B Proofs for Section 5

B.1 Proof of Proposition 5

Let $\Delta = 1/K$, and recall that the message space for $\mathcal{M}(m, K)$ is

$$M_i = \{m, m + \Delta, \ldots, m + K\}.$$

Note that the highest message $\overline{m} = m + K$ is at least $\Delta^{-1}$. We shall extend the domain of the allocation and transfer rules to all of $\mathbb{R}^N$ for notational convenience. The discrete aggregate allocation sensitivity is

$$\mu(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \mathbb{I}_{m_i < \overline{m}}(q_i(m_i + \Delta, m_{-i}) - q_i(m)),$$

and the discrete aggregate excess growth is

$$\Xi(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \mathbb{I}_{m_i < \overline{m}}(t_i(m_i + \Delta, m_{-i}) - t_i(m)) - \Sigma t(m).$$

Now, define

$$\lambda(m; v) = v\mu(m) - \Xi(m) - c\overline{Q}(\Sigma m),$$

and let $\lambda(v) = \min_{m \in M} \lambda(m; v)$.

**Lemma 1.** For any information structures $\mathcal{S}$ and equilibrium $\beta$ of $(\mathcal{S}, \mathcal{M}(m, K))$, expected profit is at least $\int_{V} \lambda(v)H(dv)$.

**Proof of Lemma 1.** The equilibrium hypothesis implies that for all $i$,

$$\int_{\mathcal{S}} \sum_{m \in M} [w(s)(q_i(\min\{m_i + \Delta, \overline{m}\}, m_{-i}) - q_i(m))$$
which corresponds to the incentive constraint for deviating to \(\min \{m_i + \Delta, \bar{m}\}\). Summing across bidders, and dividing by \(\Delta\), we conclude that

\[
\int_s \sum_{m \in M} [w(s)\mu(m) - \Xi(m) - \Sigma t(m)] \beta(m|s)\pi(ds) \leq 0.
\]

Hence, expected profit is

\[
\int_s \sum_{m \in M} [\Sigma t(m) - cQ(\Sigma m)] \beta(m|s)\pi(ds)
\geq \int_s \sum_{m \in M} [\Sigma t(m) - cQ(\Sigma m) + w(s)\mu(m) - \Xi(m) - \Sigma t(m)] \beta(m|s)\pi(ds)
= \int_s \sum_{m \in M} [w(s)\mu(m) - \Xi(m) - cQ(\Sigma m)] \beta(m|s)\pi(ds)
\geq \int_s \lambda(w(s))\pi(ds)
\geq \int_V \lambda(v)H(dv),
\]

where the last line follows from the mean-preserving spread condition on \(w(s)\) and that \(\lambda\) is concave, being the infimum of linear functions. \(\square\)

**Lemma 2.** For all \(m \in M\),

\[
\mu(m) \geq \frac{1}{\Delta} \int_0^\Delta \pi(\Sigma m + y)dy - \hat{L}(m, \Delta),
\]

where

\[
\hat{L}(m, \Delta) = N(N + 1)\Delta + \frac{N(N - 1)}{\Delta} \left( \log(Nm + \Delta) + \frac{Nm}{N\bar{m} + \Delta} - \log(Nm) - 1 \right).
\]

Moreover, for all \(m > 0\), \(\hat{L}(m, \Delta) \to 0\) as \(\Delta \to 0\).

**Proof of Lemma 2.** From Lemma 12, we know that

\[
\mu(m) = \sum_{i=1}^N \frac{1}{\Delta} (q_i(m_i + \Delta, m_{-i}) - q_i(m)) - \sum_{i=1}^N \mathbb{1}_{m_i = \bar{m}} \frac{1}{\Delta} (q_i(m_i + \Delta, m_{-i}) - q_i(m))
\geq \sum_{i=1}^N \frac{1}{\Delta} (q_i(m_i + \Delta, m_{-i}) - q_i(m)) - N \frac{N + 1}{\bar{m}}
\geq \sum_{i=1}^N \frac{1}{\Delta} (q_i(m_i + \Delta, m_{-i}) - q_i(m)) - N(N + 1)\Delta.
\]
Recall that

$$\mu(x) = \frac{N-1}{x} Q(x) + \overline{Q}(x).$$

Also recall that

$$\frac{\partial q_i(m)}{\partial m_i} = \frac{\Sigma m_i}{(\Sigma m)^2} \overline{Q}((\Sigma m) + \frac{m_i}{\Sigma m} \overline{Q}'((\Sigma m)),$$

Thus,

$$\sum_{i=1}^{N} \frac{1}{\Delta} (q_i(m_i + \Delta, m_{-i}) - q_i(m))$$

$$= \frac{1}{\Delta} \sum_{i=1}^{N} \int_{y=0}^{\Delta} \frac{\partial q_i(m_i + y, m_{-i})}{\partial m_i} dy$$

$$= \frac{1}{\Delta} \sum_{i=1}^{N} \int_{y=0}^{\Delta} \left[ \frac{\Sigma m_i}{(\Sigma m + y)^2} \overline{Q}((\Sigma m + y) + \frac{m_i + y}{\Sigma m + y} \overline{Q}'((\Sigma m + y)) \right] dy$$

$$= \frac{1}{\Delta} \int_{y=0}^{\Delta} \left[ \frac{(N-1)\Sigma m}{(\Sigma m + y)^2} \overline{Q}((\Sigma m + y) + \frac{\Sigma m + Ny}{\Sigma m + y} \overline{Q}'((\Sigma m + y)) \right] dy$$

$$= \frac{1}{\Delta} \int_{y=0}^{\Delta} \overline{Q}(\Sigma m + y) dy - \frac{N-1}{\Delta} \int_{y=0}^{\Delta} \frac{y}{\Sigma m + y} \left[ \frac{\overline{Q}(\Sigma m + y)}{\Sigma m + y} - \overline{Q}'(\Sigma m + y) \right] dy.$$

We need to bound the last integral from above. If $x$ is in a non-graded interval, then $Q(x)/x - \overline{Q}'(x)$ is just $1/x$. If $x$ is in a graded interval $[a, b]$, then

$$\frac{Q(x)}{x} - \overline{Q}'(x) = \frac{C(a, b)}{N} + \frac{D(a, b)}{x^N} - \frac{C(a, b)}{N} + (N-1) \frac{D(a, b)}{x^N} = \frac{ND(a, b)}{x^N}.$$

From equation (33), $D(a, b) \leq x^{N-1}$, so that the integrand in this case is at most $N/x$, and

$$\int_{y=0}^{\Delta} \frac{y}{x+y} \left[ \frac{Q(x+y)}{x+y} - \overline{Q}'(x+y) \right] dy \leq N \int_{y=0}^{\Delta} \frac{y}{(x+y)^2} dy$$

$$= N \int_{y=0}^{\Delta} \left( \frac{1}{x+y} - \frac{x}{(x+y)^2} \right) dy$$

$$= N \left( \log(x+\Delta) + \frac{x}{x+\Delta} - \log(x) - 1 \right).$$

The derivative with respect to $x$ is

$$N \left( \frac{1}{x+\Delta} - \frac{1}{x} + \frac{\Delta}{(x+\Delta)^2} \right) = N\Delta \left( \frac{1}{(x+\Delta)^2} - \frac{1}{x(x+\Delta)} \right)$$

which is clearly negative, so subject to $x \geq Nm$, the expression is maximized with $x = Nm$, which gives us the lower bound on $\mu$. 3
Moreover, as $\Delta \to 0$, $N(N+1)\Delta \to 0$, and by L'Hôpital's rule,
\[
\lim_{\Delta \to 0} \left( \log(Nm + \Delta) + \frac{Nm}{Nm + \Delta} - \log(Nm) - 1 \right) = \lim_{\Delta \to 0} \left( \frac{1}{Nm + \Delta} - \frac{Nm}{(Nm + \Delta)^2} \right) = 0.
\]

Now let us write $\Xi_p(m) = \Xi(m) - v(\mu(m) - Q(m))$, and recall that $\Xi^p(x) = \Xi(x) - v(\bar{u}(x) - \bar{Q}(x))$. These are the excess growths for the “premium” transfers $t^p_i(m) = t_i(m) - vq_i(m)$ and $\bar{t}^p_i(m) = \bar{t}_i(m) - v\bar{q}_i(m)$, respectively. We similarly denote by $T^p(x) = T(x) - v\bar{Q}(x)$ the aggregate premium transfer, and note that $T^p$ satisfies the differential equation
\[
\left( \frac{N-1}{x} - 1 \right) T^p(x) + \frac{d}{dx} T^p(x) = \Xi^p(x),
\]
with the boundary condition $T^p(0) = 0$.

**Lemma 3.** Let $L_\Xi$ be an upper bound on $|\Xi^p|$ and let $L_T$ be an upper bound on $T^p$. Then
\[
\Xi^p(m) \leq \frac{1}{\Delta} \int_{y=0}^{\Delta} \Xi^p(\Sigma m + y)dy + L(m) \frac{\Delta}{2} + N L_p m
\]
\[
- \frac{1}{\Delta} \sum_i I_{m_i=m} [\bar{t}^p_i(m_i + \Delta, m_{-i}) - \bar{t}^p_i(m)],
\]
where
\[
\bar{L}(m) = \left( 1 + \frac{N-1}{Nm} \right) L_p + \frac{N-1}{(Nm)^2} L_T.
\]

**Proof of Lemma 3.** Recall that $T^p$ is Lipschitz with constant $L_p$. Furthermore, the function $T^p(x)(N-1)/x$ is Lipschitz on $[Nm, \infty)$, and
\[
\left| \frac{d}{dx} \left( \frac{N-1}{x} T^p(x) \right) \right| = \left| \frac{N-1}{x} \frac{d}{dx} T^p(x) - \frac{N-1}{x^2} T^p(x) \right|
\]
\[
\leq \frac{N-1}{Nm} L_p + \frac{N-1}{(Nm)^2} L_T = L_1(m).
\]

Using the differential equation for $T^p$,
\[
\frac{1}{\Delta} \int_{y=0}^{\Delta} \Xi^p(\Sigma m + y)dy
\]
\[
= \frac{1}{\Delta} \int_{y=0}^{\Delta} \left[ \left( \frac{N-1}{\Sigma m + y} - 1 \right) T^p(\Sigma m + y) + \frac{d}{dx} T^p(x) \bigg|_{x=\Sigma m+y} \right] dy
\]
\[
= \frac{1}{\Delta} \left[ \int_{y=0}^{\Delta} \left( \frac{N-1}{\Sigma m + y} - 1 \right) T^p(\Sigma m + y)dy + T^p(\Sigma m + \Delta) - T^p(\Sigma m) \right]
\]

...
Now, let us write $T^p(\Sigma m)$ for the aggregate transfer when the messages are $m$. Thus,

$$
\Xi^p(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \left[ t^p_i (m_i + \Delta, m_{-i}) - t^p_i (m) \right] - T^p(\Sigma m) - \frac{1}{\Delta} \sum_{i=1}^{N} \| t^p_i (m_i + \Delta, m_{-i}) - t^p_i (m) \|
$$

$$
= \frac{1}{\Delta} \sum_{i=1}^{N} \left[ \bar{t}^p_i (m_i + \Delta, m_{-i}) - t^p_i (m) \right] - T^p(\Sigma m) - \frac{1}{\Delta} \sum_{i=1}^{N} \| t^p_i (m_i + \Delta, m_{-i}) - t^p_i (m) \|
$$

$$
\leq \frac{1}{\Delta} \left[ \frac{\Sigma m + N \Delta}{(\Sigma m + \Delta)} T^p(\Sigma m + \Delta) - T^p(\Sigma m) \right] - T^p(\Sigma m) - \frac{1}{\Delta} \sum_{i=1}^{N} \| t^p_i (m_i + \Delta, m_{-i}) - t^p_i (m) \|.
$$

The lemma follows from combining these two inequalities, with the observation that $T^p(x) = T^p(x) - NL_v m$.

