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OPTIMAL AUCTION DESIGN WITH COMMON VALUES: AN INFORMATIONALLY ROBUST APPROACH

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A profit-maximizing seller has a single unit of a good to sell. The bidders have a pure common value that is drawn from a distribution that is commonly known. The seller does not know the bidders' beliefs about the value and thinks that beliefs are designed adversarially by Nature to minimize profit. We construct a *strong maxmin solution* to this joint mechanism design and information design problem, consisting of a mechanism, an information structure, and an equilibrium, such that neither the seller nor Nature can move profit in their respective preferred directions, even if the deviator can select the new equilibrium. The mechanism and information structure solve a family of maxmin mechanism design and minmax information design problems, regardless of how an equilibrium is selected. The maxmin mechanism takes the form of a *proportional auction*: each bidder submits a one-dimensional bid, the aggregate allocation and aggregate payment depend on the aggregate bid, and individual allocations and payments are proportional to bids. We report a number of additional properties of the maxmin mechanisms, including what happens as the number of bidders grows large and robustness with respect to the prior over the value.

KEYWORDS: Mechanism design, information design, optimal auctions, profit maximization, common value, information structure, maxmin, Bayes correlated equilibrium, direct mechanism.

1. INTRODUCTION

1.1. *Background and Motivation*

WE STUDY THE DESIGN of profit-maximizing mechanisms when the bidders have a pure common value for the good being sold, but partial and differential information about that value. Potential applications include the sale of natural resources or financial assets, where to a first order all bidders have the same preferences over the market value of the resource or the future cash flows of the asset.

Although common-value auctions have been studied since the early days of auction theory, relatively little is known about optimal common-value auctions. When bidders' signals are independent and one dimensional, Bulow and Klemperer (1996) have argued that a variation of the English auction is optimal when higher signals indicate higher values and bidders with higher signals have smaller information rents. In the perhaps more natural case where signals are correlated through the common value, such as in the mineral rights model, McAfee, McMillan, and Reny (1989) and McAfee and Reny (1992)

construct mechanisms that extract virtually all of the surplus by having the bidders bet on other bidders' information. While the full-surplus extracting mechanisms are of significant theoretical interest, there are a number of reasons why they may not be practically useful, including that the designer may not know exactly how information is correlated, and the optimal mechanism may be too complicated for bidders to use.

This discussion points to some conceptual challenges in optimal auction design. First, optimal mechanisms vary widely with the model of bidders' information, for example, whether and how signals are correlated. At the same time, it may be difficult to determine, either through measurement or introspection, which model of information is empirically relevant, and, hence, which of the potentially optimal mechanisms is appropriate. Moreover, relatively little is known about how bidders would behave in optimal mechanisms if the model is misspecified, which raises the question of whether Bayesian optimal mechanisms should be used in the presence of such model uncertainty. Note that these problems also arise in non-common-value auction design. When values are private, these concerns are partially allayed by the broad consensus that the independent private value model is a useful benchmark. In contrast, when values are common, there is no comparably canonical model.

To address these issues, we model a seller who knows the distribution of the common value but regards bidders' beliefs and higher-order beliefs about the value as ambiguous. These beliefs are modeled as a common-prior *information structure*. The seller is concerned about model misspecification, and believes that the information structure is chosen adversarially by Nature to minimize equilibrium profit.

1.2. *Main Results*

This joint mechanism design and information design problem is not a standard zero-sum game, as a given mechanism and information structure need not have a unique equilibrium. What is the resulting profit level when there are multiple equilibria or, for that matter, if no equilibrium exists? We address the issues of equilibrium multiplicity and existence by employing a new solution concept: A *strong maxmin solution* is a triple of a mechanism, an information structure, and an equilibrium strategy profile, with the property that neither the sellers nor Nature can move equilibrium profit in their preferred directions by changing the mechanism or information structure, respectively, even if the deviator can select the equilibrium. In other words, holding the information structure fixed, the mechanism and equilibrium maximize profit, and holding the mechanism fixed, the information structure and equilibrium minimize profit. Moreover, these statements remain true regardless of how an equilibrium is selected. The solution has an associated *profit guarantee*, which is expected profit in the constituent equilibrium. The profit guarantee is both a tight lower bound on equilibrium profit for the mechanism across all information structures and a tight upper bound on equilibrium profit for the information structure across all mechanisms.

One can interpret the strong maxmin solution in terms of a class of games between the seller (who chooses the mechanism to maximize profit) and an adversarial Nature (who chooses the information structure to minimize profit). Each game in the class is associated with a fixed rule for selecting a bidder equilibrium. A strong maxmin solution corresponds to a strategy profile that is a Nash equilibrium of the game between seller and Nature for all bidder-equilibrium selection rules.

Our main result (Theorem 1) is the construction of a strong maxmin solution. The maxmin mechanism is what we term a *proportional auction*: Bidders submit one-dimensional bids. The aggregate allocation and the aggregate transfer depend only on

the aggregate bid, and individual allocations and transfers are proportional to bids.[1] In benchmark cases, the aggregate allocation is linearly increasing in the aggregate bid until it hits 1 and is constant thereafter. An interpretation is that bids are "demands" for a quantity of the good, which are completely filled when the aggregate demand is less than the available supply, and otherwise the good is rationed proportionally. Bidders pay a constant price per unit that depends only on the aggregate bid.

In the minmax information structure, bidders' signals are independent and identically distributed (i.i.d.) draws from the standard exponential distribution, and the expectation of the value given the signals depends only on the aggregate signal. Finally, in equilibrium, each bidder submits a bid that is equal to their signal.

When the number of bidders is large, the profit guarantee is approximately the entire ex ante gains from trade, that is, the expectation of the value under the prior minus the cost of production (or zero if the expected value is less than the cost). The guarantee also seems to be a substantial share of surplus even when the number of bidders is small. For example, when there are two bidders, the value is standard uniform, and production is costless, the maxmin proportional auction guarantees the seller at least 56 percent of the expected value as profit.

Before presenting our main theorem, we give a heuristic derivation of the solution. First, the minmax information structure is constructed so that the seller is indifferent between a wide range of mechanisms. In particular, the seller is indifferent between *all* mechanisms with the same aggregate allocation, and whenever the optimal mechanism rations the good, the seller is indifferent between allocating and not allocating.

The mechanism is then constructed to be a profit maximizing direct mechanism on the minmax information structure with the additional property that the optimal profit at the minmax information is minimum equilibrium profit across all information structures. Importantly, messages in the maxmin mechanism are "normalized" to be signals in the minmax information structure. We refer to this as the *double revelation principle*: The maxmin mechanism is a profit-maximizing direct mechanism on the minmax information structure, and the minmax information structure is a profit-minimizing correlated equilibrium on the maxmin mechanism. The existence of a solution of this form is a nontrivial result, and it does not follow from the standard revelation arguments.

The requirement that profit be minimized at the minmax information structure reduces to a pair of differential equations involving the mechanism's allocation and transfer rules. The first equation pins down the divergence of the allocation rule, which we refer to as the *aggregate allocation sensitivity*. The second differential equation, which we refer to as *profit-incentive alignment*, links ex post profit to the bidders' local incentives. In particular, it pins down the difference between the divergence of the transfers and the aggregate transfer, which we term the *aggregate excess growth*. The proportional auction solves these two equations and also satisfies the revelation constraints at the minmax information structure.

In the definition of a strong maxmin solution, the profit guarantee is only compared with profit in other equilibria, where we hold fixed the mechanism and vary the information structure, or vice versa. It is possible that there are other strong maxmin solutions with different profit guarantees, but this can only happen if either the seller or Nature cannot

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[1]Proportional auctions can be viewed as a generalization of the Tullock contest, which corresponds to the case where the good is always allocated and the aggregate transfer is linear in the aggregate bid. The closest real-world auctions that we can find are the "voucher auctions" used to privatize Soviet state assets in the 1990s (Krishna (2009), p. 184).

deviate to the objects we construct because it would result in a game with no equilibria. A distinct but related concern is that minor modifications to the maxmin mechanism, such as discretizing or bounding the message space, would lead to qualitatively new equilibria that dramatically change the profit guarantee.

We address these questions of uniqueness and robustness of the strong maxmin solution in the following manner. A strong maxmin solution is *finitely approximable* if there are finite mechanisms and finite information structures with associated profit guarantees that are arbitrarily close to the solution's profit guarantee. We show that our solution is finitely approximable (Theorem 3). Moreover, any finitely approximable solution must have the same value (Theorem 4). In fact, the approximating finite mechanisms are simply discrete proportional auctions, and the finite information structures correspond to finite partitions of signals in the minmax information structure. If we restrict the seller and Nature to finite mechanisms and information structures, respectively, then these approximations attain the sup-inf and inf-sup of profit, regardless of the equilibrium selection rule (Corollary 1). Thus, the strong maxmin solution we construct is a limit of ϵ equilibria of the zero-sum game in which the seller and Nature choose finite mechanisms and finite information structures, respectively.

As a last topic, we consider the behavior of maxmin proportional auctions as the number of bidders grows large and the value distribution and cost are held fixed. In the many-bidder limit, the optimal profit guarantee converges to the ex ante gains from trade. This generalizes a result of Du (2018) to the case where there may not be common knowledge of gains from trade. Moreover, this limit obtains even if the good is always sold, and at the same optimal rate of $O(1/\sqrt{N})$. Finally, we show that the profit guarantee converges to the ex ante gains from trade *even if the prior is misspecified*.

The maxmin modeling approach allows us to identify new mechanisms with desirable theoretical properties that hold uniformly across information structures and equilibria. There is a conceptual tension, however, between the extreme ambiguity aversion of the seller and the common knowledge of the information structure among the agents. In particular, why does the seller not simply ask the agents to report the information structure, and use these reports to further improve the mechanism? In our view, the information structure and Bayes Nash equilibrium are an as-if description of behavior, which we hope is a reasonable approximation. We do not want to interpret these objects literally as something that either the seller or the bidders could fully articulate. The maxmin mechanism does not require the bidders to report higher-order beliefs, nor does the seller need to specify a model of beliefs in order to compute the maxmin mechanism. In that sense, it is consistent with real-world limitations on knowledge and communication.[2] That being said, the assumption of large ambiguity is as extreme as the assumption that the seller knows the information structure exactly. We view it as a benchmark and a starting point for future work on informationally-robust optimal mechanisms. We return to this point in the conclusion of the paper.

1.3. *Related Literature*

This paper lies at the intersection of the literatures on mechanism design and information design. We build on the seminal paper of Myerson (1981) on optimal auction design,

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[2]To be sure, some features of the information structure are relatively easy to express, such as first-order expectations, but we are skeptical as to whether real-world bidders or auction designers can describe the fine details of higher-order beliefs.

and the subsequent work by Bulow and Klemperer (1996). We also draw heavily from the literature on informationally robust predictions (Bergemann and Morris (2013, 2016)).

