N(\Delta)\overline{v}}{1-\exp(-\Delta)}.$$

An application of L'Hôpital's rule shows that the last term converges to zero as $\Delta \to 0$ and $\Delta/(1-\exp(-\Delta)) \to 1$, this implies the lemma. □

Proof of Proposition 6. Combining Lemmas 7 and 8, we can pick Δ sufficiently small so that $\Pi \leq \tilde{\Pi} + \epsilon/2 \leq \overline{\Pi} + \epsilon$. This completes the proof of the proposition. $\square$

Note that every step of the proof of Proposition 6 goes through in the must-sell case, where we replace $\overline{w}$ with $\widehat{w}$, and we skip the step in Lemma 8 of proving that the discrete virtual value is non-negative.

C Proofs for Section 6

Proof of Lemma 9. The left-tail assumption could equivalently be stated as: there exists some $\overline{\alpha} > 0$ and $\varphi > 1$ such that for all $0 \le \alpha' < \alpha \le \overline{\alpha}$

$$H^{-1}(\alpha) - \underline{v} \le G_N^{-1}(\alpha)^{\varphi}$$

and if $\underline{v} > c$,

$$\frac{H^{-1}(\alpha) - c}{H^{-1}(\alpha') - c} \le \exp(G_N^{-1}(\alpha) - G_N^{-1}(\alpha')).$$

The following Lemma 10 implies that if the above two conditions hold for N, they hold for all $N' > N$ as well. $\square$

Lemma 10. *For any $N \ge 1$ and $N' > N$, there exists $\overline{\alpha} > 0$ such that $G_N^{-1}(\alpha) - G_N^{-1}(\alpha') \le G_{N'}^{-1}(\alpha) - G_{N'}^{-1}(\alpha')$ for all $0 \le \alpha' < \alpha \le \overline{\alpha}$.*

Proof of Lemma 10. Clearly it suffices to prove the lemma for $N' = N + 1$. Let us extend the definition of G_N to any real number N:

$$G_N(x) = \int_{y=0}^{x} e^{-y} \frac{y^{N-1}}{\Gamma(N)}\, dy,$$

where

$$\Gamma(N) = \int_{y=0}^{\infty} e^{-y} y^{N-1}\, dy.$$

(We have $\Gamma(N) = (N-1)!$ when $N \ge 1$ is an integer.)

By definition, we have

$$\int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{x^{N-1}}{\Gamma(N)}\, dx = \alpha.$$

Differentiating the above equation with respect to N gives:

$$\frac{\partial G_N^{-1}(\alpha)}{\partial N} \frac{e^{-G_N^{-1}(\alpha)} G_N^{-1}(\alpha)^{N-1}}{\Gamma(N)} + \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{\partial \left(\frac{x^{N-1}}{\Gamma(N)}\right)}{\partial N}\, dx = 0.$$

i.e.,

$$\begin{aligned}
\frac{\partial G_N^{-1}(\alpha)}{\partial N} =& \frac{\Gamma(N) e^{G_N^{-1}(\alpha)}}{G_N^{-1}(\alpha)^{N-1}} \left(- \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{\partial \left(\frac{x^{N-1}}{\Gamma(N)}\right)}{\partial N}\, dx \right) \\
=& \frac{e^{G_N^{-1}(\alpha)}}{\Gamma(N) G_N^{-1}(\alpha)^{N-1}} \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} [-x^{N-1} \log(x) \Gamma(N) + x^{N-1} \Gamma'(N)]\, dx \\
=& \frac{e^{G_N^{-1}(\alpha)}}{\Gamma(N)} f(G_N^{-1}(\alpha), N),
\end{aligned}$$

where
$$f(z,N) = \frac{1}{z^{N-1}} \int_{x=0}^{z} e^{-x}[-x^{N-1}\log(x)\Gamma(N) + x^{N-1}\Gamma'(N)]\,dx.$$

We compute:

$$\begin{aligned}\frac{\partial f(z,N)}{\partial z} =& \frac{1}{z^{2(N-1)}}\Bigg(z^{N-1}e^{-z}[-z^{N-1}\log(z)\Gamma(N) + z^{N-1}\Gamma'(N)] \\ & - (N-1)z^{N-2}\int_{x=0}^{z} e^{-x}[-x^{N-1}\log(x)\Gamma(N) + x^{N-1}\Gamma'(N)]\,dx \Bigg) \\ =& e^{-z}[-\log(z)\Gamma(N) + \Gamma'(N)] - (N-1)z^{-N}\int_{x=0}^{z} e^{-x}[-x^{N-1}\log(x)\Gamma(N) + x^{N-1}\Gamma'(N)]\,dx.\end{aligned}$$

For any $z \leq 1$, we have

$$\begin{aligned}\frac{\partial f(z,N)}{\partial z} \geq& e^{-z}[-\log(z)\Gamma(N) + \Gamma'(N)] - (N-1)z^{-N}\int_{x=0}^{z} [-x^{N-1}\log(x)\Gamma(N) + x^{N-1}\Gamma'(N)]\,dx \\ =& e^{-z}[-\log(z)\Gamma(N) + \Gamma'(N)] - (N-1)z^{-N}\left[\Gamma(N)\left(\frac{z^N}{N^2} - \frac{z^N\log z}{N}\right) + \Gamma'(N)\frac{z^N}{N}\right] \\ =& e^{-z}[-\log(z)\Gamma(N) + \Gamma'(N)] - \frac{N-1}{N}\left[\Gamma(N)\left(\frac{1}{N} - \log z\right) + \Gamma'(N)\right] \\ =& \left(e^{-z} - \frac{N-1}{N}\right)[-\log(z)\Gamma(N) + \Gamma'(N)] - \frac{N-1}{N^2}\Gamma(N).\end{aligned}$$

Since the last line goes to infinity as z goes to zero, for any fixed $N \geq 1$ we can choose $\overline{z} \in (0,1]$ such that $\partial f(z,\widehat{N})/\partial z \geq 0$ for all $z \in [0,\overline{z}]$ and $\widehat{N} \in [N, N+1]$. Let $\overline{\alpha} = G_{N+1}(\overline{z})$.

Suppose $0 \leq \alpha' < \alpha \leq \overline{\alpha}$. We have

$$[G_{N+1}^{-1}(\alpha) - G_{N+1}^{-1}(\alpha')] - [G_N^{-1}(\alpha) - G_N^{-1}(\alpha')] = \int_{\widehat{N}=N}^{N+1}\left(\frac{\partial G_{\widehat{N}}^{-1}(\alpha)}{\partial \widehat{N}} - \frac{\partial G_{\widehat{N}}^{-1}(\alpha')}{\partial \widehat{N}}\right)d\widehat{N}.$$

Since $d\left(e^z f(z,\widehat{N})/\Gamma(\widehat{N})\right)/dz \geq 0$ for all $z \in [0,\overline{z}]$ and $\widehat{N} \in [N, N+1]$, we have $\partial G_{\widehat{N}}^{-1}(\alpha)/\partial\widehat{N} - \partial G_{\widehat{N}}^{-1}(\alpha')/\partial\widehat{N} \geq 0$, which proves the lemma. □

Let us now define

$$\begin{aligned}G_N^C(x) &= G_N\left(\sqrt{N-1}x + N - 1\right); \\ g_N^C(x) &= \sqrt{N-1}\, g_N\left(\sqrt{N-1}x + N - 1\right).\end{aligned}$$

To prove Proposition 7, we first need a number of technical results.