**Lemma 4.** For all $\epsilon > 0$, there exists a $K$ such that for all $m$ such that $\Sigma m > K$ and for all $i$,

$$
\frac{1}{\Delta} | \bar{t}^p_i (m_i + \Delta, m_{-i}) - t^p_i (m) | < \epsilon.
$$

**Proof of Lemma 4.** Since $\lim_{x \to \infty} T^p(x) = -\Xi^p(\infty)$, we can find a $K$ large enough so that for $x > K$, $| T^p(x) + \Xi^p(\infty) | < \epsilon/4$ and $L_T/K < \epsilon/4$, and thus $| dT^p(x)/dx | < \epsilon/2$. Thus, when $\Sigma m > K$, then using $\Delta = K^{-1}$,

$$
\frac{1}{\Delta} \left[ \bar{t}^p_i (m_i + \Delta, m_{-i}) - t^p_i (m) \right] = \frac{1}{\Delta} \int_{y=0}^{\Delta} \frac{\partial \bar{t}^p_i (m_i + y, m_{-i})}{\partial m_i} dy
$$

$$
= \frac{1}{\Delta} \int_{y=0}^{\Delta} \left[ \frac{\Sigma m_i - y}{(\Sigma m + y)^2} \bar{T}^p(\Sigma m + y) + \frac{m_i + y}{\Sigma m + y} \frac{d\bar{T}^p(x)}{dx} \right]_{x=\Sigma m+y} dy
$$

$$
\leq \frac{L_T}{K} + \frac{\epsilon}{2}
$$

$$
< \epsilon.
$$

**Proof of Proposition 5.** We first argue that there exists $\bar{m}$ and a $K$ such that $\lambda(m; v) \geq \inf_{m' \in \mathbb{R}^N} \lambda(m'; v) - \epsilon$ for all $m \in M$ and $v \in [\bar{v}, \bar{v}]$, where

$$
\lambda(m; v) = (v - \bar{v})\bar{\pi}(\Sigma m) - \Xi^p(\Sigma m) + (v - c)\bar{Q}(\Sigma m).
$$
From Lemma 12, we know that $|\overline{Q}(x + y) - \overline{Q}(x)| \leq y(N - 1)/m$. Thus,
\[
\left| \overline{Q}(x) - \frac{1}{\Delta} \int_{y=0}^{\Delta} \overline{Q}(x + y) dy \right| \leq \frac{1}{\Delta} \int_{y=0}^{\Delta} |\overline{Q}(x + y) - Q(x)| dy \leq \frac{1}{\Delta} \int_{y=0}^{\Delta} y \frac{N - 1}{m} dy = \Delta \frac{N - 1}{2m}.
\]
Combining this inequality with Lemmas 2 and 3, we get that
\[
\lambda(m; v) = (v - \nu)\mu(m) - \Xi'(m) + (v - c)\overline{Q}(\Sigma m)
\geq \frac{1}{\Delta} \int_{y=0}^{\Delta} [(v - \nu)\overline{m}(\Sigma m + \Delta) - \Xi'(\Sigma m + y) + (v - c)\overline{Q}(\Sigma m + y)] dy
- (v - \nu)\hat{L}(m, \Delta) - v\Delta \frac{N - 1}{2m} - \frac{\Delta}{2} \hat{L}(m) - NL_{p} m
- \frac{1}{\Delta} \sum_{i} \|m_{i} = m\| \left| \overline{p}_{i}(m_{i} + \Delta, m_{-i}) - \overline{p}_{i}(m) \right|
\geq \inf_{m' \mid \Sigma m = \Sigma m' = \Sigma m + \Delta} \overline{\lambda}(m'; v)
- (v - \nu)\hat{L}(m, \Delta) - v\Delta \frac{N - 1}{2m} - \frac{\Delta}{2} \hat{L}(m) - NL_{p} m
- \frac{1}{\Delta} \sum_{i} \|m_{i} = m\| \left| \overline{p}_{i}(m_{i} + \Delta, m_{-i}) - \overline{p}_{i}(m) \right|.
\]
We can first pick $m > 0$ so that $NL_{p} m < \epsilon/2$. We can then pick $K$ large enough (and $\Delta$ small enough) such that the remaining terms in the last two lines sum to less than $\epsilon/2$ (where for the first term in the middle line and last line, this follows from Lemmas 2 and 4, respectively). We then conclude that
\[
\lambda(m; v) \geq \inf_{m' \in \mathbb{R}_{+}^{N}} \overline{\lambda}(m'; v) - \epsilon \geq \overline{\lambda}(v) - \epsilon.
\]
Hence, $\lambda(v) \geq \overline{\lambda}(v) - \epsilon$, and Lemma 1 and Lemma 6 give the result.

This proof goes through verbatim with the maxmin must-sell mechanism $\hat{M}$.

**B.2 Proof of Proposition 6**

Recall the definition of $\tilde{S}(K)$. Let $\Delta = 1/K$. We subsequently choose $K$ sufficiently large (and equivalently $\Delta$ sufficiently small) to attain the desired $\epsilon$. Note that the signal space can be written
\[
S_{i} = \{0, \Delta, \ldots, K^{2}\Delta\},
\]
and the highest message is simply $\Delta^{-1}$. The probability mass function of $s_{i}$ is
\[
f_{i}(s_{i}) = \begin{cases} (1 - \exp(-\Delta)) \exp(-s_{i}) & \text{if } s_{i} < \Delta^{-1}; \\ \exp(-\Delta^{-1}) & \text{if } s_{i} = \Delta^{-1}. \end{cases}
\]
As a result, $s_i/\Delta$ is a censored geometric random variable with arrival rate $1 - \exp(-\Delta)$. We write $f(s) = \times_{i=1}^{N} f_i(s_i)$ for the joint probability, and

$$F_i(s_i) = \sum_{s'_i \leq s_i} f_i(s'_i) = \begin{cases} 1 - \exp(-s_i - \Delta) & \text{if } s_i < \Delta^{-1}; \\ 1 & \text{otherwise}, \end{cases}$$

for the cumulative distribution. The value function is

$$w(s) = \frac{1}{f(s)} \int_{\{s' \in \mathbb{R}_+^N \mid r(s'_i) = s_i \forall i\}} \overline{\pi}(\Sigma s') \exp(-\Sigma s') ds',$$

where

$$\tau(x) = \begin{cases} \Delta \lceil x/\Delta \rceil & \text{if } x < \Delta^{-1}; \\ \Delta^{-1} & \text{otherwise}. \end{cases}$$

An interpretation is that we draw “true” signals $s'$ for the bidders from $\overline{S}$ and agent $i$ observes $s_i = \min\{\Delta \lceil \Delta^{-1} s'_i \rceil, \Delta^{-1}\}$, i.e., signals above $\Delta^{-1}$ are censored and otherwise they are rounded down to the nearest multiple of $\Delta$, and $w$ is the conditional expectation of $\overline{w}$ given the noisy observations $s$. Thus, the distribution of $\overline{w}$ is a mean-preserving spread of the distribution of $w$, so that $H$ is a mean-preserving spread of the distribution of $w$ as well.

**Lemma 5.** If $s_i < \Delta^{-1}$ for all $i$, then $w(s)$ only depends on the sum of the signals $l = \Sigma s$ and

$$w(s) = \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \rho(x - l) \exp(-x) dx,$$

where $\rho(y)$ is the $N - 1$-dimensional volume of the set $\{s \in [0, \Delta]^N \mid \Sigma s = y\}$.

**Proof of Lemma 5.** First observe that

$$f(s) = (1 - \exp(-\Delta))^N \exp(-\Sigma s) = (1 - \exp(-\Delta))^N \exp(-l).$$

Thus,

$$w(s) = \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{\{s' \in \mathbb{R}_+^N \mid r_i(s'_i) = s_i \forall i\}} \overline{\pi}(\Sigma s') \exp(-\Sigma s') ds'$$

$$= \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \int_{\{s' \in \mathbb{R}_+^N \mid r_i(s'_i) = s_i \forall i, \Sigma s' = x\}} \overline{\pi}(\Sigma s') \exp(-\Sigma s') ds' dx$$

$$= \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \exp(-x) \int_{\{s' \in \mathbb{R}_+^N \mid r_i(s'_i - s_i) = 0 \forall i, \Sigma s' = x\}} ds' dx$$

$$= \frac{\exp(l)}{(1 - \exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \exp(-x) \int_{\{s' \in \mathbb{R}_+^N \mid r_i(s'_i - s_i) = 0 \forall i, \Sigma s' = l\}} ds' dx,$$

where the inner integral is just $\rho(x - l)$. 

\[\Box\]
We now abuse notation slightly by writing $w(l)$ for the value when $l = \Sigma s$, and let $\gamma(l) = w(l) - c$.