The most closely related papers are Du (2018) and Bergemann, Brooks, and Morris (2016). Du (2018) solves our maxmin mechanism design problem in the limit as the number of bidders goes to infinity and the production cost is zero. Specifically, Du constructs a sequence of mechanisms and associated lower bounds on profit that converge to the expected surplus in the many-bidder limit. The mechanisms from Du (2018) do not achieve the optimal profit guarantee when the number of bidders is finite and more than 1.[3] In contrast, Bergemann, Brooks, and Morris (2016) construct what is essentially a strong maxmin solution for the special case of two bidders and two possible values. Interestingly, the minmax information structure they identify coincides with the one we construct, but the maxmin mechanisms are different. We discuss this further in Section 5.

Chung and Ely (2007), Yamashita (2018), and Chen and Li (2018) also study maxmin mechanism design when the seller does not know the information structure but when values are private and when the seller preferred equilibrium is selected. In contrast, we focus on a common value environment. Other conceptually related studies of robust mechanism design are Neeman (2003), Brooks (2013), Yamashita (2015), Carroll (2017), Bergemann, Brooks, and Morris (2019), and the literature on algorithmic mechanism design (e.g., Hartline and Roughgarden (2009)).

The rest of the paper proceeds as follows. Section 2 describes our model and solution concept. Section 3 informally derives the strong maxmin solution. Section 4 presents the main result. Section 5 discusses uniqueness of the profit guarantee. Section 6 explores the many-bidder limit. Section 7 is a conclusion. Appendix A appears at the end of the article and Appendices B–D are provided in the Supplemental Material (Brooks and Du (2021)).

2. MODEL

2.1. *Primitives*

A seller has a unit of a good that can be sold to $N \geq 2$ bidders. The bidders have a pure common value for the good v, which is distributed according to the cumulative distribution function H on $\mathbb{R}_+ = [0, \infty)$. The support of H, denoted V, is assumed to be bounded, with $\underline{v}$ and $\overline{v}$ denoting the minimum and maximum, respectively. We also assume that $\underline{v} < \overline{v}$.[4]

Bidders' preferences over probabilities of receiving the good, q_i, and the amount they pay for it, t_i, are represented by the state-dependent utility index $vq_i - t_i$.

The good costs $c \geq 0$ to produce. The seller's profit from the profiles of allocations $q = (q_1, \ldots, q_N)$ and transfers $t = (t_1, \ldots, t_N)$ is $\sum_{i=1}^{N}(t_i - cq_i)$. We assume that the expected value is strictly larger than c.[5]

For technical reasons, we assume that the left tail of H is not too thin. To state the precise condition, we need the following definition: For a cumulative distribution F on $\mathbb{R}$, the associated *quantile function* is

$$F^{-1}(\alpha) = \min\{x \mid F(x) \geq \alpha\}.$$

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[3]Du (2018) also solves the present problem in the case of one bidder. With one bidder and binary values, our model reduces to that of Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2018).

[4]If not, then the value is common knowledge and the seller can easily extract all of the gains from trade.

[5]Otherwise, the solution is trivial: It is possible that bidders have no information, in which case the seller prefers to keep the good, and the profit guarantee is zero.

Because F is increasing and right-continuous, the set of values with cumulative probability higher than α is closed, so this minimum is well defined. Now let G_N denote the distribution of the sum of N independent draws from the exponential distribution with unit arrival rate, also known as an Erlang distribution, which is a special case of the Gamma distribution. (This object features prominently in our analysis and is given explicitly in equation (15) below.) The first part of the left-tail assumption is that there exist $\epsilon > 0$ and $\varphi > 1$ such that, for all $x \in [0, \epsilon]$,

$$H^{-1}\big(G_N(x)\big) - \underline{v} \le x^{\varphi}.$$

The second part of the left-tail assumption is that if $\underline{v} > c$, then there exists an $\epsilon > 0$ such that, for all $x', x \in [0, \epsilon]$ such that $x' < x$,

$$\frac{H^{-1}\big(G_N(x)\big) - c}{H^{-1}\big(G_N(x')\big) - c} \le \exp(x - x').$$

These assumptions are satisfied if there is a mass point at $\underline{v}$ (which implies $H^{-1}(G_N(x)) = \underline{v}$ for x sufficiently small) or if H has a density that is bounded away from zero around $\underline{v}$.[6]

2.2. *Information*

Fix cumulative distributions F_1 and F_2. Recall that F_1 is a *mean-preserving spread* of F_2 if there exist a probability space and random variables X_1 and X_2 such that X_1 has distribution F_1, X_2 has distribution F_2, and $\mathbb{E}[X_1|X_2] = X_2$. Equivalently, for all $x \in \mathbb{R}$,

$$\int_{y=-\infty}^{x} \big(F_1(y) - F_2(y)\big)\, dy \ge 0, \tag{1}$$

with equality when $x = \infty$ (Blackwell and Girshick (1954), Rothschild and Stiglitz (1970)).

An *information structure* $\mathcal{S}$ consists of (i) a measurable set S_i of signals for each bidder i, (ii) a joint distribution $\pi \in \Delta(S)$, where $S = \times_{i=1}^{N} S_i$, and (iii) a function $w : S \to \mathbb{R}$ such that H is a mean-preserving spread of the distribution of $w(s)$. For a profile of signals s, $w(s)$ represents the interim expectation of v conditional on s.[7]

2.3. *Mechanisms*

A *mechanism* $\mathcal{M}$ consists of measurable sets of messages M_i for each i and measurable mappings $q_i : M \to [0, 1]$ and $t_i : M \to \mathbb{R}$ for each i, where $M = \times_{i=1}^{N} M_i$ is the set of

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[6]If $\underline{v} > c$ and H has a density $h(v) \ge b > 0$ for $v \in [\underline{v}, \underline{v} + \epsilon']$, then

$$\frac{dH^{-1}\big(G_N(x)\big)}{dx} = \frac{g_N(x)}{h\big(H^{-1}(G_N(x))\big)} \le \frac{g_N(x)}{b},$$

$$\frac{d\log\big(H^{-1}(G_N(x)) - c\big)}{dx} = \frac{g_N(x)}{\big(H^{-1}(G_N(x)) - c\big) \cdot h\big(H^{-1}(G_N(x))\big)} \le \frac{g_N(x)}{(\underline{v} - c)b}$$

whenever $G_N(x) \le H(\underline{v} + \epsilon')$, where g_N is the density for G_N. As is evident from the formula for g_N in (14) below, $g_N(x) \to 0$ at a rate of x (or faster) as $x \to 0$. Thus, there exists an $\epsilon > 0$ such that for every $x \in [0, \epsilon]$, $H^{-1}(G_N(x)) - \underline{v} \le x^{\varphi}$ for $\varphi \in (1, 2)$ and $d(\log(H^{-1}(G_N(x)) - c))/dx \le 1$. This implies the left-tail assumption.

[7]Thus, our notion of an information structure does not fully specify the correlation structure between s and v. Since the interim expectation is the key object in our analysis, this formulation simplifies notation.

message profiles, such that $\sum_{i=1}^{N} q_i(m) \leq 1$. For technical reasons, we assume that t_i is bounded below (although it may be negative).

We further restrict attention to mechanisms that satisfy a condition we call *participation security*: For every i, there exists $0 \in M_i$ such that $v q_i(0, m_{-i}) - t_i(0, m_{-i}) \geq 0$ for every $v \in V$ and every $m_{-i} \in M_{-i}$. By sending this message, bidder i is assured a nonnegative payoff ex post, no matter what messages are sent by the other bidders.

2.4. *Equilibrium*

A mechanism $\mathcal{M}$ and an information structure $\mathcal{S}$ comprise a game of incomplete information. A *(behavioral) strategy* for bidder i is a transition kernel $\beta_i : S_i \to \Delta(M_i)$. A profile of strategies $\beta = (\beta_1, \ldots, \beta_N)$ is identified with a transition kernel that associates to each $s \in S$ the product measure $\beta_1(s_1) \times \cdots \times \beta_N(s_N)$ on $\Delta(M)$.

Given a strategy profile β, bidder i's payoff is

$$U_i(\mathcal{M}, \mathcal{S}, \beta) = \int_S \int_M \big(w(s)q_i(m) - t_i(m)\big)\beta(dm|s)\pi(ds).$$

Note that since w, q, and $-t$ are all bounded above, this integral is always well defined. A strategy profile β is a *(Bayes Nash) equilibrium* if for all i and strategies β_i',

$$U_i(\mathcal{M}, \mathcal{S}, \beta) \geq U_i\big(\mathcal{M}, \mathcal{S}, \big(\beta_i', \beta_{-i}\big)\big).$$

The set of equilibria is denoted by $B(\mathcal{M}, \mathcal{S})$. Expected profit is

$$\Pi(\mathcal{M}, \mathcal{S}, \beta) = \int_S \int_M \sum_{i=1}^{N} \big(t_i(m) - cq_i(m)\big)\beta(dm|s)\pi(ds).$$

2.5. *Solution Concept*

We will shortly introduce the solution concept employed in this paper. This concept is motivated by the following simultaneous-move game between seller and Nature: Fix a measurable set X. Define $\mathbf{M}(X)$ to be the set of participation-secure mechanisms in which each bidder's message space is of the form $M_i \cup \{1, \ldots, k_i\}$ for some measurable $M_i \subseteq X$ and some nonnegative integer k_i. Similarly define $\mathbf{S}(X)$ to be the set of information structures where signal spaces are of the form $S_i \cup \{1, \ldots, k_i\}$ for some measurable $S_i \subseteq X$ and some nonnegative integer k_i.[8] Let $\mathcal{B}(X)$ denote the set of all selections from the equilibrium correspondence B on the subset of $\mathbf{M}(X) \times \mathbf{S}(X)$ for which an equilibrium exists. Given a selection $\beta^* \in \mathcal{B}(X)$, we define the game $\mathcal{G}(X, \beta^*)$ where seller and Nature simultaneously choose actions in $\mathbf{M}(X)$ and $\mathbf{S}(X)$. The seller's payoff is $\Pi(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S}))$ and Nature's payoff is $-\Pi(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S}))$ if $B(\mathcal{M}, \mathcal{S}) \neq \emptyset$, and both parties' payoffs are minus infinity if $B(\mathcal{M}, \mathcal{S}) = \emptyset$.

By fixing the equilibrium selection, we have formulated the joint mechanism design and information design problem as a standard non-cooperative game. A Nash equilibrium of this game $(\mathcal{M}, \mathcal{S})$ is nontrivial if a bidder equilibrium exists for the game $(\mathcal{M}, \mathcal{S})$. Such an equilibrium can be understood as a pair of an informationally robust mechanism and

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[8]The finitely many extra messages and signals allow us to add messages to direct revelation mechanisms in order to make them participation secure. This construction is used in the proof of Proposition 1.

a worst-case informational environment, which rationalize one another as optimal, given the equilibrium selection rule.[9]

A concern with this modeling approach is that whether $(\mathcal{M}, \mathcal{S})$ is a Nash equilibrium may depend on the particular equilibrium selection rule. This motivates us to consider pairs $(\mathcal{M}, \mathcal{S})$, which are nontrivial Nash equilibria for all selections β^*.[10] As Proposition 1 below shows, this notion of a "selection-invariant" nontrivial Nash equilibrium is equivalent to the following solution concept:

A *strong maxmin solution* is a triple $(\mathcal{M}, \mathcal{S}, \beta)$ of a mechanism, an information structure, and a strategy profile, with profit $\Pi = \Pi(\mathcal{M}, \mathcal{S}, \beta)$, such that the following conditions are satisfied:

C1. For any mechanism $\mathcal{M}'$ and any equilibrium β' of $(\mathcal{M}', \mathcal{S})$, $\Pi \geq \Pi(\mathcal{M}', \mathcal{S}, \beta')$.