Lemma 11. *As N goes to infinity, g_N^C and G_N^C converge pointwise to ϕ and Φ, respectively.*

Proof of Lemma 11. Note that

$$\begin{aligned} g^C_{N+1}(x) &= \sqrt{N} g_{N+1}(\sqrt{N}x + N) \\ &= \sqrt{N} \frac{(\sqrt{N}x + N)^N}{N!} \exp(-\sqrt{N}x - N). \end{aligned}$$

Stirling's Approximation says that

$$\lim_{N\to\infty} \frac{N!}{\sqrt{2\pi N} \left(\frac{N}{e}\right)^N} = 1.$$

Moreover, for all N, the ratio inside the limit is greater than 1.

Thus, when N is large, $g^C_{N+1}(x)$ is approximately

$$\frac{1}{\sqrt{2\pi}} \left(1 + \frac{x}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x),$$

and hence

$$\log(g^C_{N+1}(x)) \approx \log(1/\sqrt{2\pi}) + N \log\left(1 + \frac{x}{\sqrt{N}}\right) - \sqrt{N}x.$$

Using the mean-value formulation of Taylor's Theorem centered around 0, for every y, there exists a $z \in [0, y]$ such that

$$\log(1 + y) = y - \frac{y^2}{2} + \frac{1}{(1+z)^3} y^3.$$

Plugging in $y = x/\sqrt{N}$, we conclude that

$$\begin{aligned} \log(g^C_{N+1}(x)) &\approx \log(1/\sqrt{2\pi}) + N\frac{x}{\sqrt{N}} - N\frac{1}{2}\left(\frac{x}{\sqrt{N}}\right)^2 + N\frac{1}{(1+z)^3}\left(\frac{x}{\sqrt{N}}\right)^3 - \sqrt{N}x \\ &= \log(1/\sqrt{2\pi}) - \frac{1}{2}x^2 + \frac{1}{(1+z)^3}\frac{x^3}{\sqrt{N}}, \end{aligned}$$

which converges to $\log(1/\sqrt{2\pi}) - \frac{1}{2}x^2$ as N goes to infinity, so $g^C_{N+1}(x)$ converges to $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Pointwise convergence of G^C_N to Φ follows from Scheffé's lemma. □

Let us define

$$\tilde{g}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) & \text{if } x < 0; \\ \frac{1}{\sqrt{2\pi}}(1+x)\exp(-x) & \text{otherwise.} \end{cases}$$

Lemma 12. *The function $\tilde{g}(x)|x|$ is integrable, and for all N and x, $|g^C_N(x)| \leq \tilde{g}(x)$.*

Proof of Lemma 12. Note that

$$\int_{x=-\infty}^{\infty} \tilde{g}(x)|x|dx = \int_{x=-\infty}^{0} \phi(x)|x|dx + \frac{1}{\sqrt{2\pi}} \int_{x=0}^{\infty} (1+x)x\exp(-x)dx,$$

which is clearly finite, since the half-normal distribution has finite expectation.

Next, Stirling's Approximation implies that

$$g_{N+1}^C(x) \le \frac{1}{\sqrt{2\pi}} \left(1 + \frac{x}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x) \equiv \tilde{g}_N(x).$$

Now,

$$\frac{d}{dN}\log(\tilde{g}_N(x)) = \log\left(1 + \frac{x}{\sqrt{N}}\right) - \frac{1}{2}\frac{x}{\sqrt{N}+x} - \frac{x}{2\sqrt{N}},$$

which is clearly zero when $x = 0$, and

$$\begin{aligned}\frac{d}{dx}\frac{d}{dN}\log(\tilde{g}_N(x)) &= \frac{1}{\sqrt{N}+x} - \frac{\sqrt{N}}{2(\sqrt{N}+x)^2} - \frac{1}{2\sqrt{N}} \\ &= \frac{2N + 2\sqrt{N}x}{2\sqrt{N}(\sqrt{N}+x)^2} - \frac{N}{2\sqrt{N}(\sqrt{N}+x)^2} - \frac{N + 2\sqrt{N}x + x^2}{2\sqrt{N}(\sqrt{N}+x)^2} \\ &= \frac{-x^2}{2\sqrt{N}(\sqrt{N}+x)^2},\end{aligned}$$

which is non-positive and strictly negative when $x \neq 0$. As a result, $\tilde{g}_N(x)$ is increasing in N when $x < 0$ and decreasing in N when $x > 0$. Since it converges to $\phi(x)$ in the limit as N goes to infinity, we conclude that for $x < 0$, $g_{N+1}^C(x) \le \tilde{g}_N(x) \le \phi(x) = \tilde{g}(x)$, and for $x > 0$, $g_{N+1}^C(x) \le \tilde{g}_N(x) \le \tilde{g}_1(x) = \tilde{g}(x)$ as desired. □

Lemma 13. *As N goes to infinity,* $\widehat{\gamma}_N^C$ *converges almost surely to* $\widehat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x))$ *and* $\widehat{\Gamma}_N^C$ *converges pointwise to*

$$\widehat{\Gamma}_\infty^C(x) = \int_{y=-\infty}^{x} \widehat{\gamma}_\infty^C(y)\phi(y)dy.$$

The latter convergence is uniform on any bounded interval.

Proof of Lemma 13. Note that $\widehat{\gamma}_N^C(x) = H^{-1}(G_N^C(x)) - c$. By Lemma 11, $G_N^C(x)$ converges to $\Phi(x)$ pointwise. Thus, if H^{-1} is continuous at $\Phi(x)$, then as N goes to infinity, we must have $\widehat{\gamma}_N^C(x) \to H^{-1}(\Phi(x)) - c = \widehat{\gamma}_\infty^C(x)$. Since H^{-1} is monotonic, the set of discontinuities has Lebesgue measure zero, so that the pointwise convergence is almost everywhere.

Pointwise convergence of $\widehat{\Gamma}_N^C$ follows from almost sure convergence of $\widehat{\gamma}_N^C$, combined with the fact that $\widehat{\gamma}_N^C$ is uniformly bounded by $|\overline{v}|$, so that we can apply the dominated convergence theorem. Moreover, $\widehat{\Gamma}_N^C(x)$ is uniformly Lipschitz continuous across N and x. As a result, the family $\{\widehat{\Gamma}_N^C(\cdot)\}_{N=2}^\infty$ is uniformly bounded and uniformly equicontinuous. The conclusion about uniform convergence is then a consequence of the Arzela-Ascoli theorem. □

Recall that x^* is the largest solution to $\widehat{\Gamma}^C_\infty(x^*) = 0$ (which may be $-\infty$). Also, let us define x_N so that $\overline{\Gamma}^C_N$ has a graded interval $[-\sqrt{N-1}, x_N]$. (If there is no graded interval with left end point $-\sqrt{N-1}$, then we let $x_N = -\sqrt{N-1}$.)

Lemma 14. *As N goes to infinity, x_N converges to x^*.*

Proof of Lemma 14. By a change of variables $y = (G^C_N)^{-1}(\Phi(x))$, we conclude that

$$\widehat{\Gamma}^C_\infty(x^*) = \int_{x=-\infty}^{x^*} \widehat{\gamma}^C_\infty(x)\phi(x)dx = \int_{x=-\sqrt{N-1}}^{(G^C_N)^{-1}(\Phi(x^*))} \widehat{\gamma}^C_N(x) g^C_N(x)dx = \widehat{\Gamma}^C_N\left((G^C_N)^{-1}(\Phi(x^*))\right).$$

This integral must be zero by the definition of x^*, so that $x_N \geq (G^C_N)^{-1}(\Phi(x^*))$. Since the latter converges to x^* as $N \to \infty$, we conclude $\liminf_{N\to\infty} x_N \geq x^*$.