**Lemma 6.** If $l > \Delta$, then $\gamma(l) \leq \text{exp}(\Delta)\gamma(l - \Delta)$.

*Proof of Lemma 6.* From Lemma 5, we know that

$$\gamma(l) = \frac{\text{exp}(l)}{(1 - \text{exp}(-\Delta))^{N}} \int_{x=l}^{l+N\Delta} \gamma(x) \text{exp}(-x) \rho(x - l)dx$$

$$= \frac{\text{exp}(l)}{(1 - \text{exp}(-\Delta))^{N}} \int_{x=l-\Delta}^{l+(N-1)\Delta} \gamma(x + \Delta) \text{exp}(-x - \Delta) \rho(x - l + \Delta)dx$$

$$\leq \frac{\text{exp}(l - \Delta)}{(1 - \text{exp}(-\Delta))^{N}} \int_{x=l-\Delta}^{l+(N-1)\Delta} \gamma(x) \text{exp}(\Delta) \text{exp}(-x) \rho(x - l + \Delta)dx$$

$$= \text{exp}(\Delta)\gamma(l - \Delta),$$

where the inequality follows from Lemma 2. \qed

**Lemma 7.** If the direct allocation $q_i(s)$ is Nash implemented by a participation secure mechanism, profit is at most

$$\sum_{s \in S} f(s) \sum_{i=1}^{N} q_i(s) \left[ \gamma(\Sigma s) - \frac{1 - F_i(s)}{f_i(s)}(\gamma(\Sigma s + \Delta) - \gamma(\Sigma s)) \right]. \quad (1)$$

*Proof of Lemma 7.* This follows from standard revenue equivalence arguments: If we write $U_i(s_i, s'_i)$ for the utility of a signal $s_i$ that reports $s'_i$, with $U_i(s_i) = U_i(s_i, s_i)$, then

$$U_i(s_i) \geq U_i(s_i, s'_i) = U_i(s'_i) + \sum_{s_{-i} \in S_{-i}} f_{-i}(s_{-i}) q_i(s'_i, s_{-i}) \left( \gamma(s_i + \Sigma s_{-i}) - \gamma(s'_i + \Sigma s_{-i}) \right).$$

Thus, for $s_i \geq \Delta$,

$$U_i(s_i) \geq U_i(0) + \sum_{k=0}^{s_i/\Delta-1} \sum_{s_{-i} \in S_{-i}} f_{-i}(s_{-i}) q_i(k\Delta, s_{-i}) \left( \gamma((k + 1)\Delta + \Sigma s_{-i}) - \gamma(k\Delta + \Sigma s_{-i}) \right).$$

The expectation of $U_i(s_i)$ across $s_i$ is therefore bounded below by

$$\sum_{s \in S} f(s) \sum_{k=0}^{s_i/\Delta-1} q_i(k\Delta, s_{-i}) \left( \gamma((k + 1)\Delta + \Sigma s_{-i}) - \gamma(k\Delta + \Sigma s_{-i}) \right)$$

$$= \sum_{s \in S} f(s) q_i(s) \frac{\gamma(\Sigma s + \Delta) - \gamma(\Sigma s)}{\gamma(\Sigma s + \Delta) - \gamma(\Sigma s))} \frac{1 - F_i(s_i)}{f_i(s_i)}.$$

The formula then follows from subtracting the bound on bidder surplus from total surplus. \qed

Let $\Pi$ denote the maximum of the profit bound (1) across all $q$. Let $\bar{\Pi}$ denote the profit bound when we set $q_i(s) = 1$ and $q_j(s) = 0$ for all $j \neq 1$. 

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Lemma 8. $\Pi \leq \tilde{\Pi} + (1 - (1 - \exp(-\Delta^{-1}))^N)v$.

Proof of Lemma 8. When signals are all less than $\Delta^{-1}$, the bidder-independent virtual value is

$$\gamma(l) - \frac{1}{\exp(\Delta) - 1} (\gamma(l + \Delta) - \gamma(l))$$

$$\geq \gamma(l) - \frac{\exp(-\Delta)}{1 - \exp(-\Delta)} (\gamma(l) \exp(\Delta) - \gamma(l)) = 0,$$

where the inequality follows from Lemma 6. Thus, the virtual value is maximized pointwise by allocating with probability one to, say, bidder 1. With probability $1 - (1 - \exp(-\Delta^{-1}))^N$, one of the signals is above $\Delta^{-1}$, in which case $v$ is an upper bound on the virtual value.

Lemma 9. $\lim_{\Delta \to 0} \tilde{\Pi} \leq \Pi$.

Proof of Lemma 9. Plugging in $q_1 = 1$, we find that

$$\tilde{\Pi} = \sum_{s_{-1} \in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1 \in S_1} \left[ f_1(s_1) \gamma(\Sigma s) - \sum_{s'_1 > s_1} f_1(s'_1) (\gamma(\Sigma s + \Delta) - \gamma(\Sigma s)) \right]$$

$$= \sum_{s_{-1} \in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1 \in S_1} \left[ f_1(s_1) \left( \gamma(\Sigma s) + \sum_{s'_1 < s_1} (\gamma(s'_1 + \Sigma s_{-1}) - \gamma(s'_1 + \Sigma s_{-1} + \Delta)) \right) \right]$$

$$= \sum_{s_{-1} \in S_{-1}} f_{-1}(s_{-1}) \gamma(\Sigma s_{-1}).$$

Using the definition of $\gamma$, this is

$$\tilde{\Pi} = \frac{1}{1 - \exp(-\Delta)} \int_{y=0}^{\Delta} \int_{x=0}^{\infty} \tilde{\gamma}(x+y) g_{N-1}(x) \exp(-y) dxdy$$

$$= \frac{1}{1 - \exp(-\Delta)} \int_{x=0}^{\infty} \tilde{\gamma}(x) \int_{y=0}^{\min\{x,\Delta\}} g_{N-1}(x-y) \exp(-y) dydx$$

$$\leq \frac{1}{1 - \exp(-\Delta)} \left[ \int_{x=\Delta}^{\infty} \tilde{\gamma}(x) \int_{y=0}^{\Delta} g_{N-1}(x-y) \exp(-y) dydx + G_N(\Delta)\tilde{v} \right].$$

Now, observe that

$$\int_{y=0}^{\Delta} g_{N-1}(x-y) \exp(-y) dy = \frac{x^{N-1} - (x - \Delta)^{N-1}}{(N-1)!} \exp(-x)$$

$$\leq \frac{\Delta(N-1)x^{N-2}}{(N-1)!} \exp(-x) = \Delta g_{N-1}(x),$$

where we have used convexity of $x^{N-1}$. Thus,

$$\tilde{\Pi} \leq \frac{\Delta}{1 - \exp(-\Delta)} \int_{x=0}^{\infty} \tilde{\gamma}(x) g_{N-1}(x) dx + \frac{G_N(\Delta)\tilde{v}}{1 - \exp(-\Delta)}.$$

An application of L'Hôpital's rule shows that the last term converges to zero as $\Delta \to 0$ and $\Delta/(1 - \exp(-\Delta)) \to 1$, this implies the lemma. □
Proof of Proposition 6. Combining Lemmas 7 and 8, we can pick $\Delta$ sufficiently small so that $\Pi \leq \tilde{\Pi} + \epsilon/2 \leq \Pi + \epsilon$. This completes the proof of the proposition.

Note that every step of the proof of Proposition 6 goes through in the must-sell case, where we replace $w$ with $\hat{w}$, and we skip the step in Lemma 8 of proving that the discrete virtual value is non-negative.
Proof of Lemma 9. The left-tail assumption could equivalently be stated as: there exists some $\alpha > 0$ and $\varphi > 1$ such that for all $0 \leq \alpha' < \alpha \leq \overline{\alpha}$

$$H^{-1}(\alpha) - v \leq G_N^{-1}(\alpha)^\varphi$$

and if $v > c$,

$$\frac{H^{-1}(\alpha) - c}{H^{-1}(\alpha') - c} \leq \exp(G_N^{-1}(\alpha) - G_N^{-1}(\alpha')).$$

The following Lemma 10 implies that if the above two conditions hold for $N$, they hold for all $N' > N$ as well.

Lemma 10. For any $N \geq 1$ and $N' > N$, there exists $\overline{\alpha} > 0$ such that $G_N^{-1}(\alpha) - G_N^{-1}(\alpha') \leq G_N^{-1}(\alpha) - G_N^{-1}(\alpha')$ for all $0 \leq \alpha' < \alpha \leq \overline{\alpha}$.

Proof of Lemma 10. Clearly it suffices to prove the lemma for $N' = N + 1$. Let us extend the definition of $G_N$ to any real number $N$:

$$G_N(x) = \int_{y=0}^{x} e^{-y} y^{N-1} \frac{1}{\Gamma(N)} dy,$$

where

$$\Gamma(N) = \int_{y=0}^{\infty} e^{-y} y^{N-1} dy.$$

(We have $\Gamma(N) = (N-1)!$ when $N \geq 1$ is an integer.)

By definition, we have

$$\int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} x^{N-1} \frac{1}{\Gamma(N)} dx = \alpha.$$

Differentiating the above equation with respect to $N$ gives:

$$\frac{\partial G_N^{-1}(\alpha)}{\partial N} e^{-G_N^{-1}(\alpha)} G_N^{-1}(\alpha)^{N-1} \frac{1}{\Gamma(N)} + \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{\partial}{\partial N} \left( \frac{x^{N-1}}{\Gamma(N)} \right) dx = 0.$$

i.e.,

$$\frac{\partial G_N^{-1}(\alpha)}{\partial N} = \frac{\Gamma(N) e^{G_N^{-1}(\alpha)}}{G_N^{-1}(\alpha)^{N-1}} \left( - \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{\partial}{\partial N} \left( \frac{x^{N-1}}{\Gamma(N)} \right) dx \right)$$

$$= \frac{e^{G_N^{-1}(\alpha)}}{\Gamma(N) G_N^{-1}(\alpha)^{N-1}} \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \left[ -x^{N-1} \log(x) \Gamma(N) + x^{N-1} \Gamma'(N) \right] dx$$

$$= \frac{e^{G_N^{-1}(\alpha)}}{\Gamma(N)} f(G_N^{-1}(\alpha), N),$$
where

\[ f(z, N) = \frac{1}{z^{N-1}} \int_{x=0}^{z} e^{-x} [-x^{N-1} \log(x) \Gamma(N) + x^{N-1} \Gamma'(N)] \, dx. \]

We compute:

\[
\frac{\partial f(z, N)}{\partial z} = \frac{1}{z^{2(N-1)}} \left( z^{N-1} e^{-z} [-z^{N-1} \log(z) \Gamma(N) + z^{N-1} \Gamma'(N)] 
- (N-1)z^{N-2} \int_{x=0}^{z} e^{-x} [-x^{N-1} \log(x) \Gamma(N) + x^{N-1} \Gamma'(N)] \, dx \right)
\]

\[= e^{-z} [- \log(z) \Gamma(N) + \Gamma'(N)] - (N-1)z^{-N} \int_{x=0}^{z} e^{-x} [-x^{N-1} \log(x) \Gamma(N) + x^{N-1} \Gamma'(N)] \, dx. \]

For any \( z \leq 1 \), we have

\[
\frac{\partial f(z, N)}{\partial z} \geq e^{-z} [- \log(z) \Gamma(N) + \Gamma'(N)] - (N-1)z^{-N} \int_{x=0}^{z} [-x^{N-1} \log(x) \Gamma(N) + x^{N-1} \Gamma'(N)] \, dx
\]

\[= e^{-z} [- \log(z) \Gamma(N) + \Gamma'(N)] - (N-1)z^{-N} \left[ \Gamma(N) \left( \frac{z}{N^2} - \frac{z \log z}{N} \right) + \Gamma'(N) \frac{z^N}{N} \right]
\]

\[= e^{-z} [- \log(z) \Gamma(N) + \Gamma'(N)] - \frac{N-1}{N} \left[ \Gamma(N) \left( \frac{1}{N} - \log z \right) + \Gamma'(N) \right]
\]

\[= \left( e^{-z} - \frac{N-1}{N} \right) [- \log(z) \Gamma(N) + \Gamma'(N)] - \frac{N-1}{N^2} \Gamma(N). \]

Since the last line goes to infinity as \( z \) goes to zero, for any fixed \( N \geq 1 \) we can choose \( \bar{z} \in (0, 1] \) such that \( \partial f(z, \bar{N})/\partial z \geq 0 \) for all \( z \in [0, \bar{z}] \) and \( \bar{N} \in [N, N+1] \). Let \( \bar{\alpha} = G_{N+1}(\bar{z}) \).