C2. For any information structure $\mathcal{S}'$ and any equilibrium β' of $(\mathcal{M}, \mathcal{S}')$, $\Pi \leq \Pi(\mathcal{M}, \mathcal{S}', \beta')$.

C3. Strategy β is an equilibrium of $(\mathcal{M}, \mathcal{S})$.

We refer to Π as the *profit guarantee* of the solution.[11]

C1 and C2 say that the seller and Nature cannot improve their payoff by deviating, even if the deviator selects the equilibrium. C3 says that the profit guarantee is not vacuous, and there exists an equilibrium at which Π is attained. In fact, the definition implies that for a strong maxmin solution $(\mathcal{M}, \mathcal{S}, \beta)$, all equilibria of $(\mathcal{M}, \mathcal{S})$ result in the same profit.

The following result connects the strong maxmin solution to Nash equilibria of the previously defined non-cooperative game.

PROPOSITION 1: *Fix a pair $(\mathcal{M}, \mathcal{S})$. Then the following statements are equivalent*:
 (i) *There exist strategies β such that $(\mathcal{M}, \mathcal{S}, \beta)$ is a strong maxmin solution.*
 (ii) *There exists an X such that $(\mathcal{M}, \mathcal{S})$ is a nontrivial Nash equilibrium of $\mathcal{G}(X, \beta^*)$ for all $\beta^* \in \mathcal{B}(X)$.*

The proof is a straightforward application of the revelation principles for mechanism design and information design, and is relegated to the Appendix.

The main result of our paper is the construction of a strong maxmin solution. Proposition 1 shows that we can equivalently interpret this solution as a Nash equilibrium of the game between seller and Nature, regardless of how we select an equilibrium.[12] How we should model equilibrium selection depends on a number of considerations. On the one hand, β^* could select the profit-minimizing equilibrium if the seller is concerned for robustness with respect to equilibrium selection. On the other hand, the literature on partial implementation typically assumes that the mechanism designer can use their prominence

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[9]By assuming that the payoff from bidder-equilibrium nonexistence is minus infinity, we implicitly restrict the seller to only consider mechanisms for which an equilibrium exists on $\mathcal{S}$, and correspondingly for Nature. We can view this as capturing a belief of the seller that the information structure $\mathcal{S}$ is possible, and that they must ensure the mechanism is well behaved in that environment.

[10]Many mechanisms with multiple equilibria can be perturbed to select a particular equilibrium, with a negligible effect on profit. For this reason, it is not surprising that Nash equilibrium payoffs in $\mathcal{G}(X, \beta^*)$ are invariant to β^*. Nonetheless, the equilibrium selection rule could have a significant impact on which mechanisms and information structures are part of Nash equilibria.

[11]This definition nominally depends on qualifiers over *all* mechanisms and information structures, which are of course not well defined. However, it is clearly without loss to restrict attention to the set of incentive compatible and participation-secure direct mechanisms on $\mathcal{S}$ and truthful equilibria in C1, and to restrict attention to Bayes correlated equilibria on $\mathcal{M}$ and obedient strategies in C2.

[12]This invariance criterion is reminiscent of Govindan and Wilson (2009), who study sequential equilibrium outcomes that are invariant to the extensive form representation of the underlying reduced normal form.

to coordinate bidders on the designer's preferred equilibrium. It is therefore normatively appealing that the solution we construct does not depend on additional assumptions about equilibrium selection.

3. A ROAD MAP TO THE SOLUTION

We rigorously construct a strong maxmin solution at the beginning of Section 4, and Theorem 1 verifies that the construction is indeed a solution. This section gives an informal derivation. To be clear, our purpose is to develop intuition, and the proof of Theorem 1 does not depend on the present discussion.

3.1. *The Structure of the Solution*

The strong maxmin solution we construct is denoted $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$. The high level structure is as follows. Signals in the information structure and messages in the mechanism are elements of $\overline{M}_i = \overline{S}_i = [0, \infty)$. In addition, the equilibrium strategies specify that each bidder send a message that is equal to their signal: for all i and s_i, $\overline{\beta}_i(s_i)$ puts probability one on s_i. Thus, a common language is used for signals and messages. One interpretation is that the maxmin mechanism $\overline{\mathcal{M}}$ is a *direct mechanism* on the minmax information structure $\overline{\mathcal{S}}$, whereby a message is a "report" of which signal a bidder received, and bidders report truthfully in equilibrium. An equally valid interpretation is that $\overline{\mathcal{S}}$ is a *Bayes correlated equilibrium* (BCE) on $\overline{\mathcal{M}}$, whereby a signal is a "recommendation" of a message to send, and in equilibrium, bidders obey their recommendations.

If we held the information structure fixed and maximized profit across mechanisms and equilibria, then the well known revelation principle (Myerson (1981)) says that it is without loss of generality to restrict attention to direct mechanisms. Similarly, if the mechanism were fixed and we minimized profit across information structures and equilibria, then it is without loss to restrict attention to BCE, which is a kind of revelation principle for games (Bergemann and Morris (2013, 2016)). In the present model, both the mechanism and the information structure are endogenous, so the standard revelation arguments do not apply.[13] It is therefore a non-trivial result that there exists a solution that admits the same normalization. We refer to this as the *double revelation principle*.

3.2. *The Minmax Information Structure*

We next describe the rest of the minmax information structure $\overline{\mathcal{S}}$, from which we subsequently derive the maxmin mechanism. First, signals in $\overline{\mathcal{S}}$ turn out to be independently distributed. This is intuitive, for if signals were correlated, the seller could extract surplus by having bidders make bets about others' beliefs, similar to the full-surplus extracting mechanisms mentioned in the Introduction (although it may not be possible to extract the entire surplus).

Given independence, the rest of the form of $\overline{\mathcal{S}}$ can be understood using the celebrated *revenue-equivalence formula* of Myerson (1981), suitably adapted to the common value

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[13]Holding information fixed, any mechanism $\mathcal{M}$ and equilibrium β have an equivalent direct mechanism $\mathcal{M}'$ in which truth-telling is an equilibrium. But $\mathcal{M}'$ may have other equilibria with no counterpart in $\mathcal{M}$, and our solution concept considers how profit varies across *all* equilibria. Similarly, replacing a given information structure and equilibrium with the corresponding direct information may lead to a qualitatively different set of equilibria.

setting. Let us suppose that the marginal distribution of each signal s_i admits a density f_i. Revenue equivalence says that expected profit is, up to a constant, the expectation of the *virtual value* of the bidder who receives the good. When the value function is differentiable and the signal profile is s, the virtual value of bidder i is[14,15]

$$\psi_i(s) = w(s) - c - \frac{1 - F_i(s_i)}{f_i(s_i)} \frac{\partial w(s)}{\partial s_i},$$

where F_i is the cumulative distribution of bidder i's signal. Thus, the virtual value is equal to the gains from trade minus an information rent. The latter is the product of the inverse hazard rate, which is the relative measure of types who receive an information rent when s_i is allocated the good, and $\partial w(s)/\partial s_i$, which quantifies bidder i's private information.

Among independent-signal information structures, it is without loss of generality to normalize the signals to be exponential with a unitary arrival rate,[16] $F_i(x) = 1 - \exp(-x)$. As a result, the inverse hazard rate is constant and equal to 1, and drops out of the virtual value formula.

The remaining degree of freedom is the value function $w(s)$. To develop intuition for the minmax value function, we may ask, which value function would be worst for the seller? Drawing on experience from zero-sum games, we might suspect that the worst case would be associated with indifference between lots of mechanisms. This would roughly mean that $\overline{\mathcal{S}}$ is hard to respond to, in that while lots of mechanisms perform reasonably well, no mechanism stands out as exceptional.

In fact, there is a class of value functions that make the seller indifferent as to who is allocated the good for *every* signal profile, namely those of the form $w(s) = w(\Sigma s)$, where $\Sigma s = s_1 + \cdots + s_N$ is the *aggregate signal*.[17] (We maintain this convention for the sum of a vector's elements throughout the paper.) As a result, the interim expected value is equally sensitive to all signals, and all bidders have the same virtual value of $w(\Sigma s) - c - w'(\Sigma s)$.

We are still free to choose the particular function of the aggregate signal. An important variant of our model, discussed in Section 4.3, is the *must-sell case*, where the good has to be sold with probability 1. This is in contrast to the general *can-keep case*, where the seller can withhold the good. Note that Σs has the Erlang cumulative distribution G_N introduced in Section 2 and that g_N denotes the associated density.[18] All bidders have the same virtual value, so profit is

$$\int_{x=0}^{\infty} \bigl(w(x) - c - w'(x)\bigr) g_N(x) \, dx. \tag{2}$$

This formula assumes that transfers are set so that the bidder with the lowest signal receives a payoff of zero, which maximizes revenue subject to local incentive compatibil-

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[14]In the classic formulation of Myerson (1981), bidder i's virtual value is their value minus the inverse hazard rate. We obtain this formula if there are bidder-specific values $w_i(s)$ and signals are normalized so that $w_i(s) = s_i$, in which case the partial derivative is identically 1. The formula reported here is a special case of one that appears in Bulow and Klemperer (1996).

[15]Our formal arguments in Section 4 sidestep the direct calculation of virtual values, to avoid technical complications associated with whether there is an integral representation for the bidders' indirect utilities.

[16]Given any w and $F_1, \ldots, F_N$, an equivalent information structure would be one with i.i.d. exponential signals and the value function $w(F_1^{-1}(1 - \exp(-s_1)), \ldots, F_N^{-1}(1 - \exp(-s_N)))$.

[17]We hope we do not create confusion by using the same notation for the interim value as a function of the signal profile and as a function of the aggregate signal.

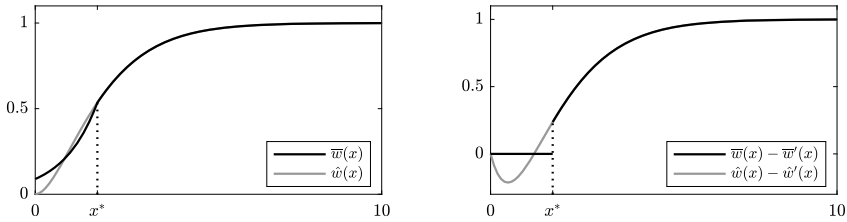[18]Both G_N and g_N have closed-form expressions, given as equations (14) and (15) below.