Next, recall that x_{N+1} solves the equation

$$\begin{aligned}
\widehat{\Gamma}^C_{N+1}(x_{N+1}) &= \widehat{\gamma}^C_{N+1}(x_{N+1}) \int_{x=-\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N}(x - x_{N+1})) g^C_{N+1}(x)dx \\
&= \widehat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N}x_{N+1} - N) \int_{x=-\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N}x + N) g^C_{N+1}(x)dx \\
&= \widehat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N}x_{N+1} - N) \int_{x=-\sqrt{N}}^{x_{N+1}} \sqrt{N}\frac{(\sqrt{N}x + N)^N}{N!}dx \\
&\leq \overline{v} \exp(-\sqrt{N}x_{N+1} - N) \frac{(\sqrt{N}x_{N+1} + N)^{N+1}}{(N+1)!} \\
&= \overline{v} g^C_{N+2}\left(\sqrt{\frac{N}{N+1}}x_{N+1} - \frac{1}{\sqrt{N+1}}\right)\frac{1}{\sqrt{N+1}} \\
&\leq \overline{v}\tilde{g}\left(\sqrt{\frac{N}{N+1}}x_{N+1} - \frac{1}{\sqrt{N+1}}\right)\frac{1}{\sqrt{N+1}},
\end{aligned}$$

where we have used Lemma 12. The last line converges to zero pointwise, so $\widehat{\Gamma}^C_N(x_N)$ must converge to zero as well.

Now, if $z = \limsup_{N\to\infty} x_N > x^*$, then since $\widehat{\Gamma}^C_\infty(z) > \widehat{\Gamma}^C_\infty(x^*) = 0$, we would contradict our earlier finding that $\widehat{\Gamma}^C_N(x_N) \to 0$. Thus, $\limsup_{N\to\infty} x_N \leq x^*$, so x_N must converge to x^* as N goes to ∞. □

Lemma 15. *For every $\epsilon > 0$, there exists $\widehat{N}$ such that for all $N > \widehat{N}$, there exists an $x \in [x^* + \epsilon, x^* + 2\epsilon]$ at which $\overline{\gamma}^C_N$ is not graded.*

Proof of Lemma 15. Suppose not. Then there exist infinitely many N such that for every $x \in [x^* + \epsilon, x^* + 2\epsilon]$, $\overline{\gamma}^C_{N+1}(x) = \exp(\sqrt{N}(x - \tilde{x}))\widehat{\gamma}^C_{N+1}(\tilde{x})$ for some $\tilde{x} \geq x^* + 2\epsilon$. Thus, for all $x \leq x^* + \epsilon$, we conclude that

$$\overline{\gamma}^C_{N+1}(x) \leq \overline{\gamma}^C_{N+1}(x^* + \epsilon) \leq \exp(-\sqrt{N}\epsilon)\overline{v}$$

which converges to zero as N goes to infinity. This implies that $\liminf_{N\to\infty} \overline{\Gamma}^C_{N+1}(x^*+\epsilon) = 0$. But $\overline{\Gamma}^C_{N+1}(x^*+\epsilon)$ must be weakly larger than $\widehat{\Gamma}^C_{N+1}(x^*+\epsilon)$, so

$$0 = \liminf_{N\to\infty} \overline{\Gamma}^C_{N+1}(x^*+\epsilon) \geq \liminf_{N\to\infty} \widehat{\Gamma}^C_{N+1}(x^*+\epsilon) = \widehat{\Gamma}^C_{\infty}(x^*+\epsilon) > 0,$$

a contradiction. □

Lemma 16. *As N goes to infinity, $\overline{\gamma}^C_N$ converges almost surely to*

$$\overline{\gamma}^C_\infty(x) = \begin{cases} 0 & \text{if } x < x^*; \\ \widehat{\gamma}^C_\infty(x) & \text{if } x \geq x^*. \end{cases}$$

Proof of Lemma 16. Let $x < x^*$. Since $x_N \to x^*$ by Lemma 14, for N sufficiently large, $x_N > (x^*+x)/2$. Since $\overline{\gamma}^C_N(x)$ is graded on $(-\infty, x_N]$, it is graded at x, and

$$\begin{aligned} \overline{\gamma}^C_N(x) &= \exp(\sqrt{N-1}(x-x_N))\overline{\gamma}^C_N(x_N) \\ &\leq \exp(\sqrt{N-1}(x-x^*)/2)\overline{v}. \end{aligned}$$

The last line clearly converges to zero pointwise. Since $\overline{\gamma}^C_N(x) \geq 0$ for all N, we conclude that $\overline{\gamma}^C_N(x) \to 0$.

Now consider $x > x^*$ at which $\widehat{\gamma}^C_\infty$ is continuous. Take ϵ so that $x > x^* + 2\epsilon$ and so that $\widehat{\gamma}^C_\infty$ is continuous at $x^*+\epsilon$. Lemma 15 says that there is a $\widehat{N}$ such that for all $N > \widehat{N}$, there exists a point in $[x^*+\epsilon, x^*+2\epsilon]$ at which the gains function is not graded. Moreover, since $\widehat{\gamma}^C_N(x^*+\epsilon)$ converges to $\widehat{\gamma}^C_\infty(x^*+\epsilon)$, we can pick $\widehat{N}$ large enough and find a constant $\underline{\gamma} > 0$ such that for $N > \widehat{N}$, $\widehat{\gamma}^C_N(x^*+\epsilon) \geq \underline{\gamma}$.

Now, suppose that $\overline{\gamma}^C_N$ is graded at x, with x in a graded interval $[a,b]$. Then $a \geq x^*+\epsilon$, and hence $\widehat{\gamma}^C_N(a) \geq \widehat{\gamma}^C_N(x^*+\epsilon) \geq \underline{\gamma}$. Recall that on $[a,b]$,

$$\overline{\gamma}^C_N(x) = \widehat{\gamma}^C_N(a)\exp(\sqrt{N-1}(x-a)).$$

Since $\widehat{\gamma}^C_N$ is bounded above by $\overline{v}$, it must be that $\widehat{\gamma}^C_N(a)\exp(\sqrt{N-1}(b-a)) \leq \overline{v}$, so

$$\begin{aligned} b - a &\leq \frac{1}{\sqrt{N-1}} \log\left(\frac{\overline{v}}{\widehat{\gamma}^C_N(a)}\right) \\ &\leq \frac{1}{\sqrt{N-1}} \log\left(\frac{\overline{v}}{\underline{\gamma}}\right) = \epsilon_N. \end{aligned}$$

Thus,

$$\widehat{\gamma}^C_N(x-\epsilon_N) \leq \overline{\gamma}^C_N(x) \leq \widehat{\gamma}^C_N(x+\epsilon_N).$$

This was true if $\overline{\gamma}^C_N(x)$ is graded at x, but clearly the inequality is also true if it is not graded at x, in which case $\overline{\gamma}^C_N(x) = \widehat{\gamma}^C_N(x)$. Now, $\widehat{\gamma}^C_N(x) = \widehat{\gamma}^C_\infty(\Phi^{-1}(G^C_N(x)))$, so

$$\widehat{\gamma}^C_\infty(\Phi^{-1}(G^C_N(x-\epsilon_N))) \leq \overline{\gamma}^C_N(x) \leq \widehat{\gamma}^C_\infty(\Phi^{-1}(G^C_N(x+\epsilon_N))).$$

As $N \to \infty$, the left and right hand sides converge to $\widehat{\gamma}^C_\infty(x)$ from the left and right, respectively. Since $\widehat{\gamma}^C_\infty$ is continuous at x, we conclude that $\overline{\gamma}^C_N(x) \to \widehat{\gamma}^C_\infty(x)$. The lemma follows from the fact that the monotonic function $\widehat{\gamma}^C_\infty$ is continuous almost everywhere. □

Proof of Proposition 7. We argue that

$$Z_{N+1} = \sqrt{N} \int_{x=0}^{\infty} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx$$

converges to a positive constant as N goes to infinity. Since this is $\sqrt{N}$ times the difference between ex ante gains from trade and profit, this proves the result.