Suppose \( 0 \leq \alpha' < \alpha \leq \bar{\alpha} \). We have

\[
[G_{N+1}^{-1}(\alpha) - G_{N+1}^{-1}(\alpha')] - [G_{N}^{-1}(\alpha) - G_{N}^{-1}(\alpha')] = \int_{\bar{N}=N}^{N+1} \left( \frac{\partial G_{\bar{N}}^{-1}(\alpha)}{\partial \bar{N}} - \frac{\partial G_{\bar{N}}^{-1}(\alpha')}{\partial \bar{N}} \right) \, d\bar{N}.
\]

Since \( d \left( e^{z} f(z, \bar{N}) / \Gamma(\bar{N}) \right) / dz \geq 0 \) for all \( z \in [0, \bar{z}] \) and \( \bar{N} \in [N, N+1] \), we have \( \partial G_{\bar{N}}^{-1}(\alpha)/\partial \bar{N} - \partial G_{\bar{N}}^{-1}(\alpha')/\partial \bar{N} \geq 0 \), which proves the lemma.

Let us now define

\[
G_{N}^{C}(x) = G_{N} \left( \sqrt{N-1} x + N - 1 \right);
\]

\[
g_{N}^{C}(x) = \sqrt{N-1} g_{N} \left( \sqrt{N-1} x + N - 1 \right).
\]

To prove Proposition 7, we first need a number of technical results.

**Lemma 11.** As \( N \) goes to infinity, \( g_{N}^{C} \) and \( G_{N}^{C} \) converge pointwise to \( \phi \) and \( \Phi \), respectively.
Proof of Lemma 11. Note that
\[ g_{N+1}^C(x) = \sqrt{N}g_{N+1}(\sqrt{N}x + N) = \sqrt{N}(\sqrt{N}x + N)^N \frac{N!}{N^N} \exp(-\sqrt{N}x - N). \]

Stirling’s Approximation says that
\[ \lim_{N \to \infty} \frac{N!}{\sqrt{2\pi N} \left( \frac{N}{e} \right)^N} = 1. \]

Moreover, for all \( N \), the ratio inside the limit is greater than 1.

Thus, when \( N \) is large, \( g_{N+1}^C(x) \) is approximately
\[ \frac{1}{\sqrt{2\pi}} \left( 1 + \frac{x}{\sqrt{N}} \right)^N \exp(-\sqrt{N}x), \]
and hence
\[ \log(g_{N+1}^C(x)) \approx \log(1/\sqrt{2\pi}) + N \log \left( 1 + \frac{x}{\sqrt{N}} \right) - \sqrt{N}x. \]

Using the mean-value formulation of Taylor’s Theorem centered around 0, for every \( y \), there exists a \( z \in [0, y] \) such that
\[ \log(1 + y) = y - \frac{y^2}{2} + \frac{1}{(1+z)^3}y^3. \]

Plugging in \( y = x/\sqrt{N} \), we conclude that
\[ \log(g_{N+1}^C(x)) \approx \log(1/\sqrt{2\pi}) + N \frac{x}{\sqrt{N}} - N \frac{1}{2} \left( \frac{x}{\sqrt{N}} \right)^2 + N \frac{1}{(1+z)^3} \left( \frac{x}{\sqrt{N}} \right)^3 - \sqrt{N}x \]
\[ = \log(1/\sqrt{2\pi}) - \frac{1}{2} x^2 + \frac{1}{(1+z)^3} \frac{x^3}{\sqrt{N}} \]
which converges to \( \log(1/\sqrt{2\pi}) - \frac{1}{2} x^2 \) as \( N \) goes to infinity, so \( g_{N+1}^C(x) \) converges to \( \phi(x) = \exp(-x^2/2)/\sqrt{2\pi} \). Pointwise convergence of \( G_N^C \) to \( \Phi \) follows from Scheffé’s lemma.

Let us define
\[ \tilde{g}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) & \text{if } x < 0; \\ \frac{1}{\sqrt{2\pi}} (1 + x) \exp(-x) & \text{otherwise}. \end{cases} \]

Lemma 12. The function \( \tilde{g}(x)|x| \) is integrable, and for all \( N \) and \( x \), \( |g_N^C(x)| \leq \tilde{g}(x) \).
Proof of Lemma 12. Note that
\[ \int_{x=\infty}^{x=-\infty} \tilde{g}(x)|x|dx = \int_{x=\infty}^{0} \phi(x)|x|dx + \frac{1}{\sqrt{2\pi}} \int_{x=0}^{\infty} (1 + x)x \exp(-x)dx, \]
which is clearly finite, since the half-normal distribution has finite expectation.

Next, Stirling’s Approximation implies that
\[ g_{N+1}^C(x) \leq \frac{1}{\sqrt{2\pi}} \left(1 + \frac{x}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x) \equiv \tilde{g}_N(x). \]

Now,
\[ \frac{d}{dN} \log(\tilde{g}_N(x)) = \frac{d}{dx} \frac{d}{dN} \log(\tilde{g}_N(x)) = \frac{1}{\sqrt{N} + x} - \frac{\sqrt{N}}{2(\sqrt{N} + x)^2} - \frac{1}{2\sqrt{N}}, \]
which is clearly zero when \( x = 0 \), and
\[ \frac{d}{dx} \frac{d}{dN} \log(\tilde{g}_N(x)) = \frac{2N + 2\sqrt{N}x}{2\sqrt{N}(\sqrt{N} + x)^2} - \frac{N}{2\sqrt{N}(\sqrt{N} + x)^2} - \frac{N + 2\sqrt{N}x + x^2}{2\sqrt{N}(\sqrt{N} + x)^2} \]
\[ = \frac{-x^2}{2\sqrt{N}(\sqrt{N} + x)^2}, \]
which is non-positive and strictly negative when \( x \neq 0 \). As a result, \( \tilde{g}_N(x) \) is increasing in \( N \) when \( x < 0 \) and decreasing in \( N \) when \( x > 0 \). Since it converges to \( \phi(x) \) in the limit as \( N \) goes to infinity, we conclude that for \( x < 0 \), \( g_{N+1}^C(x) \leq \tilde{g}_N(x) \leq \phi(x) = \tilde{g}(x) \), and for \( x > 0 \), \( g_{N+1}^C(x) \leq \tilde{g}_N(x) \leq \tilde{g}(x) \) as desired.

Lemma 13. As \( N \) goes to infinity, \( \hat{\gamma}_N^C \) converges almost surely to \( \hat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x)) \) and \( \hat{\Gamma}_N^C \) converges pointwise to
\[ \hat{\Gamma}_\infty^C(x) = \int_{y=-\infty}^{x} \hat{\gamma}_\infty^C(y)\phi(y)dy. \]

The latter convergence is uniform on any bounded interval.

Proof of Lemma 13. Note that \( \hat{\gamma}_N^C(x) = H^{-1}(G_N^C(x)) - c \). By Lemma 11, \( G_N^C(x) \) converges to \( \Phi(x) \) pointwise. Thus, if \( H^{-1} \) is continuous at \( \Phi(x) \), then as \( N \) goes to infinity, we must have \( \hat{\gamma}_N^C(x) \to H^{-1}(\Phi(x)) - c = \hat{\gamma}_\infty^C(x) \). Since \( H^{-1} \) is monotonic, the set of discontinuities has Lebesgue measure zero, so that the pointwise convergence is almost everywhere.

Pointwise convergence of \( \hat{\Gamma}_N^C \) follows from almost sure convergence of \( \hat{\gamma}_N^C \), combined with the fact that \( \hat{\gamma}_N^C \) is uniformly bounded by \( |\tilde{v}| \), so that we can apply the dominated convergence theorem. Moreover, \( \hat{\Gamma}_N^C(x) \) is uniformly Lipschitz continuous across \( N \) and \( x \). As a result, the family \( \{\hat{\Gamma}_N^C(\cdot)\}_{N=2}^{\infty} \) is uniformly bounded and uniformly equicontinuous. The conclusion about uniform convergence is then a consequence of the Arzela-Ascoli theorem.
Recall that $x^*$ is the largest solution to $\hat{\Gamma}^C_\infty(x^*) = 0$ (which may be $-\infty$). Also, let us define $x_N$ so that $\Gamma_N^C$ has a graded interval $[-\sqrt{N-1}, x_N]$. (If there is no graded interval with left end point $-\sqrt{N-1}$, then we let $x_N = -\sqrt{N-1}$.)

**Lemma 14.** As $N$ goes to infinity, $x_N$ converges to $x^*$.

**Proof of Lemma 14.** By a change of variables $y = (G_N^C)^{-1}(\Phi(x))$, we conclude that

$$\hat{\Gamma}^C_\infty(x^*) = \int_{x = -\infty}^{x^*} \gamma^C_\infty(x) \phi(x) dx = \int_{x = -\sqrt{N-1}}^{(G_N^C)^{-1}(\Phi(x^*))} \gamma^C_N(x) g^C_N(x) dx = \hat{\Gamma}^C_N((G_N^C)^{-1}(\Phi(x^*))).$$

This integral must be zero by the definition of $x^*$, so that $x_N \geq (G_N^C)^{-1}(\Phi(x^*))$. Since the latter converges to $x^*$ as $N \to \infty$, we conclude $\liminf_{N \to \infty} x_N \geq x^*$.

Next, recall that $x_{N+1}$ solves the equation

$$\hat{\Gamma}^C_{N+1}(x_{N+1}) = \hat{\gamma}^C_{N+1}(x_{N+1}) \int_{x = -\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N}(x - x_{N+1})) g^C_{N+1}(x) dx$$

$$= \hat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N} x_{N+1} - N) \int_{x = -\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N} x + N) g^C_{N+1}(x) dx$$

$$= \hat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N} x_{N+1} - N) \int_{x = -\sqrt{N}}^{x_{N+1}} \frac{\sqrt{N} x + N}{N!} dx$$

$$\leq \bar{v} \exp(-\sqrt{N} x_{N+1} - N) \frac{(\sqrt{N} x_{N+1} + N)^{N+1}}{(N+1)!}$$

$$\leq \bar{v} \hat{g}_{N+2} \left( \sqrt{\frac{N}{N+1}} x_{N+1} - \frac{1}{\sqrt{N+1}} \right) \frac{1}{\sqrt{N+1}}$$

where we have used Lemma 12. The last line converges to zero pointwise, so $\hat{\Gamma}^C_N(x_N)$ must converge to zero as well.

Now, if $z = \limsup_{N \to \infty} x_N > x^*$, then since $\hat{\Gamma}^C_\infty(z) > \hat{\Gamma}^C_\infty(x^*) = 0$, we would contradict our earlier finding that $\hat{\Gamma}^C_N(x_N) \to 0$. Thus, $\limsup_{N \to \infty} x_N \leq x^*$, so $x_N$ must converge to $x^*$ as $N$ goes to $\infty$. \hfill $\square$

**Lemma 15.** For every $\epsilon > 0$, there exists $\hat{N}$ such that for all $N > \hat{N}$, there exists an $x \in [x^* + \epsilon, x^* + 2\epsilon]$ at which $\pi^C_N$ is not graded.