FIGURE 1.—Value functions and virtual value functions when $N = 2$, $v \sim U[0, 1]$, and $c = 0$.

ity and participation security (since the latter implies that interim payoffs are nonnegative). Note that the expectation of $w(x)$ must equal the ex ante expected value under H. Thus, to minimize profit, the value function should maximize the expected slope. This is achieved by the *fully revealing value function* $\widehat{w}(x) = H^{-1}(G_N(x))$, where H^{-1} is the quantile function for H. This value function matches aggregate signals and values comonotonically, so that their percentiles are perfectly correlated. It is fully revealing in that there is no uncertainty about the value, conditional on the join of the bidders' information. It is intuitive that $\widehat{w}$ minimizes profit, since it maximizes the bidders' private information about the value. Figure 1 illustrates the fully revealing value and virtual value functions when $N = 2$, $c = 0$, and v is standard uniform, so that $\widehat{w}(x) = G_2(x)$.

For some value distributions, $\widehat{w}$ is also the minmax value function when the seller can keep the good. This is not the case for the uniform distribution. The right-hand panel of Figure 1 shows that the virtual value is strictly negative when the aggregate signal is low, so that the seller strictly prefers to withhold the good. The seller can be made strictly worse off by adding noise to the bidders' information so that the seller is indifferent between selling and not selling. This requires that the virtual value is zero, that is, $w(x) - c - w'(x) = 0$. Equivalently, the *gains function* $\gamma(x) = w(x) - c$ (for interim expected gains from trade) is of the form $k \exp(x)$ for some $k \in \mathbb{R}_+$.

In the uniform example, we can replace the fully revealing gains function $\widehat{\gamma}(x) = \widehat{w}(x) - c$ on an interval $[0, x^*]$ with an exponential segment to obtain

$$\overline{\gamma}(x) = \begin{cases} \overline{\gamma}(0) \exp(x) & \text{if } x \leq x^*, \\ \widehat{\gamma}(x) & \text{if } x > x^*. \end{cases}$$

We choose $\overline{\gamma}(0)$ and x^* so that H remains a mean-preserving spread of the distribution of the interim expected value and so that the exponential curve connects continuously with the fully revealing gains function. This is the black curve in Figure 1, which is the minmax gains function when the seller can keep the good.

More generally, the sign of the fully revealing virtual value might switch back and forth, and there could be many exponential segments. In Section 4.1, we describe a general procedure that transforms the fully revealing gains function so that the resulting virtual value is everywhere nonnegative. We refer to this as *grading the gains function*, meaning we decrease the derivative of the gains function so that it does not grow faster than exponential. The graded gains and value functions are denoted by $\overline{\gamma}$ and $\overline{w}$, respectively, and the resulting information structure is $\overline{\mathcal{S}}$. Proposition 2 shows that profit on $\overline{\mathcal{S}}$ is at most

$$\overline{\Pi} = \int_{x=0}^{\infty} \overline{\gamma}(x) g_{N-1}(x) \, dx. \tag{3}$$

This formula can be obtained from (2) via integration by parts, using the fact that $dg_N(x)/dx = g_{N-1}(x) - g_N(x)$. The term $\overline{\Pi} + c$ denotes the largest posted price at which all bidders would be willing to purchase the good, which is the expectation of the value given a signal $s_i = 0$. Thus, $\overline{\Pi}$ is exactly optimal profit on $\overline{\mathcal{S}}$, and a posted price is an optimal mechanism (although it is not a maxmin mechanism).

3.3. *Sufficient Conditions for an Optimal Profit Guarantee*

We now derive a maxmin mechanism $\overline{\mathcal{M}}$ from $\overline{\mathcal{S}}$. At first glance, it seems that we do not learn very much from the requirement that $\overline{\mathcal{M}}$ maximize profit on $\overline{\mathcal{S}}$, because so many mechanisms are optimal. However, we learn a great deal from the requirements that (i) $\overline{\mathcal{S}}$ and $\overline{\beta}$ minimize equilibrium profit on $\overline{\mathcal{M}}$, and (ii) minimum profit is $\overline{\Pi}$, as we now explain.

Fix a mechanism $\mathcal{M}$ with message space $M_i = \overline{S}_i$ for all i. As reviewed in the Introduction, the problem of minimizing profit in $\mathcal{M}$ across information structures and equilibria is equivalent to minimizing profit across BCE.[19] We briefly review this solution concept. An *outcome* of $\mathcal{M}$ is a joint distribution over values and message profiles $\sigma \in \Delta(V \times M)$ such that the marginal distribution of v is H. The associated profit is the expectation of $\Sigma(t - cq)$ under σ. A BCE is an outcome that is *obedient*: for all i and m_i, m_i is a best response in M_i to the distribution of (v, m_{-i}) under σ and conditional on m_i. An outcome σ is *consistent with an information structure* $\mathcal{S}$ *and strategies* β if there exists a kernel $K : S \to \Delta(V)$ such that $w(s)$ is the expectation of V under the measure $K(dv|s)$ and

$$\sigma(dv, dm) = \int_S \pi(ds)\beta(dm|s)K(dv|s),$$

that is, σ is the marginal on values and messages obtained by integrating out signals. Note that if σ is consistent with $\mathcal{S}$ and β, then they have the same expected profit. It is a result of Bergemann and Morris (2013, 2016) that an outcome is a BCE if and only if it is consistent with some $\mathcal{S}$ and equilibrium β.[20] A fortiori, minimum profit across all BCE is equal to minimum profit across all information structures and equilibria.

Let us therefore examine the problem of minimizing profit in $\mathcal{M}$ across BCE. It turns out that the only obedience constraints that are relevant for our problem are those associated with local obedience, namely, that for all i and m_i,[21]

$$\int_{V \times M_{-i}} \left(v \frac{\partial q_i(m_i, m_{-i})}{\partial m_i} - \frac{\partial t_i(m_i, m_{-i})}{\partial m_i} \right) \sigma(dv, dm_{-i}|m_i) = 0.$$

The problem of minimizing profit subject to the constraint on the marginal distribution of v and local obedience is an infinite-dimensional linear program, for which the associated

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[19]Note that we do not make explicit use of BCE in Section 4 and in the proof of Theorem 1. Nonetheless, these ideas are at work "under the hood."

[20]Strictly speaking, our setup differs from that of Bergemann and Morris in that there are infinitely many states and actions, and we use a different notion of an information structure. The equivalence in our setting can be shown by analogous arguments (although we do not provide such an argument as part of our informal derivation).

[21]This is suggested by the fact that only local incentive constraints were used in the revenue equivalence argument that motivated $\overline{\mathcal{S}}$.

Lagrangian is

$$\mathcal{L}\big(\sigma, \{\alpha_i\}, \lambda\big) = \sum_{i=1}^{N} \int_{V \times M} \big(t_i(m) - cq_i(m)\big) \sigma(dv, dm)$$

$$+ \sum_{i=1}^{N} \int_{V \times M} \alpha_i(m_i) \left(v \frac{\partial q_i(m)}{\partial m_i} - \frac{\partial t_i(m)}{\partial m_i} \right) \sigma(dv, dm)$$

$$+ \int_{V \times M} \lambda(v) \big(H(dv) - \sigma(dv, dm) \big). \tag{4}$$

This Lagrangian has three terms: profit induced by the BCE, the sum of local obedience constraints times their corresponding multipliers (the functions α_i), and the sum of marginal constraints times their corresponding multipliers (the function λ).

Requirement (i) above is equivalent to saying that an outcome that is consistent with $(\overline{\mathcal{S}}, \overline{\beta})$, denoted by $\overline{\sigma}$, minimizes (4). The key properties of $\overline{\sigma}$ are that messages are i.i.d. standard exponential and $\overline{w}(\Sigma m)$ is the conditional expectation of v given m. A necessary condition for $\overline{\sigma}$ to be the profit-minimizing BCE is that for all (v, m),

$$\sum_{i=1}^{N} \left[t_i(m) - cq_i(m) + \alpha_i(m_i) \left(v \frac{\partial q_i(m)}{\partial m_i} - \frac{\partial t_i(m)}{\partial m_i} \right) \right] - \lambda(v) \geq 0, \tag{5}$$

with the constraint holding as an equality for (v, m) in the support of $\overline{\sigma}$.

We motivated (5) by treating (q, t) as fixed and σ as endogenous. But evaluated at the putative minimizer $\overline{\sigma}$, equation (5) becomes a constraint on the maxmin allocation and transfer rules, involving the as-yet unspecified multipliers λ and $\{\alpha_i\}$.

In fact, the correct multipliers can be deduced from (i) and (ii). Based on the envelope theorem, we can guess that $\lambda(v)$ is the derivative of minimum profit in the maxmin mechanism with respect to the prior probability of v. From (ii), this should coincide with the derivative of $\overline{\Pi}$ with respect to the probability of v, denoted $\overline{\lambda}(v)$. If not, then by making v either more or less likely, we could make minimum profit from the maxmin mechanism increase faster than $\overline{\Pi}$. The function $\overline{\lambda}$ has an explicit formula given in equation (18) below, and we will shortly use the fact that $\overline{\lambda}$ is concave.[22]

As for the multipliers on local obedience, there is an even simpler answer: $\alpha_i(m_i) = 1$ for all i and m_i. This is suggested by the fact that (4) is very similar to the Lagrangian for the linear program of *maximizing* profit given $\overline{\mathcal{S}}$, where we fix $\sigma = \overline{\sigma}$ and optimize over (q_i, t_i), and obedience is reinterpreted as incentive compatibility. As is well known, local incentive constraints bind at the solution, and the optimal multiplier on local incentive compatibility is the inverse hazard rate, which we have normalized to 1.

Substituting in these multipliers and letting $Q(m) = \Sigma q(m)$ denote the aggregate allocation, equation (5) reduces to, for all (v, m),

$$\nabla \cdot t(m) - \Sigma t(m) \leq v \nabla \cdot q(m) - \overline{\lambda}(v) - cQ(m), \tag{6}$$

---

[22]For each v, the optimal $\lambda(v)$ must satisfy (5) with equality for some m. As a result, λ is the minimum of a collection of linear functions, indexed by m.

where $\nabla\cdot$ is the divergence operator and the constraint holds with equality on the support of $\overline{\sigma}$, namely, pairs (v, m) such that $v = \widehat{w}(\Sigma m)$.[23] For a fixed m, the value $v = \widehat{w}(\Sigma m)$ must minimize the right-hand side of (6), and since $\overline{\lambda}$ is concave and, hence, right-differentiable, it must be that

$$\nabla \cdot q(m) = \overline{\lambda}'\big(\widehat{w}(\Sigma m)\big). \tag{7}$$

We refer to the left-hand side of (7) as the *aggregate allocation sensitivity*. In fact, $\overline{\lambda}'\big(\widehat{w}(x)\big)$ can be computed in closed form, and we denote it by $\overline{\mu}(x)$. When the value function is fully revealing, $\overline{\mu}(x) = (N-1)/x$, and on an interval where the value function is graded, $\overline{\mu}$ is a constant that depends on the endpoints of the interval. The exact formula is given in equation (17) below.