To that end, observe that

$$Z_{N+1} = \sqrt{N} \int_{x=0}^{N/2} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx + \int_{x=-\sqrt{N}/2}^{\infty} \overline{\gamma}^C_{N+1}(x) g^C_{N+1}(x) \frac{Nx}{\sqrt{N}x + N} dx.$$

We claim that the first integral converges to zero as $N \to \infty$. Note that $g_{N+1}(x) \leq g_N(x)$ if and only if $x \leq N$. Therefore,

$$\begin{aligned}
\left| \sqrt{N} \int_{x=0}^{N/2} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx \right| &\leq (\overline{v} + c)\sqrt{N} \int_{x=0}^{N/2} (g_N(x) - g_{N+1}(x))dx \\
&= (\overline{v} + c)\sqrt{N}(G_N(N/2) - G_{N+1}(N/2)) \\
&= (\overline{v} + c)\sqrt{N} g_{N+1}(N/2) \\
&= (\overline{v} + c)\sqrt{N} \frac{(N/2)^N \exp(-N/2)}{N!} \\
&\approx (\overline{v} + c)\sqrt{N} \frac{(N/2)^N \exp(-N/2)}{\sqrt{2\pi N}(N/e)^N} \\
&= (\overline{v} + c)\frac{1}{\sqrt{2\pi}} \exp(-N(\log(2) - 1/2)),
\end{aligned}$$

where we have again used Stirling's Approximation between the third-to-last and second-to-last lines. The last line converges to zero as N goes to infinity.

Now consider the second integral in the formula for Z_{N+1}. By Lemma 12, the integrand is bounded above in absolute value by the integrable function $\overline{v}\tilde{g}(x)|x|$. Moreover, from Lemmas 11 and 16, we know that the integrand converges pointwise to $\overline{\gamma}^C_{\infty}(x)\phi(x)x$. The dominated convergence theorem then implies that as N goes to infinity, Z_N converges to

$$\int_{x=-\infty}^{\infty} \overline{\gamma}^C_{\infty}(x)\phi(x)x dx,$$

which is strictly positive because $\overline{\gamma}^C_{\infty}$ is strictly increasing.

The proof goes through for the must-sell guarantee, if we replace $\overline{\gamma}^C_N$ with $\widehat{\gamma}^C_N$. □

To prove Proposition 9, we need a few more intermediate results. Let $\overline{G}_N(x) = G_N(Nx)$ be the cumulative distribution for the mean of N independent standard exponential random variables. Define $\overline{F}_N(x) = \exp(N(1 - x + \log(x)))$. Clearly, $\overline{F}_N(x)$ is a cumulative distribution for $x \in [0, 1]$, $\overline{F}_N(0) = 0$ and $\overline{F}_N(1) = 1$. Finally, define the function $D_N(\alpha)$:

$$D_N(\alpha) = \begin{cases} \frac{1}{\overline{F}_N^{-1}(\alpha)} & \text{if } \alpha \in [0, 0.4]; \\ 1.1 & \text{if } \alpha \in (0.4, 1]. \end{cases}$$

The choices of 0.4 and 1.1 in $D_N(\alpha)$ are arbitrary: any numbers work that are less than 1/2 and more than 1, respectively.

Lemma 17. *When $\widehat{N}$ is sufficiently large, $\overline{\mu}_N(G_N^{-1}(\alpha)) \leq D_{\widehat{N}}(\alpha)$ for all $N \geq \widehat{N}$ and $\alpha \in [0,1]$.*

Proof of Lemma 17. We first apply the theory of large deviations to the exponential distribution. Let $\Lambda(t)$ be the logarithmic moment generating function for the exponential distribution:

$$\Lambda(t) = \log\left(\int_{x=0}^{\infty} \exp(xt - x)\,dx\right) = \begin{cases} \infty & \text{if } t \geq 1; \\ -\log(1-t) & \text{if } t < 1. \end{cases}$$

Let $\Lambda^*(x)$ be the Legendre transform of $\Lambda(t)$:

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}}\{xt - \Lambda(t)\} = \begin{cases} \infty & x \leq 0, \\ x - 1 - \log x & x > 0. \end{cases}$$

Cramér's theorem (or the Chernoff bound; see Theorem 1.3.12 in Stroock, 2011) then states that for any N,

$$\overline{G}_N(x) \leq \exp(-N\Lambda^*(x)) = \overline{F}_N(x)$$

for every $x \in [0,1]$; or equivalently, $\overline{F}_N^{-1}(\alpha) \leq \overline{G}_N^{-1}(\alpha)$ for every $\alpha \in [0, \overline{G}_N(1)]$.

By the law of large numbers, when $\widehat{N}$ is sufficiently large, we have $\overline{G}_N(1) \geq 0.4$ and $1/\overline{G}_N^{-1}(0.4) \leq 1.1$ and for all $N \geq \widehat{N}$. The claim of the lemma then follows from two cases:

If $\alpha \in [0, 0.4]$, then we have

$$\overline{\mu}_N(G_N^{-1}(\alpha)) \leq \frac{N}{G_N^{-1}(\alpha)} = \frac{1}{\overline{G}_N^{-1}(\alpha)} \leq \frac{1}{\overline{F}_N^{-1}(\alpha)} \leq \frac{1}{\overline{F}_{\widehat{N}}^{-1}(\alpha)} = D_{\widehat{N}}(\alpha),$$

where we have used the bound $\overline{\mu}_N(x) \leq N/x$ (equation (21)), and the facts that $\overline{G}_N(1) \geq 0.4$ when $N \geq \widehat{N}$ (so $\overline{F}_N^{-1}(\alpha) \leq \overline{G}_N^{-1}(\alpha)$ for $\alpha \leq 0.4 \leq \overline{G}_N(1)$) and that $\overline{F}_N(x) \leq \overline{F}_{\widehat{N}}(x)$ for all $N \geq \widehat{N}$ and $x \in [0,1]$ (so $\overline{F}_{\widehat{N}}^{-1}(\alpha) \leq \overline{F}_N^{-1}(\alpha)$ for all α).

If $\alpha \in (0.4, 1]$, then

$$\overline{\mu}_N(G_N^{-1}(\alpha)) \leq \frac{1}{\overline{G}_N^{-1}(\alpha)} \leq \frac{1}{\overline{G}_N^{-1}(0.4)} \leq 1.1 = D_{\widehat{N}}(\alpha),$$

since $\overline{G}_N^{-1}(\alpha)$ is increasing in α, and $1/\overline{G}_N^{-1}(0.4) \leq 1.1$ when $N \geq \widehat{N}$. □

Lemma 18. *When N is sufficiently large,*

$$\int_{\alpha=0}^{1} D_N(\alpha)\,dH^{-1}(\alpha) < \infty.$$

Proof of Lemma 18. Since $G_N(x) = 1 - \sum_{k=1}^{N} g_k(x)$, we have:

$$\begin{aligned}\overline{G}_N(x) &= 1 - \sum_{k=1}^{N} \exp(-Nx)\frac{(Nx)^{k-1}}{(k-1)!} \\ &= 1 - \exp(-Nx)\left(\exp(Nx) - \sum_{k=N}^{\infty}\frac{(Nx)^k}{k!}\right) \geq \exp(-Nx)\frac{(Nx)^N}{N!}.\end{aligned}$$