**Proof of Lemma 15.** Suppose not. Then there exist infinitely many $N$ such that for every $x \in [x^* + \epsilon, x^* + 2\epsilon]$, $\pi^C_{N+1}(x) = \exp(\sqrt{N}(x - \tilde{x})) \gamma^C_{N+1}(\tilde{x})$ for some $\tilde{x} \geq x^* + 2\epsilon$. Thus, for all $x \leq x^* + \epsilon$, we conclude that

$$\pi^C_{N+1}(x) \leq \pi^C_{N+1}(x^* + \epsilon) \leq \exp(-\sqrt{N\epsilon}) \bar{v}$$
which converges to zero as $N$ goes to infinity. This implies that $\liminf_{N \to \infty} \Gamma_{N+1}^C(x^* + \epsilon) = 0$. But $\Gamma_{N+1}^C(x^* + \epsilon)$ must be weakly larger than $\Gamma_{N+1}^C(x^* + \epsilon)$, so

$$0 = \lim_{N \to \infty} \inf \Gamma_{N+1}^C(x^* + \epsilon) \geq \lim_{N \to \infty} \inf \Gamma_{N+1}^C(x^* + \epsilon) = \Gamma_{\infty}^C(x^* + \epsilon) = 0,$$

a contradiction. \qed

**Lemma 16.** As $N$ goes to infinity, $\gamma_N^C$ converges almost surely to

$$\gamma_\infty^C(x) = \begin{cases} 0 & \text{if } x < x^*; \\ \gamma_\infty^C(x) & \text{if } x \geq x^*. \end{cases}$$

**Proof of Lemma 16.** Let $x < x^*$. Since $x_N \to x^*$ by Lemma 14, for $N$ sufficiently large, $x_N > (x^* + x)/2$. Since $\gamma_N^C(x)$ is graded on $(-\infty, x_N]$, it is graded at $x$, and

$$\gamma_N^C(x) = \exp(\sqrt{N-1}(x - x_N))\gamma_N^C(x_N) \leq \exp(\sqrt{N-1}(x - x^*)/2)v.$$ 

The last line clearly converges to zero pointwise. Since $\gamma_N^C(x) \geq 0$ for all $N$, we conclude that $\gamma_N^C(x) \to 0$.

Now consider $x > x^*$ at which $\gamma_\infty^C$ is continuous. Take $\epsilon$ so that $x > x^* + 2\epsilon$ and so that $\gamma_\infty^C$ is continuous at $x^* + \epsilon$. Lemma 15 says that there is a $N$ such that for all $N > N$, there exists a point in $[x^* + \epsilon, x^* + 2\epsilon]$ at which the gains function is not graded. Moreover, since $\gamma_N^C(x^* + \epsilon)$ converges to $\gamma_\infty^C(x^* + \epsilon)$, we can pick $N$ large enough and find a constant $\gamma > 0$ such that for $N > N$, $\gamma_N^C(x^* + \epsilon) \geq \gamma$.

Now, suppose that $\gamma_N^C$ is graded at $x$, with $x$ in a graded interval $[a, b]$. Then $a \geq x^* + \epsilon$, and hence $\gamma_N^C(a) \geq \gamma_N^C(x^* + \epsilon) \geq \gamma$. Recall that on $[a, b]$,

$$\gamma_N^C(x) = \gamma_N^C(a) \exp(\sqrt{N-1}(x - a)).$$

Since $\gamma_N^C$ is bounded above by $\overline{v}$, it must be that $\gamma_N^C(a) \exp(\sqrt{N-1}(b - a)) \leq \overline{v}$, so

$$b - a \leq \frac{1}{\sqrt{N-1}} \log \left( \frac{\overline{v}}{\gamma_N^C(a)} \right) \leq \frac{1}{\sqrt{N-1}} \log \left( \frac{\overline{v}}{\gamma} \right) = \epsilon_N.$$ 

Thus,

$$\gamma_N^C(x - \epsilon_N) \leq \gamma_N^C(x) \leq \gamma_N^C(x + \epsilon_N).$$

This was true if $\gamma_N^C(x)$ is graded at $x$, but clearly the inequality is also true if it is not graded at $x$, in which case $\gamma_N^C(x) = \gamma_N^C(x)$. Now, $\gamma_N^C(x) = \gamma_\infty^C(\Phi^{-1}(G_N^C(x)))$, so

$$\gamma_\infty^C(\Phi^{-1}(G_N^C(x - \epsilon_N))) \leq \gamma_N^C(x) \leq \gamma_\infty^C(\Phi^{-1}(G_N^C(x + \epsilon_N))).$$

As $N \to \infty$, the left and right hand sides converge to $\gamma_\infty^C(x)$ from the left and right, respectively. Since $\gamma_\infty^C$ is continuous at $x$, we conclude that $\gamma_N^C(x) \to \gamma_\infty^C(x)$. The lemma follows from the fact that the monotonic function $\gamma_\infty^C$ is continuous almost everywhere. \qed
Proof of Proposition 7. We argue that
\[ Z_{N+1} = \sqrt{N} \int_{x=0}^{\infty} \tau_{N+1}(x)(g_{N+1}(x) - g_N(x))dx \]
converges to a positive constant as \( N \) goes to infinity. Since this is \( \sqrt{N} \) times the difference between ex ante gains from trade and profit, this proves the result.

To that end, observe that
\[ Z_{N+1} = \sqrt{N} \int_{x=0}^{N/2} \tau_{N+1}(x)(g_{N+1}(x) - g_N(x))dx + \int_{x=-\sqrt{N}/2}^{\infty} \tau_{N+1}(x) g_{N+1}(x) \frac{Nx}{\sqrt{N}x + N} dx. \]
We claim that the first integral converges to zero as \( N \to \infty \). Note that \( g_{N+1}(x) \leq g_N(x) \) if and only if \( x \leq N \). Therefore,
\[
\left| \sqrt{N} \int_{x=0}^{N/2} \tau_{N+1}(x)(g_{N+1}(x) - g_N(x))dx \right| \leq (\nu + c) \sqrt{N} \int_{x=0}^{N/2} (g_N(x) - g_{N+1}(x))dx
\]
\[
= (\nu + c) \sqrt{N} (G_N(N/2) - G_{N+1}(N/2))
\]
\[
= (\nu + c) \sqrt{N} g_{N+1}(N/2)
\]
\[
= (\nu + c) \sqrt{N} \frac{(N/2)^N \exp(-N/2)}{N!}
\]
\[
\approx (\nu + c) \sqrt{N} \frac{(N/2)^N \exp(-N/2)}{\sqrt{2\pi N} (N/e)^N}
\]
\[
= (\nu + c) \frac{1}{\sqrt{2\pi}} \exp(-N(\log(2) - 1/2)),
\]
where we have again used Stirling’s Approximation between the third-to-last and second-to-last lines. The last line converges to zero as \( N \) goes to infinity.

Now consider the second integral in the formula for \( Z_{N+1} \). By Lemma 12, the integrand is bounded above in absolute value by the integrable function \( \tilde{\tau}_N(x)|x| \). Moreover, from Lemmas 11 and 16, we know that the integrand converges pointwise to \( \tau_{C}^{\infty}(x) \phi(x) \). The dominated convergence theorem then implies that as \( N \) goes to infinity, \( Z_N \) converges to
\[
\int_{x=-\infty}^{\infty} \frac{\tau_{\infty}^{C}(x) \phi(x)}{\tau_{\infty}^{C}(x)} dx,
\]
which is strictly positive because \( \tau_{\infty}^{C} \) is strictly increasing.

The proof goes through for the must-sell guarantee, if we replace \( \tau_N^{C} \) with \( \gamma_N^{C} \).

To prove Proposition 9, we need a few more intermediate results. Let \( \overline{G}_N(x) = G_N(Nx) \) be the cumulative distribution for the mean of \( N \) independent standard exponential random variables. Define \( \overline{F}_N(x) = \exp(N(1 - x + \log(x))) \). Clearly, \( \overline{F}_N(x) \) is a cumulative distribution for \( x \in [0, 1] \), \( \overline{F}_N(0) = 0 \) and \( \overline{F}_N(1) = 1 \). Finally, define the function \( D_N(\alpha) \):
\[
D_N(\alpha) = \begin{cases} 
\frac{1}{\overline{F}_N(\alpha)} & \text{if } \alpha \in [0, 0.4]; \\
1.1 & \text{if } \alpha \in (0.4, 1]. 
\end{cases}
\]
The choices of 0.4 and 1.1 in $D_N(\alpha)$ are arbitrary: any numbers work that are less than $1/2$ and more than 1, respectively.

**Lemma 17.** When $\hat{N}$ is sufficiently large, $\bar{\mu}_N(G_N^{-1}(\alpha)) \leq D_{\hat{N}}(\alpha)$ for all $N \geq \hat{N}$ and $\alpha \in [0, 1]$.

**Proof of Lemma 17.** We first apply the theory of large deviations to the exponential distribution. Let $\Lambda(t)$ be the logarithmic moment generating function for the exponential distribution:

$$\Lambda(t) = \log \left( \int_{x=0}^{\infty} \exp(xt - x) \, dx \right) = \begin{cases} \infty & \text{if } t \geq 1; \\ -\log(1-t) & \text{if } t < 1. \end{cases}$$

Let $\Lambda^*(x)$ be the Legendre transform of $\Lambda(t)$:

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}} \{ xt - \Lambda(t) \} = \begin{cases} \infty & x \leq 0, \\ x - 1 - \log x & x > 0. \end{cases}$$

Cramér’s theorem (or the Chernoff bound; see Theorem 1.3.12 in Stroock, 2011) then states that for any $N$,

$$G_N(x) \leq \exp(-N\Lambda^*(x)) = \overline{F}_N(x)$$

for every $x \in [0, 1]$; or equivalently, $\overline{F}_N^{-1}(\alpha) \leq G_N^{-1}(\alpha)$ for every $\alpha \in [0, \overline{G}_N(1)]$.

By the law of large numbers, when $\hat{N}$ is sufficiently large, we have $\overline{G}_N(1) \geq 0.4$ and $1/\overline{G}_N^{-1}(0.4) \leq 1.1$ and for all $N \geq \hat{N}$. The claim of the lemma then follows from two cases:

If $\alpha \in [0, 0.4]$, then we have

$$\bar{\mu}_N(G_N^{-1}(\alpha)) \leq \frac{N}{G_N^{-1}(\alpha)} = \frac{1}{\overline{G}_N^{-1}(\alpha)} \leq \frac{1}{\overline{F}_N^{-1}(\alpha)} = D_{\hat{N}}(\alpha),$$

where we have used the bound $\bar{\mu}_N(x) \leq N/x$ (equation (21)), and the facts that $\overline{G}_N(1) \geq 0.4$ when $N \geq \hat{N}$ (so $\overline{F}_N^{-1}(\alpha) \leq \overline{G}_N^{-1}(\alpha)$ for $\alpha \leq 0.4 \leq \overline{G}_N(1)$) and that $\overline{F}_N(x) \leq \overline{F}_{\hat{N}}(x)$ for all $N \geq \hat{N}$ and $x \in [0, 1]$ (so $\overline{F}_{\hat{N}}^{-1}(\alpha) \leq \overline{F}_N^{-1}(\alpha)$ for all $\alpha$).