Substituting (7) into (6), we obtain the following condition on transfers:

$$\nabla \cdot t(m) - \Sigma t(m) = \widehat{w}(\Sigma m)\overline{\mu}(\Sigma m) - \overline{\lambda}\big(\widehat{w}(\Sigma m)\big) - cQ(m). \tag{8}$$

The left-hand side of (8) is the *aggregate excess growth*, that is, the difference between how fast the bidders' transfers grow in their own messages relative to exponential growth. We refer to equation (8) as *profit-incentive alignment*, since it links ex post profit, $\Sigma(t - cq)$, to the bidders' local incentive constraints, $v\nabla \cdot q - \nabla \cdot t$. This ensures that as long as bids are locally optimal, profit cannot fall below $\overline{\Pi}$.

We have been using the profit-minimization program to derive necessary conditions on a maxmin mechanism. But as we argue in Proposition 3, these conditions are actually *sufficient* for a mechanism to guarantee profit of at least $\overline{\Pi}$.[24] Specifically, if a mechanism is such that the aggregate allocation sensitivity is $\overline{\mu}$, and the aggregate excess growth and the aggregate allocation satisfy (8), then profit is at least $\overline{\Pi}$ in all information structures and all equilibria. The proof is essentially an application of weak duality.

3.4. *Construction of a Maxmin Mechanism*

The last step is to construct a mechanism that satisfies (7) and (8), and such that truth-telling is an equilibrium at $\overline{\mathcal{S}}$. Note that the latter condition is logically separate from the profit lower bound of Proposition 3.

Let us start with the allocation. Consider the case with two bidders. In the must-sell case, we have $Q(m) = 1$, so $q_2(m_1, m_2) = 1 - q_1(m_1, m_2)$, and the aggregate allocation sensitivity reduces to

$$\frac{\partial q_1(m_1, m_2)}{\partial m_1} - \frac{\partial q_1(m_1, m_2)}{\partial m_2} = \frac{1}{m_1 + m_2}. \tag{9}$$

Now consider a level curve where $m_1 + m_2 = x$. We can then view the left-hand side of (9) as the total derivative of q_1 with respect to m_1 along the parametric curve $m_2(m_1) = x - m_1$, so that integrating both sides, we obtain $q_1(m_1, m_2) = m_1/x + C(x)$ for some function $C(\cdot)$. In order to have $q_1 \in [0, 1]$, we must have $C(x) = 0$, so the allocation probability is simply the bidder's share of the aggregate bid.

---

[23]Recall that $\overline{w}(\Sigma m)$ is just the conditional expectation of v. It may be that $\overline{w}(\Sigma m)$ is not even in the support of H. It is always the case, however, that $\widehat{w}(\Sigma m)$ is in the support of the conditional distribution of v given Σm.

[24]This is true as long as local incentive compatibility (25) is satisfied in any equilibrium. See Lemma 8 below.

More generally, equation (7) is satisfied by the *proportional allocation rule*

$$\overline{q}_i(m) = \begin{cases} \dfrac{1}{N}\overline{Q}(0) & \text{if } \Sigma m = 0, \\[2mm] \dfrac{m_i}{\Sigma m}\overline{Q}(\Sigma m) & \text{if } \Sigma m > 0, \end{cases} \tag{10}$$

where the aggregate allocation $\overline{Q}$ is a function of the aggregate bid and is chosen so that induced aggregate allocation sensitivity is $\overline{\mu}$:

$$\nabla \cdot \overline{q}(m) = \frac{N-1}{\Sigma m}\overline{Q}(\Sigma m) + \overline{Q}'(\Sigma m) = \overline{\mu}(\Sigma m).$$

In equation (16), we give the explicit solution to this differential equation.

This leaves the transfers. Since $\overline{Q}$ has been specified, we can denote by $\overline{\Xi}$ the target aggregate excess growth, which is equal to the right-hand side of (8) and just depends on the aggregate bid. Any solution to (8) must be associated with an apportionment of $\overline{\Xi}$ among the bidders. Indeed, given such an apportionment, it is straightforward to integrate a bidder's excess growth to obtain the implied transfer (see also the discussion in footnote 30).

At first glance, there seems to be tremendous flexibility in how we divide the aggregate excess growth. The danger lurking here is that there is no guarantee, for an arbitrary solution of (8), that an equilibrium exists on *any* information structure, let alone $\overline{\mathcal{S}}$. As a result, the profit lower bound implicit in (8) may be vacuous.

It turns out that there is a subtle connection between the incentive compatibility of $\overline{\beta}$ and boundedness of the transfers. In particular, given that the allocation is $\overline{q}$ and that transfers satisfy (8), boundedness implies incentive compatibility, and they are equivalent when $N = 2$. Some obvious solutions to (8), such as equal sharing of the excess growth, result in transfers that sometimes diverge to minus infinity as m_i grows large.[25] We comment further on this connection in the proof of Proposition 4 and in footnote 30.

Thus, the last step to complete the strong maxmin solution is the construction of a bounded transfer rule that satisfies (8). These conditions are satisfied by the *proportional transfer rule*[26]

$$\overline{t}_i(m) = \begin{cases} \dfrac{1}{N}\overline{T}(0) & \text{if } \Sigma m = 0, \\[2mm] \dfrac{m_i}{\Sigma m}\overline{T}(\Sigma m) & \text{if } \Sigma m > 0, \end{cases} \tag{11}$$

---

[25]Equal sharing means that $\partial t_i(m)/\partial m_i - t_i(m) = \overline{\Xi}(\Sigma m)/N$. Together with the boundary condition $t_i(0, m_{-i}) = 0$, this implies the transfer rule $t_i(m) = \exp(m_i)\int_{x=0}^{m_i} \overline{\Xi}(x + \Sigma m_{-i})\exp(-x)\,dx/N$. Since the ex ante expectation of $\overline{\Xi}$ is zero (Lemma 11), it must be that $\int_{x=0}^{\infty} \overline{\Xi}(x + \Sigma m_{-i})\exp(-x)\,dx$ is zero on average across m_{-i}. But this integral is nonconstant (and generally strictly decreasing in m_{-i}), so sometimes it must be positive and sometimes negative, in which case the transfer tends to $\pm\infty$ as $m_i \to \infty$.

[26]This functional form is suggested by the fact that the aggregate excess growth only depends on the aggregate bid, and as we have seen with the allocation rule, the divergence of a proportional rule is similarly only a function of the aggregate bid.

where $\overline{T}(x)$ is the aggregate transfer

$$\overline{T}(x) = \begin{cases} \underline{v}\,\overline{Q}(0) & \text{if } x = 0, \\ \dfrac{1}{g_N(x)} \displaystyle\int_{y=0}^{x} \overline{\overline{\Xi}}(y) g_N(y)\, dy & \text{if } x > 0. \end{cases} \tag{12}$$

Note that with this functional form for the transfers, when $\Sigma m > 0$, (8) reduces to

$$\left(\frac{N-1}{x} - 1 \right) \overline{T}(x) + \overline{T}'(x) = \overline{\overline{\Xi}}(x).$$

It is easily verified that (12) satisfies this differential equation, so that profit-incentive alignment is satisfied.

In addition, boundedness of the transfers is equivalent to boundedness of $\overline{T}(x)$. Using l'Hôpital's rule and the fact that the ex ante expectation of $\overline{\overline{\Xi}}$ is zero (established in Lemma 11), we conclude that

$$\lim_{x \to \infty} \overline{T}(x) = \lim_{x \to \infty} \frac{\overline{\overline{\Xi}}(x)}{\dfrac{N-1}{x} - 1} = -\lim_{x \to \infty} \overline{\overline{\Xi}}(x). \tag{13}$$

We show below that $\overline{\overline{\Xi}}$ is bounded as $x \to \infty$, thus verifying that the transfers are also bounded.

Finally, by construction, participation security is satisfied with $m_i = 0$. Moreover, we show below that $\overline{\overline{\Xi}}(x)$ is bounded as $x \to 0$ when $\overline{Q}(0) = 0$, and it is approximately $(N-1)/x$ when $\overline{Q}(x) = 1$. Thus, a similar calculation as (13) shows that $\lim_{x \to 0} \overline{T}(x) = \underline{v}\,\overline{Q}(x)$. As a result, transfers are continuous at 0 and profit-incentive alignment is satisfied everywhere.

We refer to the mechanism comprised of $\overline{q}$ and $\overline{t}$ as a *proportional auction*. A key feature of this mechanism that makes it informationally robust is that it equalizes ex post profit and aggregate ex post local incentives across lots of message profiles. In particular, all message profiles that have the same aggregate message also have the same revenue, cost, and divergences of the allocation and transfer rules, so that they all contribute equally to the weighted sum of profit and local incentives. Thus, just as $\overline{S}$ induces indifference on the part of the seller as to how to allocate the good, $\overline{\mathcal{M}}$ induces indifference on the part of Nature, as to which message profiles should be played among those with a given aggregate message.

The optimal aggregate allocation and aggregate transfer functions are plotted for the uniform example in Figure 2. Theorem 1 shows that the proportional auction $\overline{\mathcal{M}}$, the additive-exponential information structure $\overline{S}$, and the truthful strategies $\overline{\beta}$ together comprise a strong maxmin solution.

4. A STRONG MAXMIN SOLUTION

We now formally construct and characterize a strong maxmin solution. We first completely construct the solution in Section 4.1. We then present our main theorem in Section 4.2, which asserts that the constructed triple is indeed a strong maxmin solution. The proof immediately follows. Sections 4.3 and 4.4 discuss two special cases, when the good must be sold and when the value distribution is single crossing, respectively.
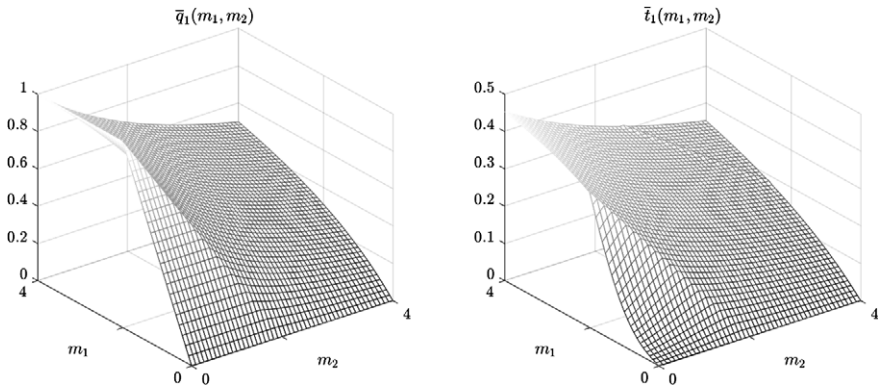
FIGURE 2.—The minmax aggregate allocation and aggregate transfer rules for $m_i \in [0, 5]$, when $N = 2$, $v \sim U[0, 1]$, and $c = 0$.