Clearly, there exists an $\overline{x} \in (0,1)$ such that

$$\overline{F}_{N+1}(x) = \exp((N+1)(1-x))x^{N+1} \leq \exp(-Nx)\frac{(Nx)^N}{N!} \leq \overline{G}_N(x)$$

for all $x \in [0, \overline{x}]$. We therefore have $D_{N+1}(\alpha) = 1/\overline{F}_{N+1}^{-1}(\alpha) \leq 1/\overline{G}_N^{-1}(\alpha)$ for all $\alpha \in [0, \overline{\alpha}]$, where $\overline{\alpha} = \min\{\overline{F}_{N+1}(\overline{x}), 0.4\}$. As a result,

$$\int_{\alpha=0}^{1} D_{N+1}(\alpha)\, dH^{-1}(\alpha) \leq \int_{\alpha=0}^{\overline{\alpha}} \frac{1}{\overline{G}_N^{-1}(\alpha)}\, dH^{-1}(\alpha) + \int_{\alpha=\overline{\alpha}}^{1} \max\left(\frac{1}{\overline{F}_{N+1}^{-1}(\overline{\alpha})}, 1.1\right) dH^{-1}(\alpha) < \infty$$

whenever we have

$$\int_{\alpha=0}^{1} \frac{1}{\overline{G}_N^{-1}(\alpha)}\, dH^{-1}(\alpha) = \int_{x=0}^{\infty} \frac{N}{x}\, d\widehat{w}_N(x) < \infty.$$

Finiteness of the last integral follows from part one of the left-tail assumption. □

Lemma 19. *Suppose* $\lim_{N\to\infty} y_N \in (-\infty, \infty)$. *Then* $\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y_N + N) = 1$.

Proof of Lemma 19. We first argue that for almost every y, $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ tends to 1 as $N \to \infty$. For this we recall x^* and x_N from Lemmas 14–16.

Consider first $y < x^*$. By Lemma 14, for N sufficiently large, the gains function is graded at y, and hence

$$\overline{\mu}_{N+1}(\sqrt{N}y + N) = C(0, \sqrt{N}x_{N+1} + N) = \frac{N+1}{\sqrt{N}x_{N+1} + N}.$$

Since we have already shown that $x_N \to x^*$ (Lemma 14), we conclude that $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ goes to 1.

Now consider $y > x^*$ at which $\widehat{\gamma}_\infty^C$ is continuous. If the gains function is not graded at y, then $\overline{\mu}_{N+1}(\sqrt{N}y + N) = N/(\sqrt{N}y + N)$. If the gains function is graded at y, then the length of the graded interval $[a, b] \ni y$ in the central limit units is less than $\epsilon_N = \overline{v}/(\underline{\gamma}\sqrt{N})$ for some $\underline{\gamma} > 0$ independent of N (see Lemma 16). Since $\overline{\mu}$ is decreasing (Lemma 3), we have

$$\frac{N}{\sqrt{N}(y + \epsilon_N) + N} \leq \overline{\mu}_{N+1}(\sqrt{N}y + N) \leq \frac{N}{\sqrt{N}(y - \epsilon_N) + N},$$

since $\lim_{z \nearrow a} \overline{\mu}_{N+1}(\sqrt{N}z + N) = N/(\sqrt{N}a + N)$ and $\lim_{z \searrow b} \overline{\mu}_{N+1}(\sqrt{N}z + N) = N/(\sqrt{N}b + N)$. As a result, $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ is squeezed to 1 as N goes to infinity.

We conclude that $\overline{\mu}_{N+1}(\sqrt{N}y+N)$ goes to 1 for $y > x^*$ at which $\widehat{\gamma}_\infty^C$ is continuous. Since $\widehat{\gamma}_\infty^C(y)$ is a monotone function of y, it is continuous at almost every y, so the convergence $\overline{\mu}_N \to 1$ is almost everywhere.

Finally, suppose $\lim_{N\to\infty} y_N = y \in (-\infty, \infty)$. Choose y' and y'' such that $y \in (y', y'')$ and such that

$$\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y' + N) = 1 = \lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y'' + N).$$

When N is sufficiently large, we have $y_N \in (y', y'')$, so

$$\overline{\mu}_{N+1}(\sqrt{N}y'' + N) \le \overline{\mu}_{N+1}(\sqrt{N}y_N + N) \le \overline{\mu}_{N+1}(\sqrt{N}y' + N).$$

Taking the limit as $N \to \infty$, we conclude $\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y_N + N) = 1$. □

Proof of Proposition 9. We first prove that

$$\lim_{N\to\infty} \overline{\lambda}_N(v; H) \to v - c \tag{2}$$

for every $v \in [\underline{v}, \overline{v}]$.

Replacing $\overline{\mu}_N$ by 1 in equation (18), the definition of $\overline{\lambda}_N(v; H)$, we have

$$\begin{aligned}\overline{\Pi}_N(H) + \int_{y=0}^{\infty} G_N(y)\, d\widehat{w}_N(y) - \int_{\nu=v}^{\overline{v}} d\nu &= \overline{\Pi}_N(H) + \left(\overline{v} - \int_{y=0}^{\infty} g_N(y)\widehat{w}_N(y)\, dy\right) - (\overline{v} - v) \\ &= \overline{\Pi}_N(H) - \int_{v'=\underline{v}}^{\overline{v}} v'\, dH(v') + v.\end{aligned}$$

Since by Proposition 7 $\lim_{N\to\infty} \overline{\Pi}_N(H) \to \int_{v'=\underline{v}}^{\overline{v}} v'\, dH(v') - c$, to prove (2), it suffices to prove that

$$\lim_{N\to\infty} \int_{y=0}^{\infty} |1 - \overline{\mu}_N(y)|\, d\widehat{w}_N(y) = 0.$$

Changing variables, we can rewrite the above equation as:

$$\lim_{N\to\infty} \int_{\alpha=0}^{1} |1 - \overline{\mu}_N(G_N^{-1}(\alpha))|\, dH^{-1}(\alpha) = 0. \tag{3}$$

We note that Stieltjes integration with respect to $dH^{-1}(\alpha)$ is equivalent to a Lebesgue integration with respect to the finite measure ω on $[0, 1]$ satisfying $\omega([s, t)) = H^{-1}(t) - H^{-1}(s)$, $0 \le s \le t \le 1$, and $\omega(\{1\}) = 0$. Part one of the left-tail assumption implies that

$$\omega(\{0\}) = \lim_{\alpha\to 0} \omega([0, \alpha)) = \lim_{\alpha\to 0} H^{-1}(\alpha) - H^{-1}(0) \le \lim_{\alpha\to 0} G_N^{-1}(\alpha)^{\varphi} = 0$$

for some $\varphi > 1$. Therefore, $\omega(\{0, 1\}) = 0$.

The central limit theorem implies that $\lim_{N\to\infty}(G_N^{-1}(\alpha) - (N-1))/\sqrt{N-1} = \Phi^{-1}(\alpha)$ for every $\alpha \in (0, 1)$. Therefore, Lemma 19 implies $\lim_{N\to\infty} \overline{\mu}_N(G_N^{-1}(\alpha)) = 1$ for every $\alpha \in (0, 1)$. Moreover, Lemmas 17 and 18 imply that there exists a $\widehat{N}$ such that for all $N \ge \widehat{N}$, the integrand $|1 - \overline{\mu}_N(G_N^{-1}(\alpha))|$ in (3) is dominated by $1 + D_{\widehat{N}}(\alpha)$ which is integrable with respect to ω. Therefore, equation (3) follows from the dominated convergence theorem, from which equation (2) follows.