If $\alpha \in (0.4, 1]$, then

$$\bar{\mu}_N(G_N^{-1}(\alpha)) \leq \frac{1}{G_N^{-1}(\alpha)} \leq \frac{1}{G_N^{-1}(0.4)} \leq 1.1 = D_{\hat{N}}(\alpha),$$

since $\overline{G}_N^{-1}(\alpha)$ is increasing in $\alpha$, and $1/\overline{G}_N^{-1}(0.4) \leq 1.1$ when $N \geq \hat{N}$.

**Lemma 18.** When $N$ is sufficiently large,

$$\int_{\alpha=0}^{1} D_N(\alpha) \, dH^{-1}(\alpha) < \infty.$$
Proof of Lemma 18. Since \( G_N(x) = 1 - \sum_{k=1}^{N} g_k(x) \), we have:

\[
\overline{G}_N(x) = 1 - \sum_{k=1}^{N} \exp(-Nx) \frac{(Nx)^{k-1}}{(k-1)!} \\
= 1 - \exp(-Nx) \left( \exp(Nx) - \sum_{k=N}^{\infty} \frac{(Nx)^k}{k!} \right) \geq \exp(-Nx) \frac{(Nx)^N}{N!}.
\]

Clearly, there exists an \( \bar{x} \in (0,1) \) such that

\[
\overline{F}_{N+1}(x) = \exp((N + 1)(1 - x))x^{N+1} \leq \exp(-Nx) \frac{(Nx)^N}{N!} \leq \overline{G}_N(x)
\]

for all \( x \in [0, \bar{x}] \). We therefore have \( D_{N+1}(\alpha) = 1/\overline{F}_{N+1}^{-1}(\alpha) \leq 1/\overline{G}_N^{-1}(\alpha) \) for all \( \alpha \in [0, \bar{\alpha}] \), where \( \bar{\alpha} = \min\{\overline{F}_{N+1}(\bar{x}), 0.4\} \). As a result,

\[
\int_{\alpha=0}^{1} D_{N+1}(\alpha) \, dH^{-1}(\alpha) \leq \int_{\alpha=0}^{\bar{\alpha}} \frac{1}{\overline{G}_N^{-1}(\alpha)} \, dH^{-1}(\alpha) + \int_{\alpha=\bar{\alpha}}^{1} \max \left( \frac{1}{\overline{F}_{N+1}^{-1}(\bar{x})}, 1.1 \right) \, dH^{-1}(\alpha) < \infty
\]

whenever we have

\[
\int_{\alpha=0}^{1} \frac{1}{\overline{G}_N^{-1}(\alpha)} \, dH^{-1}(\alpha) = \int_{x=0}^{\infty} \frac{N}{x} \, d\hat{w}_N(x) < \infty.
\]

Finiteness of the last integral follows from part one of the left-tail assumption.

Lemma 19. Suppose \( \lim_{N \to \infty} y_N \in (-\infty, \infty) \). Then \( \lim_{N \to \infty} \overline{\mu}_{N+1}(\sqrt{N}y_N + N) = 1 \).

Proof of Lemma 19. We first argue that for almost every \( y_0 \), \( \overline{\mu}_{N+1}(\sqrt{N}y_N + N) \) tends to 1 as \( N \to \infty \). For this we recall \( x^* \) and \( x_N \) from Lemmas 14–16.

Consider first \( y < x^* \). By Lemma 14, for \( N \) sufficiently large, the gains function is graded at \( y \), and hence

\[
\overline{\mu}_{N+1}(\sqrt{N}y + N) = C(0, \sqrt{N}x_N + N) = \frac{N + 1}{\sqrt{N}x_N + N}.
\]

Since we have already shown that \( x_N \to x^* \) (Lemma 14), we conclude that \( \overline{\mu}_{N+1}(\sqrt{N}y + N) \) goes to 1.

Now consider \( y > x^* \) at which \( \hat{\gamma}_C^y \) is continuous. If the gains function is not graded at \( y \), then \( \overline{\mu}_{N+1}(\sqrt{N}y + N) = N/(\sqrt{N}y + N) \). If the gains function is graded at \( y \), then the length of the graded interval \( [a, b] \ni y \) in the central limit units is less than \( \epsilon_N = \bar{\gamma}/(\gamma \sqrt{N}) \) for some \( \gamma > 0 \) independent of \( N \) (see Lemma 16). Since \( \overline{\mu} \) is decreasing (Lemma 3), we have

\[
\frac{N}{\sqrt{N}(y + \epsilon_N) + N} \leq \overline{\mu}_{N+1}(\sqrt{N}y + N) \leq \frac{N}{\sqrt{N}(y - \epsilon_N) + N},
\]

since \( \lim_{z \nearrow a} \overline{\mu}_{N+1}(\sqrt{N}z + N) = N/(\sqrt{N}a + N) \) and \( \lim_{z \searrow b} \overline{\mu}_{N+1}(\sqrt{N}z + N) = N/(\sqrt{N}b + N) \). As a result, \( \overline{\mu}_{N+1}(\sqrt{N}y + N) \) is squeezed to 1 as \( N \) goes to infinity.
We conclude that \( \bar{\mu}_{N+1}(\sqrt{N}y + N) \) goes to 1 for \( y > x^* \) at which \( \hat{\gamma}_C^\infty \) is continuous. Since \( \hat{\gamma}_C^\infty(y) \) is a monotone function of \( y \), it is continuous at almost every \( y \), so the convergence \( \bar{\mu}_N \to 1 \) is almost everywhere.

Finally, suppose \( \lim_{N \to \infty} y_N = y \in (-\infty, \infty) \). Choose \( y' \) and \( y'' \) such that \( y \in (y', y'') \) and such that
\[
\lim_{N \to \infty} \bar{\mu}_{N+1}(\sqrt{N}y' + N) = 1 = \lim_{N \to \infty} \bar{\mu}_{N+1}(\sqrt{N}y'' + N).
\]
When \( N \) is sufficiently large, we have \( y_N \in (y', y'') \), so
\[
\bar{\mu}_{N+1}(\sqrt{N}y'' + N) \leq \bar{\mu}_{N+1}(\sqrt{N}y_N + N) \leq \bar{\mu}_{N+1}(\sqrt{N}y' + N).
\]
Taking the limit as \( N \to \infty \), we conclude \( \lim_{N \to \infty} \bar{\mu}_{N+1}(\sqrt{N}y_N + N) = 1. \)

**Proof of Proposition 9.** We first prove that
\[
\lim_{N \to \infty} \lambda_N(v; H) \to v - c
\]
for every \( v \in [\underline{v}, \overline{v}] \).

Replacing \( \bar{\mu}_N \) by 1 in equation (18), the definition of \( \lambda_N(v; H) \), we have
\[
\Pi_N(H) + \int_{y=0}^{\infty} G_N(y) \, d\hat{w}_N(y) - \int_{v=v}^{\overline{v}} d\nu = \Pi_N(H) + \left( \overline{v} - \int_{y=0}^{\infty} g_N(y) \hat{w}_N(y) \, dy \right) - (\overline{v} - v)
\]
\[
= \Pi_N(H) - \int_{v'=v}^{\overline{v}} v' \, dH(v') + v.
\]
Since by Proposition 7 \( \lim_{N \to \infty} \Pi_N(H) \to \int_{v'=v}^{\overline{v}} v' \, dH(v') - c \), to prove (2), it suffices to prove that
\[
\lim_{N \to \infty} \int_{y=0}^{\infty} |1 - \bar{\mu}_N(y)| \, d\hat{w}_N(y) = 0.
\]
Changing variables, we can rewrite the above equation as:
\[
\lim_{N \to \infty} \int_{\alpha=0}^{1} |1 - \bar{\mu}_N(G_N^{-1}(\alpha))| \, dH^{-1}(\alpha) = 0.
\]

We note that Stieltjes integration with respect to \( dH^{-1}(\alpha) \) is equivalent to a Lebesgue integration with respect to the finite measure \( \omega \) on \([0, 1]\) satisfying \( \omega([s, t]) = H^{-1}(t) - H^{-1}(s) \), \( 0 \leq s \leq t \leq 1 \), and \( \omega(\{1\}) = 0 \). Part one of the left-tail assumption implies that
\[
\omega(\{0\}) = \lim_{\alpha \to 0} \omega([0, \alpha]) = \lim_{\alpha \to 0} H^{-1}(\alpha) - H^{-1}(0) \leq \lim_{\alpha \to 0} G_N^{-1}(\alpha)^\varphi = 0
\]
for some \( \varphi > 1 \). Therefore, \( \omega(\{0, 1\}) = 0 \).

The central limit theorem implies that \( \lim_{N \to \infty} (G_N^{-1}(\alpha) - (N - 1)/\sqrt{N - 1}) = \Phi^{-1}(\alpha) \) for every \( \alpha \in (0, 1) \). Therefore, Lemma 19 implies \( \lim_{N \to \infty} \bar{\mu}_N(G_N^{-1}(\alpha)) = 1 \) for every \( \alpha \in (0, 1) \). Moreover, Lemmas 17 and 18 imply that there exists a \( \tilde{N} \) such that for all \( N \geq \tilde{N} \), the
integrand $|1 - \overline{\mu}_N(G_N^{-1}(\alpha))|$ in (3) is dominated by $1 + D_N(\alpha)$ which is integrable with respect to $\omega$. Therefore, equation (3) follows from the dominated convergence theorem, from which equation (2) follows.

Finally, using the definition of $\overline{\lambda}_N(v; H)$, we have

$$\overline{\lambda}_N(v; H) \leq \overline{\Pi}_N(H) + \int_{y=0}^{\infty} \overline{\mu}_N(y)(1 + G_N(y)) d\hat{\omega}_N(y) \leq (\overline{c} - c) + 2 \int_{x=0}^{1} D_N(\alpha) dH^{-1}(\alpha) < \infty,$$

for all $v \in [v, \overline{v}]$ and $N \geq \hat{N}$, where the last two inequalities follow from Lemmas 17 and 18, respectively. Thus

$$\lim_{N \to \infty} \int_V \overline{\lambda}_N(v; H) dH'(v) = \int_V v dH'(v) - c$$

follows the dominated convergence theorem using (2).

The proof for the must-sell $\hat{\lambda}_N(v; H)$ is identical, after replacing $\overline{\mu}_N(x)$ with $\hat{\mu}_N(x) = (N - 1)/x$ and $\overline{\Pi}_N(H)$ with $\hat{\Pi}_N(H)$.

**Lemma 20.** Suppose the condition on $H$ in Lemma 10 holds. For any $\epsilon > 0$, there exists an $\hat{N}$ such that for all $N > \hat{N}$, we have

$$\hat{\gamma}_N(x) \leq \hat{\gamma}_N(y) \exp(x - y).$$

for all $x \geq y$ such that $\hat{\gamma}_N(y) \geq \epsilon$.