4.1. *Construction of the Solution*

4.1.1. *Minmax Information*

The minmax information structure $\overline{\mathcal{S}}$ is defined as follows. The bidders have signal spaces $\overline{S}_i = [0, \infty)$ and the signal distribution is $\overline{\pi}(ds) = \exp(-\Sigma s)\, ds$. That is, signals are independent draws from the exponential distribution with arrival rate 1.

The aggregate signal $x = \Sigma s$ has a probability density function

$$g_N(x) = \frac{x^{N-1}}{(N-1)!} \exp(-x) \tag{14}$$

and cumulative distribution function

$$G_N(x) = 1 - \sum_{n=1}^{N} g_n(x). \tag{15}$$

The value function is defined according to the following *grading procedure*. Recall that $\widehat{w}(x) = H^{-1}(G_N(x))$ is the fully revealing value function and that $\widehat{\gamma}(x) = \widehat{w}(x) - c$ is the fully revealing gains function. Let

$$\widehat{\Gamma}(x) = \int_{y=0}^{x} \widehat{\gamma}(y) g_N(y)\, dy.$$

Also let

$$E(x) = \int_{y=0}^{x} \exp(y) g_N(y)\, dy,$$

which is strictly increasing and, hence, it has a continuous inverse E^{-1}. Let $\mathrm{cav}(\widehat{\Gamma} \circ E^{-1})$ denote the smallest concave function that is everywhere above $\widehat{\Gamma} \circ E^{-1}$. We then set $\overline{\Gamma} = \mathrm{cav}(\widehat{\Gamma} \circ E^{-1}) \circ E$, and define

$$\overline{\gamma}(x) = \frac{1}{g_N(x)} \frac{d}{dx} \overline{\Gamma}(x),$$

where the derivative is taken from the right. We further define $\overline{w}(x) = \overline{\gamma}(x) + c$. We refer to $\overline{\gamma}$ and $\overline{w}$ as the *graded gains function* and the *graded value function*, respectively.[27]

4.1.2. *Maxmin Mechanism*

We next construct the maxmin mechanism $\overline{\mathcal{M}}$. The message space is $\overline{M}_i = [0, \infty)$.

We define a *graded interval* to be an interval $[a, b]$ with $a < b$ such that $\overline{\Gamma}(x) = \widehat{\Gamma}(x)$ for $x \in \{a, b\}$ and $\overline{\Gamma}(x) > \widehat{\Gamma}(x)$ for $x \in (a, b)$. As discussed in Section 3, the allocation and transfers are proportional, satisfying (10) and (11). The aggregate allocation function is given by[28]

$$\overline{Q}(x) = \begin{cases} 0 & \text{if } x = 0 \text{ and } [0, b] \text{ is a graded interval for some } b > 0, \\ C(a, b)\dfrac{x}{N} + D(a, b)\dfrac{1}{x^{N-1}} \\ & \text{if } x \in [a, b], \text{ where } [a, b] \text{ is a graded interval and } a > 0, \\ 1 & \text{otherwise} \end{cases} \tag{16}$$

and

$$C(a, b) = N\frac{b^{N-1} - a^{N-1}}{b^N - a^N}, \qquad D(a, b) = \frac{b - a}{b^N - a^N} a^{N-1} b^{N-1}.$$

The corresponding aggregate allocation sensitivity is

$$\overline{\mu}(x) = \begin{cases} C(a, b) & \text{if } x \in [a, b], \text{ where } [a, b] \text{ is a graded interval}, \\ \dfrac{N - 1}{x} & \text{otherwise.} \end{cases} \tag{17}$$

The aggregate transfer $\overline{T}$ is defined as follows. First, define

$$\overline{\lambda}(v) = \overline{\Pi} + \int_{x=0}^{\infty} \overline{\mu}(x) G_N(x)\, d\widehat{w}(x) - \int_{v=v}^{\overline{v}} \overline{\mu}\big(G_N^{-1}(H(v))\big)\, dv, \tag{18}$$

where $\overline{\Pi}$ is the profit guarantee defined in (3) and

$$\overline{\Xi}(x) = \overline{\mu}(x)\widehat{w}(x) - \overline{\lambda}\big(\widehat{w}(x)\big) - c\,\overline{Q}(x). \tag{19}$$

The aggregate transfer is then given by (12).

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[27]This procedure is evocative of "ironing" in Myerson (1981) and concavification in Kamenica and Gentzkow (2011). Grading is used to construct the bidders' information that minimizes the seller's profit, subject to the seller being always willing to allocate the good and subject to a mean-preserving spread constraint. In Myerson (1981), ironing is used to construct the mechanism that maximizes the seller's profit, subject to global incentive compatibility. In Kamenica and Gentzkow (2011), concavification is used to construct a receiver's information to maximize a sender's payoff, subject to a mean-preserving spread constraint. We can find no tight link between these problems beyond the very high-level connection of optimization subject to monotonicity and/or mean-preserving spread constraints.

[28]The functional form of $\overline{Q}$ on a graded interval $[a, b]$ can be derived from the hypotheses that $\overline{Q}(a) = \overline{Q}(b) = 1$ and that the aggregate allocation sensitivity is constant for all message profiles m with $a \le \Sigma m \le b$.
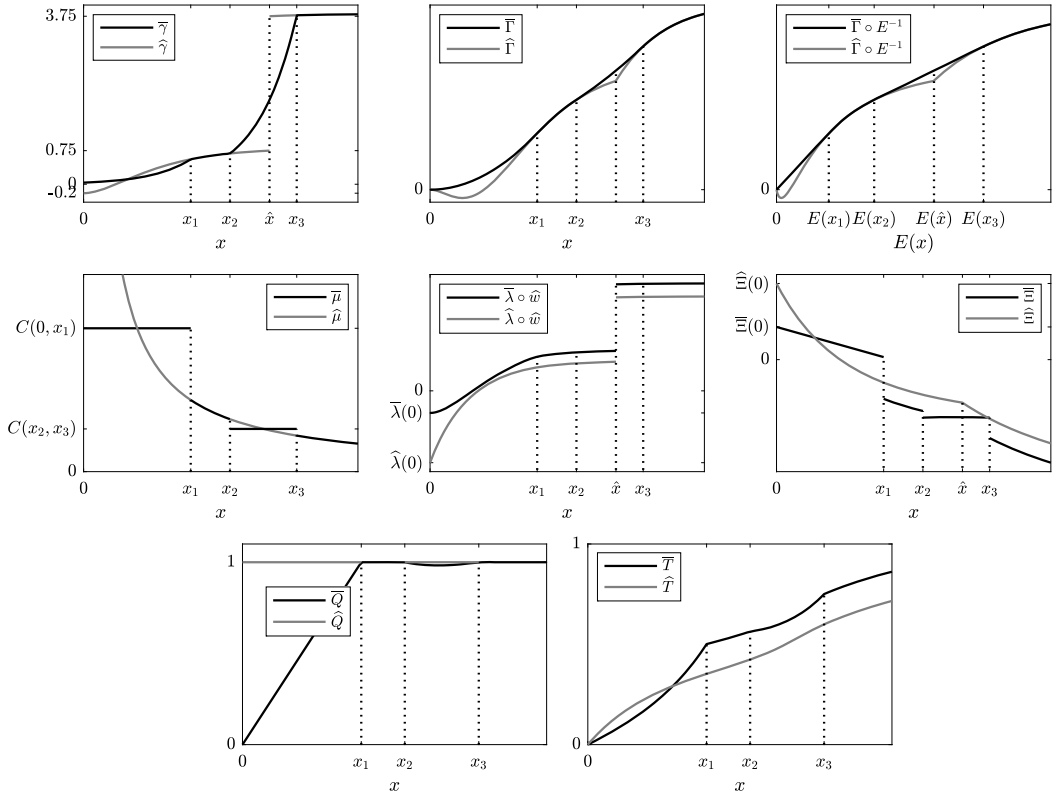
FIGURE 3.—Objects used in the construction of the solution when $N = 2$, a value that is uniform on $[0, 0.95] \cup [3.95, 4]$, and $c = 0.2$. Overbars correspond to the can-keep case and carets correspond to the must-sell case discussed in Section 4.3.

4.1.3. *Strategies*

Finally, let $\overline{\beta}_i$ be the truthful strategy in the mechanism $\overline{\mathcal{M}}$ under the information structure $\overline{\mathcal{S}}$: For all i and s_i, $\overline{\beta}_i(s_i)$ puts probability one on s_i. This completes the construction of the solution.

4.1.4. *Illustration*

Various objects in the construction are illustrated in Figure 3 for an example in which $N = 2$, the value is uniformly distributed on $[0, 0.95] \cup [3.95, 4]$, and $c = 0.2$.

The top row depicts the construction of the gains function: From left to right are the gains functions, integrated gains functions, and rescaled integrated gains functions. The fully revealing versions are in light gray and the graded versions are in black. There are two graded intervals, which are denoted $[0, x_1]$ and $[x_2, x_3]$.

The middle row shows the aggregate allocation sensitivity, value multiplier, and aggregate excess growth. Again, fully revealing objects are in gray and graded counterparts are in black (see Section 4.3 for the discussion of the fully revealing objects).

The bottom row shows the optimal aggregate allocation and transfer functions.

4.2. *Main Result*

The main result of the paper is the following.

THEOREM 1: $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ *is a strong maxmin solution with a profit guarantee of* $\overline{\Pi}$.

The theorem follows from Propositions 2–4. Proposition 2 shows that $\overline{\mathcal{S}}$ is a well defined information structure for which equilibrium profit is at most $\overline{\Pi}$. Proposition 3 shows that $\overline{\mathcal{M}}$ is a well defined mechanism for which equilibrium profit is at least $\overline{\Pi}$. Finally, Proposition 4 shows that $\overline{\beta}$ is an equilibrium of $(\overline{\mathcal{M}}, \overline{\mathcal{S}})$.

4.2.1. *Upper Bound on Profit for* $\overline{\mathcal{S}}$

We first establish C1 in the definition of a strong maxmin solution.

PROPOSITION 2: $\overline{\mathcal{S}}$ *is a well defined information structure. For all mechanisms* $\mathcal{M}$ *and equilibria* β *of* $(\mathcal{M}, \overline{\mathcal{S}})$, $\Pi(\mathcal{M}, \overline{\mathcal{S}}, \beta) \leq \overline{\Pi}$.

We first argue that $\overline{\mathcal{S}}$ is in fact an information structure consistent with the value distribution H.

LEMMA 1: *The gains function* $\overline{\gamma}$ *is a well defined and increasing function. The distribution H is a mean-preserving spread of the distribution of* $\overline{w}(\Sigma s)$.

PROOF: Since $\overline{\Gamma} \circ E^{-1}$ is a concave function, it is continuously differentiable at all but countably many points, and we can extend the derivative by right continuity. Since E is also differentiable, we conclude that $\overline{\Gamma}$ has a right derivative as well. We can therefore define $\overline{\gamma}$ as specified.