Finally, using the definition of $\overline{\lambda}_N(v; H)$, we have

$$\overline{\lambda}_N(v;H) \leq \overline{\Pi}_N(H) + \int_{y=0}^{\infty} \overline{\mu}_N(y)(1+G_N(y))\, d\widehat{w}_N(y) \leq (\overline{v}-c) + 2\int_{\alpha=0}^{1} D_{\widehat{N}}(\alpha)\, dH^{-1}(\alpha) < \infty,$$

for all $v \in [\underline{v}, \overline{v}]$ and $N \geq \widehat{N}$, where the last two inequalities follow from Lemmas 17 and 18, respectively. Thus

$$\lim_{N\to\infty} \int_V \overline{\lambda}_N(v;H)\, dH'(v) = \int_V v\, dH'(v) - c$$

follows the dominated convergence theorem using (2).

The proof for the must-sell $\widehat{\lambda}_N(v;H)$ is identical, after replacing $\overline{\mu}_N(x)$ with $\widehat{\mu}_N(x) = (N-1)/x$ and $\overline{\Pi}_N(H)$ with $\widehat{\Pi}_N(H)$. □

Lemma 20. *Suppose the condition on H in Lemma 10 holds. For any $\epsilon > 0$, there exists an $\widehat{N}$ such that for all $N > \widehat{N}$, we have*

$$\widehat{\gamma}_N(x) \leq \widehat{\gamma}_N(y) \exp(x - y).$$

for all $x \geq y$ such that $\widehat{\gamma}_N(y) \geq \epsilon$.

Proof of Lemma 20. The condition on H implies that the support of H has no gap on $[\underline{v}, \overline{v}]$, so H^{-1} is continuous on $[0,1]$. We can partition $[0,1]$ into a countable collection of intervals $\{[\alpha_i, \beta_i] : i \in I\}$ such that $\alpha_i < \beta_i$, and either H^{-1} is strictly increasing on $[\alpha_i, \beta_i]$, or H^{-1} is constant on $[\alpha_i, \beta_i]$ (i.e., H has a mass point at v, where $v = H^{-1}(p)$ for all $p \in [\alpha_i, \beta_i]$). If H^{-1} is strictly increasing on $[\alpha_i, \beta_i]$, then

$$H^{-1}(q) - H^{-1}(p) \leq \frac{q-p}{C}. \tag{4}$$

for any $p, q \in (\alpha_i, \beta_i)$ such that $p \leq q$, since in this case we have $H(H^{-1}(q)) = q$ and $H(H^{-1}(p)) = p$. By continuity of H^{-1} we can extend (4) to any $p, q \in [\alpha_i, \beta_i]$ such that $p \leq q$.

If H^{-1} is constant on $[\alpha_i, \beta_i]$, then clearly (4) also holds for any $p, q \in [\alpha_i, \beta_i]$ such that $p \leq q$. Since $\{[\alpha_i, \beta_i] : i \in I\}$ is a partition of $[0,1]$, we conclude that (4) holds for any $p, q \in [0,1]$ such that $p < q$.

With the substitution $q = G_N^C(x)$ and $p = G_N^C(y)$, with $x > y$, equation (4) becomes

$$\widehat{\gamma}_N^C(x) - \widehat{\gamma}_N^C(y) \leq \frac{G_N^C(x) - G_N^C(y)}{C}.$$

Thus,

$$\frac{\widehat{\gamma}_N^C(x)}{\widehat{\gamma}_N^C(y)} \leq 1 + \frac{1}{\widehat{\gamma}_N^C(y)} \frac{G_N^C(x) - G_N^C(y)}{C}.$$

The log-1 Lipschitz condition that we want to prove is equivalent to

$$\frac{\widehat{\gamma}_N^C(x)}{\widehat{\gamma}_N^C(y)} \leq \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))).$$

Thus, it is sufficient to show that for large N,

$$1 + \frac{1}{\widehat{\gamma}_N^C(y)} \frac{G_N^C(x) - G_N^C(y)}{C} \leq \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))).$$

Both sides are equal to one when $x = y$, and the derivatives of the left- and right-hand sides with respect to x are, respectively

$$\frac{g_N^C(x)}{\widehat{\gamma}_N^C(y)C}, \tag{5}$$

and

$$\begin{aligned} &\frac{g_N^C(x)}{g_N(G_N^{-1}(G_N^C(x)))} \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))) \\ &= \sqrt{N-1} \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))) \geq \sqrt{N-1}. \end{aligned} \tag{6}$$

We now show that (5) is always less than (6). Note that g_N attains its maximum when $g_N = g_{N-1}$, i.e., when $x = N-1$, at a value of $\frac{(N-1)^{N-1}}{(N-1)!} \exp(-(N-1))$. Multiplied by $\sqrt{N-1}$, this upper bound converges to $\phi(0)$. Hence, when N is sufficiently large, $g_N^C(x) \leq 2\phi(0)$ for all x. Since $\widehat{\gamma}_N^C(z) > 0$, then there is an N large enough such that

$$\frac{g_N^C(x)}{\widehat{\gamma}_N^C(y)C} \leq \frac{2\phi(0)}{\epsilon C} \leq \sqrt{N-1}$$

which proves the lemma. □

Proof of Lemma 10. If $\underline{v} > c$, then we can take $\epsilon = \underline{v} - c$ in the statement of Lemma 20, in which case the statement of the Lemma follows immediately.

If $\underline{v} < c$, then $\widehat{\gamma}_N^C(-\sqrt{N-1}) < 0$, so that $\widehat{\Gamma}_N^C(x)$ is non-positive for x close to $-\sqrt{N-1}$. Hence, there must be a graded interval at the bottom of the form $[-\sqrt{N-1}, x_N]$. By Lemma 14, x_N converges to x^*. Moreover, by Lemma 16, $\overline{\gamma}_N^C$ converges almost surely to $\overline{\gamma}_\infty^C$. Thus, there exists an $\widehat{N}$ such that for all $N > \widehat{N}$, $\widehat{\gamma}_N^C(x_N) \geq \epsilon$. If we take $\epsilon = \widehat{\gamma}_\infty^C(x^*)/2$ in Lemma 20, then there exists a $\widehat{N}' \geq \widehat{N}$ so that for all $N > \widehat{N}'$, the log-1 Lipschitz condition is satisfied for all $x \geq x_N$. This implies that there is exactly one graded interval, and the conclusion of the Lemma follows. □

Proof of Proposition 10. We first derive the allocation. When $\underline{v} > c$, we have $x^* = -\infty$ and the gains function $\overline{\gamma}$ is not graded when N is sufficiently large. In this case $\overline{Q}_N^C(x)$ is always exactly 1.

When $\underline{v} < c$, $x^* \in (-\infty, \infty)$, and the gains function $\overline{\gamma}$ is single crossing (Section 4.4) when N is sufficiently large. Then $\overline{Q}_N^C(x) = \min((x\sqrt{N} + N)/(x_N\sqrt{N} + N), 1)$. Since x_N converges to x^* as defined by equation (29), $\overline{Q}_N^C(x)$ converges to 1 as $N \to \infty$.

We now derive the transfer. From Lemma 10, we know that there is at most one graded interval of the form $[-\sqrt{N}, x_N]$, where $x_N = -\sqrt{N}$ if $\underline{v} > c$ and $x_N > -\sqrt{N}$ if $\underline{v} < c$.