**Proof of Lemma 20.** The condition on $H$ implies that the support of $H$ has no gap on $[y, \overline{v}]$, so $H^{-1}$ is continuous on $[0, 1]$. We can partition $[0, 1]$ into a countable collection of intervals $\{[\alpha_i, \beta_i] : i \in I\}$ such that $\alpha_i < \beta_i$, and either $H^{-1}$ is strictly increasing on $[\alpha_i, \beta_i]$, or $H^{-1}$ is constant on $[\alpha_i, \beta_i]$ (i.e., $H$ has a mass point at $v$, where $v = H^{-1}(p)$ for all $p \in [\alpha_i, \beta_i]$). If $H^{-1}$ is strictly increasing on $[\alpha_i, \beta_i]$, then

$$H^{-1}(q) - H^{-1}(p) \leq \frac{q - p}{C}. \tag{4}$$

for any $p, q \in (\alpha_i, \beta_i)$ such that $p \leq q$, since in this case we have $H(H^{-1}(q)) = q$ and $H(H^{-1}(p)) = p$. By continuity of $H^{-1}$ we can extend (4) to any $p, q \in [\alpha_i, \beta_i]$ such that $p \leq q$.

If $H^{-1}$ is constant on $[\alpha_i, \beta_i]$, then clearly (4) also holds for any $p, q \in [\alpha_i, \beta_i]$ such that $p \leq q$. Since $\{[\alpha_i, \beta_i] : i \in I\}$ is a partition of $[0, 1]$, we conclude that (4) holds for any $p, q \in [0, 1]$ such that $p < q$.

With the substitution $q = G_N^C(x)$ and $p = G_N^C(y)$, with $x > y$, equation (4) becomes

$$\hat{\gamma}_N^C(x) - \hat{\gamma}_N^C(y) \leq \frac{G_N^C(x) - G_N^C(y)}{C}.$$

Thus,

$$\frac{\hat{\gamma}_N^C(x)}{\hat{\gamma}_N^C(y)} \leq 1 + \frac{1}{\hat{\gamma}_N^C(y)} \frac{G_N^C(x) - G_N^C(y)}{C}.$$
The log-1 Lipschitz condition that we want to prove is equivalent to
\[
\frac{\dot{\gamma}_N^C(x)}{\dot{\gamma}_N^C(y)} \leq \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))).
\]
Thus, it is sufficient to show that for large \( N \),
\[
1 + \frac{1}{\gamma_N^C(y)} \frac{G_N^C(x) - G_N^C(y)}{C} \leq \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))).
\]
Both sides are equal to one when \( x = y \), and the gains function \( g_N \) is not graded when \( x \geq \hat{N} \), the log-1 Lipschitz condition that we want to prove is equivalent to
\[
\frac{g_N^C(x)}{\gamma_N^C(y)C}.
\]
and
\[
\frac{g_N^C(x)}{g_N(G_N^{-1}(G_N^C(x)))} \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y)))
\]
\[
= \sqrt{N-1} \exp(G_N^{-1}(G_N^C(x)) - G_N^{-1}(G_N^C(y))) \geq \sqrt{N-1}.
\]
We now show that (5) is always less than (6). Note that \( g_N \) attains its maximum when \( g_N = g_{N-1} \), i.e., when \( x = N - 1 \), at a value of \( \frac{(N-1)^{N-1}}{(N-1)!} \exp(-(N-1)) \). Multiplied by \( \sqrt{N-1} \), this upper bound converges to \( \phi(0) \). Hence, when \( N \) is sufficiently large, \( g_N^C(x) \leq 2\phi(0) \) for all \( x \). Since \( \dot{\gamma}_N^C(z) > 0 \), there is a \( N \) large enough such that
\[
\frac{g_N^C(x)}{\gamma_N^C(y)C} \leq \frac{2\phi(0)}{\epsilon C^2} \leq \sqrt{N-1}
\]
which proves the lemma. \( \square \)

**Proof of Lemma 10.** If \( \underline{\nu} > c \), then we can take \( \epsilon = \underline{\nu} - c \) in the statement of Lemma 20, in which case the statement of the Lemma follows immediately.

If \( \underline{\nu} < c \), then \( \dot{\gamma}_N^C(-\sqrt{N-1}) < 0 \), so that \( F_N^C(x) \) is non-positive for \( x \) close to \( -\sqrt{N-1} \). Hence, there must be a graded interval at the bottom of the form \([-\sqrt{N-1}, x_N] \). By Lemma 14, \( x_N \) converges to \( x^* \). Moreover, by Lemma 16, \( \tilde{\gamma}_N^C \) converges almost surely to \( \tilde{\gamma}_\infty^C \). Thus, there exists an \( \hat{N} \) such that for all \( N > \hat{N} \), \( \dot{\gamma}_N^C(x_N) \geq \epsilon \). If we take \( \epsilon = \dot{\gamma}_\infty^C(x^*)/2 \) in Lemma 20, then there exists a \( \hat{N}' \geq \hat{N} \) so that for all \( N > \hat{N}' \), the log-1 Lipschitz condition is satisfied for all \( x \geq x_N \). This implies that there is exactly one graded interval, and the conclusion of the Lemma follows. \( \square \)

**Proof of Proposition 10.** We first derive the allocation. When \( \underline{\nu} > c \), we have \( x^* = -\infty \) and the gains function \( \tilde{\gamma} \) is not graded when \( N \) is sufficiently large. In this case \( Q_N^C(x) \) is always exactly 1.

When \( \underline{\nu} < c \), \( x^* \in (-\infty, \infty) \), and the gains function \( \tilde{\gamma} \) is single crossing (Section 4.4) when \( N \) is sufficiently large. Then \( Q_N^C(x) = \min((x\sqrt{N} + N)/(x_N\sqrt{N} + N), 1) \). Since \( x_N \) converges to \( x^* \) as defined by equation (29), \( Q_N^C(x) \) converges to 1 as \( N \to \infty \).
We now derive the transfer. From Lemma 10, we know that there is at most one graded interval of the form \([-\sqrt{N}, x_N]\), where \(x_N = -\sqrt{N}\) if \(y > c\) and \(x_N > -\sqrt{N}\) if \(y < c\).

Recall that

\[
\Xi_N(x) = \frac{1}{g_N(x)} \int_{y=0}^{x} \Xi_N(y)g_N(y) \, dy,
\]

\[
\Xi_N(x) = \overline{\mu}_N(x)\overline{w}_N(x) - \overline{\lambda}_N(\overline{w}_N(x)) - c\overline{Q}_N(x),
\]

\[
\overline{\lambda}_N(\overline{w}_N(x)) = \int_{y=0}^{\infty} \overline{\gamma}_N(y)g_{N-1}(y) \, dy + \int_{y=x}^{\infty} \overline{\mu}_N(y)G_N(y)d\overline{\lambda}_N(y) - \int_{y=x}^{\infty} \mu_N(y)d\overline{\lambda}_N(y)
\]

\[
= \int_{y=0}^{\infty} \overline{\gamma}_N(y)g_{N-1}(y) \, dy + \int_{y=0}^{\infty} \mu_N(y)g_N(y)G_N(y)d\overline{\lambda}_N(y) + \mu_N(x)\overline{w}_N(x) + \int_{y=x}^{\infty} \overline{w}_N(y)d\overline{\mu}_N(y).
\]

Furthermore,

\[
\int_{y=0}^{\infty} \overline{\mu}_N(y)G_N(y)d\overline{\lambda}_N(y) = \int_{y=0}^{\infty} \overline{\mu}_N(y)G_N(y)d\overline{\gamma}_N(y)
\]

\[
= -\int_{y=0}^{\infty} \overline{\gamma}_N(y)d(\overline{\mu}_N(y)G_N(y))
\]

\[
= -\int_{y=0}^{\infty} \overline{\gamma}_N(y)G_N(y)d\overline{\mu}_N(y) - \int_{y=0}^{\infty} \overline{\gamma}_N(y)\overline{\mu}_N(y)G_N(y) \, dy
\]

\[
= -\int_{y=0}^{\infty} \overline{\gamma}_N(y)G_N(y)d\overline{\mu}_N(y) - \int_{y=0}^{\infty} \overline{\gamma}_N(y)g_{N-1}(y) \, dy,
\]

where the last inequality comes from equation (32). Thus,

\[
\overline{\lambda}_N(\overline{w}_N(x)) = -\int_{y=0}^{\infty} \overline{\gamma}_N(y)G_N(y)d\overline{\mu}_N(y) + \overline{\mu}_N(x)\overline{w}_N(x) + \int_{y=x}^{\infty} \overline{w}_N(y)d\overline{\mu}_N(y),
\]

and

\[
\Xi_N(x) = \int_{y=0}^{x} \overline{\gamma}_N(y)G_N(y)d\overline{\mu}_N(y) + \int_{y=x}^{\infty} (\overline{\gamma}_N(y)G_N(y) - \overline{w}_N(y))d\overline{\mu}_N(y) - c\overline{Q}_N(x)
\]

\[
= \int_{y=0}^{x} \overline{\gamma}_N(y)G_N(y)d\overline{\mu}_N(y) - \int_{y=x}^{\infty} \overline{\gamma}_N(y)(1 - G_N(y))d\overline{\mu}_N(y) - c(\overline{Q}_N(x) - \overline{\mu}_N(x))
\]

Let us now switch to central limit units.

\[
\Xi_N^C(x) = \Xi_N(\sqrt{N} - 1x + N - 1)
\]

\[
= \int_{y=-\sqrt{N}}^{x} \overline{\gamma}_N^C(y)G_N^C(y)d\overline{\mu}_N^C(y) - \int_{y=x}^{\infty} \overline{\gamma}_N^C(y)(1 - G_N^C(y))d\overline{\mu}_N^C(y) - c(\overline{Q}_N^C(x) - \overline{\mu}_N^C(x)).
\]

By Lemmas 11 and 13, \(\overline{\gamma}_N^C(y) \rightarrow \overline{\gamma}_\infty^C(y) = H^{-1}(\Phi(y)) - c\) and \(G_N^C(y) \rightarrow \Phi(y)\) as \(N \rightarrow \infty\).
Moreover, we have
\[
\sqrt{N - 1} d\mu_N^C(y) = \begin{cases} 
0 & \text{if } y < x_N; \\
(N - 1) \left( \frac{N-1}{x_N \sqrt{N-1+N-1}} - \frac{N}{x_N \sqrt{N-1+N-1}} \right) & \text{if } y = x_N; \\
-(N - 1) \frac{N-1}{(y \sqrt{N-1+N-1})^2} dy & \text{if } y > x_N,
\end{cases}
\]
where the mass point on $x_N$ is derived by comparing $\mu_N^C$ to the left and right of $x_N$, and
\[
\sqrt{N - 1}(\bar{Q}_N(x) - \bar{\mu}_N^C(x)) = \begin{cases} 
\sqrt{N - 1} \left( \frac{x\sqrt{N-1+N-1}}{x_N \sqrt{N-1+N-1}} - \frac{N}{x_N \sqrt{N-1+N-1}} \right) & \text{if } x < x_N; \\
\sqrt{N - 1} \left( 1 - \frac{N-1}{x \sqrt{N-1+N-1}} \right) & \text{if } x > x_N,
\end{cases}
\]
which converges to $x$ in both cases.