We next argue that $\overline{\gamma}(x)$ is continuous. Since $\overline{\Gamma} \circ E^{-1}$ is concave, its right derivative is monotonically decreasing. If the right derivative of $\overline{\Gamma} \circ E^{-1}$ had a downward discontinuity at x, which corresponds to a concave kink in $\overline{\Gamma} \circ E^{-1}$, it would have to occur at a point where $\overline{\Gamma}$ and $\widehat{\Gamma}$ coincide (since $\overline{\Gamma} \circ E^{-1}$ is linear on graded intervals where the two functions differ). This implies that $\widehat{\Gamma} \circ E^{-1}$ also has a concave kink at x, which contradicts the monotonicity of $\widehat{\gamma}$. Thus, we conclude that $\overline{\Gamma} \circ E^{-1}$ has a continuous right derivative, so that $\overline{\gamma}$ is continuous.

We next argue that $\overline{\gamma}$ is increasing. From continuity of $\overline{\gamma}$, it suffices to show that $\overline{\gamma}$ is increasing on graded intervals and on nongraded intervals. If x is such that there is an interval $[x, x + \epsilon)$ on which $\overline{\Gamma}$ coincides with $\widehat{\Gamma}$, then their right derivatives at x must coincide as well, so that $\overline{\gamma}(x) = \widehat{\gamma}(x)$, where the latter is increasing. In addition, if $[a, b]$ is a graded interval and $x \in [a, b]$, then $\overline{\gamma}$ has an exponential shape, as

$$\frac{d}{dx}\overline{\Gamma}(x) = \frac{d}{dz}\big(\Gamma\big(E^{-1}(z)\big)\big)\big|_{z=E(x)}E'(x) = \frac{\widehat{\Gamma}(b) - \widehat{\Gamma}(a)}{E(b) - E(a)}\exp(x)g_N(x).$$

The (positive) constant is pinned down from the fact that $\overline{\Gamma}(x)$ coincides with $\widehat{\Gamma}(x)$ at the endpoints of the graded interval. Thus, $\overline{\gamma}$ is increasing on graded intervals as well.

We next show that the distribution of $\widehat{\gamma}(\Sigma s)$ is a mean-preserving spread of the distribution of $\overline{\gamma}(\Sigma s)$. The lemma then follows from the observation that the distribution of $\widehat{\gamma}(\Sigma s) + c$ is H and $\overline{w}(\Sigma s) = \overline{\gamma}(\Sigma s) + c$.

Let $\overline{F}$ and $\widehat{F}$ denote the cumulative distributions of $\overline{\gamma}(x)$ and $\widehat{\gamma}(x)$, respectively, where $x \sim G_N$. Since $\overline{\gamma}$ and $\widehat{\gamma}$ are both increasing, for all $\alpha \in [0, 1]$, $\overline{\gamma}(G_N^{-1}(\alpha)) = \overline{F}^{-1}(\alpha)$ and $\widehat{\gamma}(G_N^{-1}(\alpha)) = \widehat{F}^{-1}(\alpha)$. From the change of variables $y = G_N(x)$, we conclude that

$$\int_{y=0}^{\alpha} \left(\overline{F}^{-1}(y) - \widehat{F}^{-1}(y)\right) dy = \int_{x=0}^{G_N^{-1}(\alpha)} \left(\overline{\gamma}(x) - \widehat{\gamma}(x)\right) g_N(x) \, dx$$

$$= \overline{\Gamma}\left(G_N^{-1}(\alpha)\right) - \widehat{\Gamma}\left(G_N^{-1}(\alpha)\right) \geq 0,$$

where the last line comes from the definition of $\overline{\Gamma}$. Theorem 3.8 of Sriboonchita, Wong, Dhompongsa, and Nguyen (2009) therefore implies that $\widehat{F}$ second-order stochastically dominates $\overline{F}$. Moreover, it must be that $\overline{\Gamma}(\infty) = \widehat{\Gamma}(\infty)$, since otherwise $\min\{\overline{\Gamma}(x), \widehat{\Gamma}(\infty)\} \circ E^{-1}$ would be a smaller concave function that dominates $\widehat{\Gamma} \circ E^{-1}$. Thus, $\overline{F}$ and $\widehat{F}$ have the same mean, and we conclude that $\widehat{F}$ is a mean-preserving spread of $\overline{F}$. *Q.E.D.*

Next, we establish a key property of the graded gains function.

LEMMA 2: *For all $x, y \in \mathbb{R}_+$ with $y \geq x$, $\overline{\gamma}(y) \leq \overline{\gamma}(x) \exp(y - x)$.*

PROOF: Since $\overline{\Gamma} \circ E^{-1}$ is concave, its derivative

$$\frac{\overline{\gamma}(E^{-1}(z)) g_N(E^{-1}(z))}{E'(E^{-1}(z))} = \frac{\overline{\gamma}(E^{-1}(z))}{\exp(E^{-1}(z))}$$

is decreasing. Hence, $\overline{\gamma}(x) \exp(-x)$ is decreasing, which implies the result. *Q.E.D.*

We can now complete the proof of Proposition 2.

PROOF OF PROPOSITION 2: From Lemma 1, we know that $\overline{\mathcal{S}}$ is well defined. To complete the proof, we show that $\overline{\Pi}$ is an upper bound on profit.

By the revelation principle, it suffices to verify that $\overline{\Pi}$ is an upper bound on revenue for every incentive compatible and interim individually rational direct mechanism (q, t) (since participation security implies that interim payoffs are nonnegative). Fix such a mechanism. Let us write

$$U_i(s_i, s_i') = \int_{\overline{S}_{-i}} \left(\overline{w}(s_i + \Sigma s_{-i}) q_i(s_i', s_{-i}) - t_i(s_i', s_{-i})\right) \exp(-\Sigma s_{-i}) \, ds_{-i}$$

and $U_i(s_i) = U_i(s_i, s_i)$. Incentive compatibility implies that for all i, s_i, and s_i',

$$U_i(s_i) \geq U_i(s_i, s_i') = U_i(s_i') + \int_{\overline{S}_{-i}} \left(\overline{\gamma}(s_i + \Sigma s_{-i}) - \overline{\gamma}(s_i' + \Sigma s_{-i})\right) q_i(s_i', s_{-i}) \exp(-\Sigma s_{-i}) \, ds_{-i}$$

and individual rationality implies that $U_i(s_i) \geq 0$. Thus, for all $\Delta > 0$,

$$U_i = \int_{\overline{S}_i} U_i(s_i) \exp(-s_i) \, ds_i$$

$$\geq \int_{\{s \in \overline{S} | s_i \geq \Delta\}} \left[U_i(s_i - \Delta) + \left(\overline{\gamma}(\Sigma s) - \overline{\gamma}(\Sigma s - \Delta)\right) q_i(s_i - \Delta, s_{-i})\right] \exp(-\Sigma s) \, ds$$

$$= \exp(-\Delta) \int_{\{s \in \overline{S} | s_i \geq \Delta\}} \left[U_i(s_i - \Delta) + \left(\overline{\gamma}(\Sigma s) - \overline{\gamma}(\Sigma s - \Delta) \right) q_i(s_i - \Delta, s_{-i}) \right]$$

$$\times \exp\left(-(\Sigma s - \Delta) \right) ds$$

$$= \exp(-\Delta) \left(U_i + \int_{\overline{S}} \left(\overline{\gamma}(\Sigma s + \Delta) - \overline{\gamma}(\Sigma s) \right) q_i(s_i, s_{-i}) \exp(-\Sigma s) ds \right).$$

Rearranging, we have

$$U_i \geq \frac{1}{(\exp(\Delta) - 1)} \int_{\overline{S}} \left(\overline{\gamma}(\Sigma s + \Delta) - \overline{\gamma}(\Sigma s) \right) q_i(s_i, s_{-i}) \exp(-\Sigma s) ds.$$

Now let

$$Q(x) = \frac{1}{g_N(x)} \int_{\{s \in \overline{S} | \Sigma s = x\}} \sum_{i=1}^{N} q_i(s) \exp(-\Sigma s) ds$$

be the expected probability of allocating the good conditional on $\Sigma s = x$. Then

$$\sum_{i=1}^{N} U_i \geq \frac{1}{(\exp(\Delta) - 1)} \int_{x=0}^{\infty} \left(\overline{\gamma}(x + \Delta) - \overline{\gamma}(x) \right) Q(x) g_N(x) dx.$$

Since total surplus is

$$\int_{x=0}^{\infty} \overline{\gamma}(x) Q(x) g_N(x) dx,$$

we conclude that an upper bound on profit is

$$\int_{x=0}^{\infty} \left(\overline{\gamma}(x) - \frac{\overline{\gamma}(x + \Delta) - \overline{\gamma}(x)}{\exp(\Delta) - 1} \right) Q(x) g_N(x) dx.$$

By Lemma 2, the term multiplying $Q(x)$ is positive, and since $Q(x) \leq 1$, profit is bounded above by

$$\int_{x=0}^{\infty} \left(\overline{\gamma}(x) - \frac{\overline{\gamma}(x + \Delta) - \overline{\gamma}(x)}{\exp(\Delta) - 1} \right) g_N(x) dx = \int_{x=0}^{\infty} \overline{\gamma}(x) \left(g_N(x) + \frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} \right) dx,$$

where $g_N(x) = 0$ if $x < 0$. The term in brackets converges pointwise for all positive x to $g_N(x) + g_N'(x) = g_{N-1}(x)$ as $\Delta \to 0$. To apply the dominated convergence theorem, all that remains is to present an integrable bounding function, which is done in Lemma 16 in Appendix A. As a result, as $\Delta \to 0$, the profit bound converges to $\overline{\Pi}$.   *Q.E.D.*

This argument is closely related to that of Myerson (1981), but we have used the special structure of $\overline{S}$ to sidestep the nontrivial technical question of whether the envelope theorem holds for the problem $\max_{s_i'} U_i(s_i, s_i')$ (cf. Milgrom and Segal (2002)).

4.2.2. *Lower Bound on Profit for* $\overline{\mathcal{M}}$

The next result establishes C2 in the definition of a strong maxmin solution.

PROPOSITION 3: $\overline{\mathcal{M}}$ *is a well defined mechanism. For all information structures* $\mathcal{S}$ *and equilibria* β *of* $(\overline{\mathcal{M}}, \mathcal{S})$, $\Pi(\overline{\mathcal{M}}, \mathcal{S}, \beta) \geq \overline{\Pi}$.

The first step toward proving Proposition 3 is the following.

LEMMA 3: *The aggregate allocation sensitivity* $\overline{\mu}$ *is decreasing. As a result,* $\overline{\lambda}$ *is concave.*

PROOF: On a nongraded interval, $\overline{\mu}(x) = (N - 1)/x$, which is decreasing, and on a graded interval $[a, b]$, $\overline{\mu}(x) = C(a, b)$. The fact that $\overline{\mu}$ is decreasing across graded intervals then follows from the definition of $C(a, b)$ and the well known inequality

$$\frac{N - 1}{N}\frac{1}{b}(b^N - a^N) \leq b^{N-1} - a^{N-1} \leq \frac{N - 1}{N}\frac{1}{a}(b^N - a^N)$$

(Hardy, Littlewood, and Pólya (1934), equation (2.15.2)).