Recall that

$$\overline{T}_N(x) = \frac{1}{g_N(x)} \int_{y=0}^{x} \overline{\Xi}_N(y) g_N(y)\, dy,$$

$$\overline{\Xi}_N(x) = \overline{\mu}_N(x)\widehat{w}_N(x) - \overline{\lambda}_N(\widehat{w}_N(x)) - c\overline{Q}_N(x),$$

$$\begin{aligned}
\overline{\lambda}_N(\widehat{w}_N(x)) &= \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y) dy + \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y) d\widehat{w}_N(y) - \int_{y=x}^{\infty} \overline{\mu}_N(y) d\widehat{w}_N(y) \\
&= \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y) dy + \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y) d\widehat{w}_N(y) + \overline{\mu}_N(x)\widehat{w}_N(x) + \int_{y=x}^{\infty} \widehat{w}_N(y) d\widehat{\mu}_N(y).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y) d\widehat{w}_N(y) &= \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y) d\widehat{\gamma}_N(y) \\
&= -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) d(\overline{\mu}_N(y) G_N(y)) \\
&= -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) - \int_{y=0}^{\infty} \widehat{\gamma}_N(y) \overline{\mu}(y) g_N(y) dy \\
&= -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) - \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y) dy,
\end{aligned}$$

where the last inequality comes from equation (32). Thus,

$$\overline{\lambda}_N(\widehat{w}_N(x)) = -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) + \overline{\mu}_N(x)\widehat{w}_N(x) + \int_{y=x}^{\infty} \widehat{w}_N(y) d\overline{\mu}_N(y),$$

and

$$\begin{aligned}
\overline{\Xi}_N(x) &= \int_{y=0}^{x} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) + \int_{y=x}^{\infty} (\widehat{\gamma}_N(y) G_N(y) - \widehat{w}_N(y)) d\overline{\mu}_N(y) - c\overline{Q}_N(x) \\
&= \int_{y=0}^{x} \widehat{\gamma}_N(y) G_N(y) d\overline{\mu}_N(y) - \int_{y=x}^{\infty} \widehat{\gamma}_N(y)(1 - G_N(y)) d\overline{\mu}_N(y) - c(\overline{Q}_N(x) - \overline{\mu}_N(x))
\end{aligned}$$

Let us now switch to central limit units.

$$\begin{aligned}
\Xi_N^C(x) &= \overline{\Xi}_N(\sqrt{N-1}x + N - 1) \\
&= \int_{y=-\sqrt{N}}^{x} \widehat{\gamma}_N^C(y) G_N^C(y) d\overline{\mu}_N^C(y) - \int_{y=x}^{\infty} \widehat{\gamma}_N^C(y)(1 - G_N^C(y)) d\overline{\mu}_N^C(y) - c(\overline{Q}_N^C(x) - \overline{\mu}_N^C(x)).
\end{aligned}$$

By Lemmas 11 and 13, $\widehat{\gamma}_N^C(y) \to \widehat{\gamma}_\infty^C(y) = H^{-1}(\Phi(y)) - c$ and $G_N^C(y) \to \Phi(y)$ as $N \to \infty$.

Moreover, we have

$$\sqrt{N-1}d\overline{\mu}_N^C(y) = \begin{cases} 0 & \text{if } y < x_N; \\ (N-1)\left(\frac{N-1}{x_N\sqrt{N-1}+N-1} - \frac{N}{x_N\sqrt{N-1}+N-1}\right) \to -1 & \text{if } y = x_N; \\ -(N-1)\frac{N-1}{(y\sqrt{N-1}+N-1)^2}dy \to -dy & \text{if } y > x_N, \end{cases}$$

where the mass point on x_N is derived by comparing $\overline{\mu}_N^C$ to the left and right of x_N, and

$$\sqrt{N-1}(\overline{Q}_N^C(x) - \overline{\mu}_N^C(x)) = \begin{cases} \sqrt{N-1}\left(\frac{x\sqrt{N-1}+N-1}{x_N\sqrt{N-1}+N-1} - \frac{N}{x_N\sqrt{N-1}+N-1}\right) & \text{if } x < x_N; \\ \sqrt{N-1}\left(1 - \frac{N-1}{x\sqrt{N-1}+N-1}\right) & \text{if } x > x_N, \end{cases}$$

which converges to x in both cases.

Define $F(x) = \lim_{N\to\infty} \sqrt{N-1}\,\overline{\Xi}_N^C(x)$. We have

$$F(x) = \begin{cases} -cx + \widehat{\gamma}_\infty^C(x^*)(1-\Phi(x^*)) + \int_{y=x^*}^\infty \widehat{\gamma}_\infty^C(y)(1-\Phi(y))\,dy & x < x^* \\ -cx - \widehat{\gamma}_\infty^C(x^*)\Phi(x^*) - \int_{y=x^*}^x \widehat{\gamma}_\infty^C(y)\Phi(y)\,dy + \int_{y=x}^\infty \widehat{\gamma}_\infty^C(y)(1-\Phi(y))\,dy & x > x^* \end{cases}.$$

Therefore,

$$\lim_{N\to\infty} \overline{T}_N^C(x) = \frac{1}{\phi(x)}\int_{y=0}^x F(y)\phi(y)\,dy.$$

□

D Derivation of Aggregate Transfer for Uniform Distribution

Suppose the prior H is the standard uniform distribution, so that $\widehat{w}(x) = G_N(x)$, and that $c = 0$.

D.1 Must-sell Case

In the must-sell case, $\widehat{\Xi}$ and $\widehat{T}$ are independent of c, so $c = 0$ is without loss. We have:

$$\begin{aligned}
\widehat{\lambda}(G_N(x)) &= \int_{y=0}^{\infty} G_N(y)g_{N-1}(y)\,dy + \int_{y=0}^{\infty} \frac{N-1}{y} G_N(y)g_N(y)\,dy - \int_{y=x}^{\infty} \frac{N-1}{y} g_N(y)\,dy \\
&= 2\int_{y=0}^{\infty} G_N(y)g_{N-1}(y)\,dy - (1 - G_{N-1}(x)) \\
&= 2\widehat{\Pi} - (1 - G_{N-1}(x)), \\
\widehat{\Xi}(x) &= \frac{N-1}{x} G_N(x) - G_{N-1}(x) + 1 - 2\widehat{\Pi}.
\end{aligned}$$

Next,

$$\begin{aligned}
\int_{y=0}^{x} \widehat{\Xi}(y)g_N(y)\,dy &= \int_{y=0}^{x} \left(\frac{N-1}{y} G_N(y) - G_{N-1}(y) + 1 - 2\widehat{\Pi} \right) g_N(y)\,dy \\
&= 2\int_{y=0}^{x} G_N(y)g_{N-1}(y)\,dy - G_N(x)G_{N-1}(x) + (1 - 2\widehat{\Pi})G_N(x) \\
&= G_{N-1}(x)^2 - 2\int_{y=0}^{x} g_N(y)g_{N-1}(y)\,dy - G_N(x)G_{N-1}(x) + (1 - 2\widehat{\Pi})G_N(x) \\
&= G_{N-1}(x)g_N(x) - 2\int_{y=0}^{x} g_N(y)g_{N-1}(y)\,dy + (1 - 2\widehat{\Pi})G_N(x) \\
&= G_{N-1}(x)g_N(x) - \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!} G_{2N-2}(2x) + (1 - 2\widehat{\Pi})G_N(x) \\
&= G_{N-1}(x)g_N(x) + \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!} (G_N(x) - G_{2N-2}(2x))
\end{aligned}$$

where the second line follows from integration by parts, the third and fourth lines use $G_N = G_{N-1} - g_N$, the fifth line is a direct computation using the formula for g_N in (14), and the last line follows from

$$\widehat{\Pi} = \int_{y=0}^{\infty} G_N(y)g_{N-1}(y)\,dy = \frac{1}{2} - \int_{y=0}^{\infty} g_N(y)g_{N-1}(y)\,dy = \frac{1}{2}\left(1 - \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!}\right).$$

Therefore, when $x > 0$,

$$\widehat{T}(x) = G_{N-1}(x) + \frac{\binom{2N-3}{N-1}}{2^{2N-3}} \frac{G_N(x) - G_{2N-2}(2x)}{g_N(x)}.$$

In the central limit normalization, we define

$$\widehat{T}^C(x) = \widehat{T}(N-1+\sqrt{N-1}x).$$

Lemma 11 shows that $G_N(N-1+\sqrt{N-1}x) \to \Phi(x)$ and $g_N(N-1+\sqrt{N-1}x)\sqrt{N-1} \to \phi(x)$ as $N \to \infty$, where Φ and ϕ are, respectively, the cumulative distribution and density of a standard Normal; this also implies that $G_{2N-2}(2(N-1+\sqrt{N-1}x)) \to \Phi(x\sqrt{2})$. Finally, using Stirling's approximation, it is easy to check that $\frac{\binom{2N-3}{N-1}}{2^{2N-3}}\sqrt{N-1} \to \frac{1}{\sqrt{\pi}}$ as $N \to \infty$. Therefore,

$$\lim_{N\to\infty} \widehat{T}^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}\,\phi(x)}$$

for a fixed x.