Define $F(x) = \lim_{N \to \infty} \sqrt{N - 1} \Xi_N^C(x)$. We have
\[
F(x) = \begin{cases} 
-cx + \gamma^C_\infty(x^*)(1 - \Phi(x^*)) + \int_{y=x^*}^{\infty} \gamma^C_\infty(y)(1 - \Phi(y)) dy & x < x^*; \\
-cx - \gamma^C_\infty(x^*)\Phi(x^*) - \int_{y=x^*}^{x} \gamma^C_\infty(y)\Phi(y) dy + \int_{y=x}^{\infty} \gamma^C_\infty(y)(1 - \Phi(y)) dy & x > x^*.
\end{cases}
\]

Therefore,
\[
\lim_{N \to \infty} T_N^C(x) = \frac{1}{\phi(x)} \int_{y=0}^{x} F(y)\phi(y) dy.
\]
\[\square\]
D Derivation of Aggregate Transfer for Uniform Distribution

Suppose the prior $H$ is the standard uniform distribution, so that $\hat{w}(x) = G_N(x)$, and that $c = 0$.

D.1 Must-sell Case

In the must-sell case, $\hat{\Xi}$ and $\hat{T}$ are independent of $c$, so $c = 0$ is without loss. We have:

$$\hat{\lambda}(G_N(x)) = \int_{y=0}^{\infty} G_N(y)g_{N-1}(y) \, dy + \int_{y=0}^{\infty} \frac{N-1}{y} G_N(y)g_N(y) \, dy - \int_{y=x}^{\infty} \frac{N-1}{y} g_N(y) \, dy$$

$$= 2 \int_{y=0}^{\infty} G_N(y)g_{N-1}(y) \, dy - (1 - G_{N-1}(x))$$

$$= 2\hat{\Pi} - (1 - G_{N-1}(x)),$$

$$\hat{\Xi}(x) = \frac{N-1}{x} G_N(x) - G_{N-1}(x) + 1 - 2\hat{\Pi}.$$

Next,

$$\int_{y=0}^{x} \hat{\Xi}(y)g_N(y) \, dy = \int_{y=0}^{x} \left( \frac{N-1}{y} G_N(y) - G_{N-1}(y) + 1 - 2\hat{\Pi} \right) g_N(y) \, dy$$

$$= 2 \int_{y=0}^{x} G_N(y)g_{N-1}(y) \, dy - G_N(x)G_{N-1}(x) + (1 - 2\hat{\Pi})G_N(x)$$

$$= G_{N-1}(x)^2 - 2 \int_{y=0}^{x} g_N(y)g_{N-1}(y) \, dy - G_N(x)G_{N-1}(x) + (1 - 2\hat{\Pi})G_N(x)$$

$$= G_{N-1}(x)g_N(x) - 2 \int_{y=0}^{x} g_N(y)g_{N-1}(y) \, dy + (1 - 2\hat{\Pi})G_N(x)$$

$$= G_{N-1}(x)g_N(x) - \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!} G_{2N-2}(2x) + (1 - 2\hat{\Pi})G_N(x)$$

$$= G_{N-1}(x)g_N(x) + \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!} (G_N(x) - G_{2N-2}(2x))$$

where the second line follows from integration by parts, the third and fourth lines use $G_N = G_{N-1} - g_N$, the fifth line is a direct computation using the formula for $g_N$ in (14), and the last line follows from

$$\hat{\Pi} = \int_{y=0}^{\infty} G_N(y)g_{N-1}(y) \, dy = \frac{1}{2} - \int_{y=0}^{\infty} g_N(y)g_{N-1}(y) \, dy = \frac{1}{2} \left( 1 - \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!} \right).$$

Therefore, when $x > 0$,

$$\hat{T}(x) = G_{N-1}(x) + \frac{\left( \frac{2N-3}{N-1} \right) G_N(x) - G_{2N-2}(2x)}{g_N(x)}.$$
In the central limit normalization, we define
\[ \hat{T}^C(x) = \hat{T}(N - 1 + \sqrt{N - 1}x). \]
Lemma 11 shows that \( G_N(N - 1 + \sqrt{N - 1}x) \rightarrow \Phi(x) \) and \( g_N(N - 1 + \sqrt{N - 1}x) \rightarrow \phi(x) \) as \( N \rightarrow \infty \), where \( \Phi \) and \( \phi \) are, respectively, the cumulative distribution and density of a standard Normal; this also implies that \( G_{2N-2}(2(N - 1 + \sqrt{N - 1}x)) \rightarrow \Phi(\sqrt{2}x) \). Finally, using Stirling’s approximation, it is easy to check that \( (\frac{2N-3}{2N-3} \sqrt{N - 1}) \rightarrow 1/\sqrt{\pi} \) as \( N \rightarrow \infty \). Therefore,
\[
\lim_{N \to \infty} \hat{T}^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(\sqrt{2}x)}{\sqrt{\pi} \phi(x)}
\]
for a fixed \( x \).

**D.2 Can-keep Case**

We have shown that the uniform distribution is single-crossing in Section 4.4. Let \([0, x^*] \) denote the graded interval. The cutoff \( x^* \) satisfies (cf. (28))
\[
\frac{G_N(x^*)}{2} = g_{N+1}(x^*).
\]  
(7)

This equation implies that \( G_{N+1}(x^*) = G_N(x^*) - g_{N+1}(x^*) = g_{N+1}(x^*) = G_N(x^*)/2 \).

Define the constants
\[
C = \int_{x=0}^{\infty} \pi(x)g_{N-1}(x) \, dx + \int_{x=0}^{\infty} \pi(x)G_N(x)g_N(x) \, dx
\]
\[
= \int_{x=0}^{x^*} \exp(x - x^*)G_N(x^*)g_{N-1}(x) \, dx + \int_{x=0}^{x^*} N \frac{G_N(x)}{x} g_N(x) \, dx
\]
\[
+ \int_{x=x^*}^{\infty} G_N(x)g_{N-1}(x) \, dx + \int_{x=x^*}^{\infty} \frac{N-1}{x} G_N(x)g_N(x) \, dx
\]
\[= C_1 + C_2 \]

We can simplify the constants as follows:
\[
C_1 = 2 \int_{x=0}^{x^*} \exp(x - x^*)G_N(x^*)g_{N-1}(x) \, dx
\]
\[= 2G_N(x^*)g_N(x^*)
\]
\[
C_2 = 2 \int_{x=x^*}^{\infty} G_N(x)g_{N-1}(x) \, dx
\]
\[= 1 - G_{N-1}(x^*)^2 - 2 \int_{x=x^*}^{\infty} g_N(x)g_{N-1}(x) \, dx
\]
\[= 1 - G_{N-1}(x^*)^2 - \frac{(2N-3)}{2N-3} (1 - G_{2N-2}(2x^*))
\]

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\[ \begin{align*}
C &= 2G_N(x^*)g_N(x^*) + 1 - G_{N-1}(x^*)^2 - \frac{(2N-3)(N-1)}{22N-3} (1 - G_{2N-2}(2x^*)).
\end{align*} \]

Then
\[
\lambda(G_N(x)) = C - \int_{y=x}^{\infty} \mu(y)g_N(y) \, dy
\]
\[
= \begin{cases}
C - \int_{y=x}^{x^*} \frac{N}{x^*} g_N(y) \, dy - \int_{y=x^*}^{\infty} \frac{N-1}{y} g_N(y) \, dy & x \leq x^* \\
C - \int_{y=x}^{\infty} \frac{N-1}{y} g_N(y) \, dy & x > x^*
\end{cases}
\]
\[
= \begin{cases}
C - (G_N(x^*) - G_N(x)) \frac{N}{x^*} - (1 - G_{N-1}(x^*)) & x \leq x^* \\
C - (1 - G_{N-1}(x)) & x > x^*
\end{cases}
\]

and
\[
\Xi(x) = \begin{cases}
G_N(x) \frac{N}{x^*} - C + (G_N(x^*) - G_N(x)) \frac{N}{x^*} + (1 - G_{N-1}(x^*)) & x \leq x^* \\
- C + G_N(x^*) \frac{N}{x^*} + 1 - G_{N-1}(x^*) & x > x^*
\end{cases}
\]

For \( x \leq x^* \), we have:
\[
\int_{y=0}^{x} \Xi(y)g_N(y) \, dy = \int_{y=0}^{x} \left(-C + G_N(x^*) \frac{N}{x^*} + 1 - G_{N-1}(x^*)\right) g_N(y) \, dy
\]
\[
= \left(-C + G_N(x^*) \frac{N}{x^*} + 1 - G_{N-1}(x^*)\right) G_N(x).
\]

For \( x > x^* \), we have:
\[
\int_{y=0}^{x} \Xi(y)g_N(y) \, dy = \left(-C + G_N(x^*) \frac{N}{x^*} + 1 - G_{N-1}(x^*)\right) G_N(x^*)
\]
\[
+ \int_{x^*}^{x} \left(G_N(y) \frac{N-1}{y} - C + 1 - G_{N-1}(y)\right) g_N(y) \, dy.
\]

Simplifying the second term, we get:
\[
X = (1 - C)(G_N(x) - G_N(x^*))
\]
\[
+ 2 \int_{y=x^*}^{x} G_N(y)g_{N-1}(y) \, dy - (G_N(x)G_{N-1}(x) - G_N(x^*)G_{N-1}(x^*))
\]
\[
= (1 - C)(G_N(x) - G_N(x^*))
\]
\[
- 2 \int_{y=x^*}^{x} g_N(y)g_{N-1}(y) \, dy + g_N(x)G_{N-1}(x) - g_N(x^*)G_{N-1}(x^*)
\]
\[
= (1 - C)(G_N(x) - G_N(x^*))
\]
\[ -\frac{(2N-3)}{22N-3}(G_{2N-2}(2x) - G_{2N-2}(2x^*)) + g_N(x)G_{N-1}(x) - g_N(x^*)G_{N-1}(x^*). \]

Therefore, for \( x \leq x^* \), we have:

\[ \bar{T}(x) = \left( -C + G_N(x^*) \frac{N}{x^*} + 1 - G_{N-1}(x^*) \right) \frac{G_N(x)}{g_N(x)}. \]

For \( x > x^* \) we have:

\[ T(x) = \left[ G_N(x^*)^2 \frac{N}{x^*} - G_{N-1}(x^*)^2 + (1 - C)G_N(x) - \frac{(2N-3)}{22N-3}(G_{2N-2}(2x) - G_{2N-2}(2x^*)) \right] \frac{1}{g_N(x)} + G_{N-1}(x). \]

Finally, we take the limit as \( N \to \infty \) for the central limit normalization:

\[ T^C(x) = T(N - 1 + \sqrt{N - 1}x). \]

Since \( G_N(x^*)/2 = G_{N+1}(x^*) \) by the discussion following equation (7), we must have \((x^* - (N - 1))/\sqrt{N - 1} \to -\infty \), \( G_N(x^*) \to 0 \), and \( g_N(x^*) \to 0 \) as \( N \to \infty \). Moreover, by equation (7), \( NG_N(x^*)/x^* = 2Ng_{N+1}(x^*)/x^* = 2g_N(x^*) \to 0 \) as \( N \to \infty \). Substituting these into the expressions of \( C \) and \( \bar{T} \) and simplify as in the must-sell case, we get

\[ \lim_{N \to \infty} T^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}} \phi(x). \]

References