D.2 Can-keep Case

We have shown that the uniform distribution is single-crossing in Section 4.4. Let $[0, x^*]$ denote the graded interval. The cutoff x^* satisfies (cf. (28))

$$\frac{G_N(x^*)}{2} = g_{N+1}(x^*). \tag{7}$$

This equation implies that $G_{N+1}(x^*) = G_N(x^*) - g_{N+1}(x^*) = g_{N+1}(x^*) = G_N(x^*)/2$.

Define the constants

$$\begin{aligned} C &= \int_{x=0}^{\infty} \overline{\gamma}(x) g_{N-1}(x)\,dx + \int_{x=0}^{\infty} \overline{\mu}(x) G_N(x) g_N(x)\,dx \\ &= \underbrace{\int_{x=0}^{x^*} \exp(x-x^*) G_N(x^*) g_{N-1}(x)\,dx + \int_{x=0}^{x^*} \frac{N}{x^*} G_N(x) g_N(x)\,dx}_{C_1} \\ &\quad + \underbrace{\int_{x=x^*}^{\infty} G_N(x) g_{N-1}(x)\,dx + \int_{x=x^*}^{\infty} \frac{N-1}{x} G_N(x) g_N(x)\,dx}_{C_2} \end{aligned}$$

We can simplify the constants as follows:

$$\begin{aligned} C_1 =& 2\int_{x=0}^{x^*} \exp(x-x^*) G_N(x^*) g_{N-1}(x)\,dx \\ =& 2G_N(x^*) g_N(x^*) \\ C_2 =& 2\int_{x=x^*}^{\infty} G_N(x) g_{N-1}(x)\,dx \\ =& 1 - G_{N-1}(x^*)^2 - 2\int_{x=x^*}^{\infty} g_N(x) g_{N-1}(x)\,dx \\ =& 1 - G_{N-1}(x^*)^2 - \frac{\binom{2N-3}{N-1}}{2^{2N-3}}(1 - G_{2N-2}(2x^*)) \end{aligned}$$

$$C = 2G_N(x^*)g_N(x^*) + 1 - G_{N-1}(x^*)^2 - \frac{\binom{2N-3}{N-1}}{2^{2N-3}}(1 - G_{2N-2}(2x^*)).$$

Then

$$\begin{aligned}
\overline{\lambda}(G_N(x)) &= C - \int_{y=x}^{\infty} \overline{\mu}(y) g_N(y)\, dy \\
&= \begin{cases} C - \int_{y=x}^{x^*} \frac{N}{x^*} g_N(y)\, dy - \int_{y=x^*}^{\infty} \frac{N-1}{y} g_N(y)\, dy & x \le x^* \\ C - \int_{y=x}^{\infty} \frac{N-1}{y} g_N(y)\, dy & x > x^* \end{cases} \\
&= \begin{cases} C - (G_N(x^*) - G_N(x))\frac{N}{x^*} - (1 - G_{N-1}(x^*)) & x \le x^* \\ C - (1 - G_{N-1}(x)) & x > x^* \end{cases}
\end{aligned}$$

and

$$\overline{\Xi}(x) = \begin{cases} G_N(x)\frac{N}{x^*} - C + (G_N(x^*) - G_N(x))\frac{N}{x^*} + (1 - G_{N-1}(x^*)) & x \le x^* \\ \qquad = -C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) & \\ G_N(x)\frac{N-1}{x} - C + 1 - G_{N-1}(x) & x > x^* \end{cases}$$

For $x \le x^*$, we have:

$$\begin{aligned}
\int_{y=0}^{x} \overline{\Xi}(y) g_N(y)\, dy &= \int_{y=0}^{x} \left(-C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) \right) g_N(y)\, dy \\
&= \left(-C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) \right) G_N(x).
\end{aligned}$$

For $x > x^*$, we have:

$$\begin{aligned}
\int_{y=0}^{x} \overline{\Xi}(y) g_N(y)\, dy = &\left(-C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) \right) G_N(x^*) \\
&+ \underbrace{\int_{x^*}^{x} \left(G_N(y)\frac{N-1}{y} - C + 1 - G_{N-1}(y) \right) g_N(y)\, dy}_{X}.
\end{aligned}$$

Simplifying the second term, we get:

$$\begin{aligned}
X = &(1 - C)(G_N(x) - G_N(x^*)) \\
&+ 2\int_{y=x^*}^{x} G_N(y) g_{N-1}(y) dy - (G_N(x)G_{N-1}(x) - G_N(x^*)G_{N-1}(x^*)) \\
= &(1 - C)(G_N(x) - G_N(x^*)) \\
&- 2\int_{y=x^*}^{x} g_N(y) g_{N-1}(y) dy + g_N(x)G_{N-1}(x) - g_N(x^*)G_{N-1}(x^*) \\
= &(1 - C)(G_N(x) - G_N(x^*))
\end{aligned}$$

$$- \frac{\binom{2N-3}{N-1}}{2^{2N-3}}(G_{2N-2}(2x) - G_{2N-2}(2x^*)) + g_N(x)G_{N-1}(x) - g_N(x^*)G_{N-1}(x^*).$$

Therefore, for $x \le x^*$, we have:

$$\overline{T}(x) = \left(-C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*)\right)\frac{G_N(x)}{g_N(x)}.$$

For $x > x^*$ we have:

$$\overline{T}(x)$$
$$= \left[G_N(x^*)^2\frac{N}{x^*} - G_{N-1}(x^*)^2 + (1-C)G_N(x) - \frac{\binom{2N-3}{N-1}}{2^{2N-3}}(G_{2N-2}(2x) - G_{2N-2}(2x^*))\right]\frac{1}{g_N(x)} + G_{N-1}(x).$$

Finally, we take the limit as $N \to \infty$ for the central limit normalization:

$$\overline{T}^C(x) = \overline{T}(N - 1 + \sqrt{N-1}x).$$

Since $G_N(x^*)/2 = G_{N+1}(x^*)$ by the discussion following equation (7), we must have $(x^* - (N-1))/\sqrt{N-1} \to -\infty$, $G_N(x^*) \to 0$, and $g_N(x^*) \to 0$ as $N \to \infty$. Moreover, by equation (7), $NG_N(x^*)/x^* = 2Ng_{N+1}(x^*)/x^* = 2g_N(x^*) \to 0$ as $N \to \infty$. Substituting these into the expressions of C and $\overline{T}$ and simplify as in the must-sell case, we get

$$\lim_{N\to\infty} \overline{T}^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}\,\phi(x)}.$$

References

Stroock, D. W. (2011): *Probability Theory: An Analytic View*, Cambridge University Press, 2 ed.