Concavity of $\overline{\lambda}$ then follows from the fact $\overline{\mu}$ is decreasing and equation (18). *Q.E.D.*

We next verify that $\overline{\mathcal{M}}$ is well defined.

LEMMA 4: $\overline{\mathcal{M}}$ *is a well defined mechanism, has bounded aggregate transfers* $\overline{T}$, *and satisfies participation security.*

PROOF: Three properties need to be verified: Feasibility of the allocation rule, existence and boundedness of the transfers, and participation security.

Clearly, $\overline{Q}(x) \geq 0$, since the constants $C(a, b)$ and $D(a, b)$ are positive. We now argue that $\overline{Q}(x) \leq 1$. This is clearly true at the endpoints of a graded interval. Moreover, on a graded interval $[a, b]$,

$$\overline{Q}'(x) = \frac{C(a, b)}{N} - (N - 1)\frac{D(a, b)}{x^N},$$

which is increasing. Thus, $\overline{Q}$ is convex on $[a, b]$ and $\overline{Q}(x) \leq \max\{\overline{Q}(a), \overline{Q}(b)\} = 1$.[29]

To show that $\overline{T}$ is well defined and finite, we first show that $\overline{\lambda}$ defined in (18) is bounded. The last integral in (18) is bounded above by

$$\int_{v=\underline{v}}^{\overline{v}} \overline{\mu}\big(G_N^{-1}\big(H(v)\big)\big)\, dv = \int_{y=0}^{\infty} \overline{\mu}(y)\widehat{w}(dy). \tag{20}$$

From the first part of the left-tail assumption, there exists $\epsilon > 0$ such that if $x \leq \epsilon$, $(\widehat{w}(x) - \underline{v})/x^{\varphi} \leq 1$ for some $\varphi > 1$. If the value function is not graded at x, $\overline{\mu}(x) = (N-1)/x$, and if x is in a graded interval $[a, b]$, then

$$\overline{\mu}(x) = C(a, b) = \frac{b^N - ba^{N-1}}{b^N - a^N}\frac{N}{b} \leq \frac{N}{b} \leq \frac{N}{x}. \tag{21}$$

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[29] We thank a referee for suggesting this argument.

Thus, if $x \leq \epsilon$, we can plug in the bound and integrate by parts to obtain

$$\int_{y=0}^{\infty} \overline{\mu}(y)\widehat{w}(dy) \leq \int_{y=0}^{\infty} \frac{N}{y}\widehat{w}(dy) = \int_{y=0}^{\epsilon} \frac{N}{y^2}\big(\widehat{w}(y) - \underline{v}\big)\,dy + \int_{y=\epsilon}^{\infty} \frac{N}{y^2}\big(\widehat{w}(y) - \underline{v}\big)\,dy$$

$$\leq N\int_{y=0}^{\epsilon} y^{\varphi-2}\,dy + \int_{y=\epsilon}^{\infty} \frac{N}{y^2}(\overline{v} - \underline{v})\,dy$$

$$= N\frac{1}{\varphi-1}\epsilon^{\varphi-1} + N\frac{\overline{v} - \underline{v}}{\epsilon}.$$

Hence, the last integral in the definition of $\overline{\lambda}(v)$ is bounded. The middle integral is simply the expectation of the last integral across lower bounds $v \sim H$, so we conclude that $\overline{\lambda}(v)$ is bounded.

Since $\overline{\lambda}$ is bounded and $\overline{\mu}(x) \leq N/x$, $\overline{\overline{\Xi}}(x)$ goes to infinity as $x \to 0$ at a rate no faster than $1/x$, so $\overline{\overline{\Xi}}(x)g_N(x)$ is integrable on $[0, \infty)$. This shows that $\overline{T}$ in (12) is well defined. Moreover, equation (12) clearly shows that $\overline{T}$ is continuous at $x > 0$, so to show that $\overline{T}$ is bounded, it suffices to show that $\lim_{x\to\infty} \overline{T}(x) < \infty$ and $\lim_{x\to 0} \overline{T}(x) < \infty$.

Lemma 11 in Appendix A and l'Hôpital's rule imply that

$$\lim_{x\to\infty} \overline{T}(x) = \lim_{x\to\infty} \frac{\overline{\overline{\Xi}}(x)g_N(x)}{g_{N-1}(x) - g_N(x)} = \lim_{x\to\infty} \frac{\overline{\overline{\Xi}}(x)}{\dfrac{N-1}{x} - 1} = \overline{\lambda}(\overline{v}) + c < \infty,$$

since $\lim_{x\to\infty} \overline{\mu}(x) = 0$ by (21) and $\lim_{x\to\infty} \overline{Q}(x) = 1$ by Lemma 13 in Appendix A.

For $x \to 0$, we again apply l'Hôpital's rule to obtain

$$\lim_{x\to 0} \overline{T}(x) = \lim_{x\to 0} \frac{\overline{\overline{\Xi}}(x)g_N(x)}{g_{N-1}(x) - g_N(x)} = \lim_{x\to 0} \frac{\overline{\overline{\Xi}}(x)\dfrac{x}{N-1}}{1 - \dfrac{x}{N-1}} = \underline{v}\,\overline{Q}(0) < \infty,$$

where we use the fact that $\overline{\lambda}$ and $\overline{Q}$ are bounded, and Lemma 15 in Appendix A shows that there is either a graded interval at 0, in which case $\lim_{x\to 0} \widehat{w}(x)\overline{\mu}(x)\frac{x}{N-1} = 0 = \underline{v}\,\overline{Q}(0)$, or there is a nongraded interval at 0, in which case $\lim_{x\to 0} \widehat{w}(x)\overline{\mu}(x)\frac{x}{N-1} = \underline{v} = \underline{v}\,\overline{Q}(0)$.

Thus, $\overline{T}$ is bounded. The above argument also shows that $\overline{T}$ is continuous at $x = 0$ as well.

Finally, participation security follows from the definition of $\overline{T}$, which implies that $\overline{t}_i(0, m_{-i}) = \underline{v}\,\overline{q}_i(0, m_{-i})$.   Q.E.D.

We now develop the lower bound on profit in $\overline{\mathcal{M}}$.

LEMMA 5: *At all $m \neq 0$, $\overline{q}_i(m)$ is right-differentiable in m_i and $\nabla \cdot \overline{q}(m) = \overline{\mu}(\Sigma m)$.*

PROOF: Suppose $m \neq 0$. For the right differentiability of $\overline{q}_i$, it suffices to show that $\overline{Q}(x)$ is right-differentiable at every $x > 0$. There are three cases.

Case 1. There exists an $\epsilon > 0$ such that $[x, x + \epsilon]$ is a subset of a graded interval. From the formula of $\overline{Q}$ in (16) it is immediate that $\overline{Q}$ is right-differentiable at x and, in fact,

$$\overline{Q}'(x) = \overline{\mu}(x) - \frac{N-1}{x}\overline{Q}(x). \tag{22}$$

Case 2. There exists an $\epsilon > 0$ such that $\overline{\Gamma}(y) = \widehat{\Gamma}(y)$ for all $y \in [x, x + \epsilon]$. Then $\overline{Q}(y) = 1$ and $\overline{\mu}(y) = (N - 1)/y$ for all $y \in [x, x + \epsilon]$, so $\overline{Q}$ is again right-differentiable at x, and equation (22) again holds.

Case 3. For every $\epsilon > 0$, there exist $x', x'' \in (x, x + \epsilon]$ such that x' is not graded and x'' is graded: $\overline{\Gamma}(x') = \widehat{\Gamma}(x')$ and $\overline{\Gamma}(x'') > \widehat{\Gamma}(x'')$. This implies that $\overline{Q}(x) = 1$, for otherwise x is in the interior of a graded interval and we are in Case 1. Let $\{x_n\}$ be a sequence converging to x from the right. We will show that for every $\epsilon > 0$, there exists an $\overline{n}$ such that for all $n \geq \overline{n}$, we have

$$\left| \frac{\overline{Q}(x_n) - \overline{Q}(x)}{x_n - x} \right| \leq \epsilon, \tag{23}$$

that is, the $\overline{Q}$ is right-differentiable at x, and $\overline{Q}'(x) = 0$. This again implies (22), since $\overline{\mu}(x) = (N - 1)/x$ by (17).

Given $\epsilon > 0$, choose $\epsilon' > 0$ so that

$$\frac{N - 1}{x} - \frac{N - 1}{x + \epsilon'} \left(1 - \frac{2(N - 1)}{x} \epsilon' \right) \leq \epsilon, \qquad \frac{N - 1}{x + \epsilon'} - \frac{N - 1}{x} \geq -\epsilon.$$

By assumption, there exists an $x' \in (x, x + \epsilon']$ at which the gains function is not graded. Choose $\overline{n}$ so that $x_n \in [x, x']$ for all $n \geq \overline{n}$.

Consider any $n \geq \overline{n}$. Suppose x_n is graded and $\overline{Q}(x_n) < 1$ (for otherwise (23) is trivial). Let $[a, b]$ be the graded interval containing x_n. We have $x < a < x_n < b \leq x' \leq x + \epsilon'$: we have $b \leq x'$ because x' is not graded and $x < a$ because there are other nongraded points in $(x, x_n]$. Since $\overline{Q}(a) = \overline{Q}(x) = 1$, we have

$$\frac{\overline{Q}(x_n) - \overline{Q}(x)}{x_n - x} = \frac{x_n - a}{x_n - x} \frac{\overline{Q}(x_n) - \overline{Q}(a)}{x_n - a} = \frac{x_n - a}{x_n - x} \overline{Q}'(y) = \frac{x_n - a}{x_n - x} \left(\overline{\mu}(y) - \frac{N - 1}{y} \overline{Q}(y) \right)$$

for some $y \in (a, x_n)$ by the mean value theorem and equation (22) applied to the graded interval $[a, b]$. Equation (22) also implies that $\overline{Q}'(z) \leq \frac{2(N-1)}{x}$ for any $z \in [a, b]$, so $|\overline{Q}(y) - 1| \leq \frac{2(N-1)}{x} \epsilon'$, and since both $\overline{\mu}(y)$ and $(N-1)/y$ are in the interval $((N-1)/b, (N-1)/a)$, we have

$$-\epsilon \leq \frac{N - 1}{x + \epsilon'} - \frac{N - 1}{x} \leq \frac{\overline{Q}(x_n) - \overline{Q}(x)}{x_n - x} \leq \frac{N - 1}{x} - \frac{N - 1}{x + \epsilon'} \left(1 - \frac{2(N - 1)}{x} \epsilon' \right) \leq \epsilon,$$

which proves (23).

Finally, it is easy to check that $\nabla \cdot \overline{q}(m) = \overline{\mu}(\Sigma m)$ by applying the product rule and equation (22) to the definition of $\overline{q}$. $\hspace{1cm}$ Q.E.D.

LEMMA 6: *The ex ante expectation of $\overline{\lambda}(v)$ is $\overline{\Pi}$:*

$$\int_{v=\underline{v}}^{\overline{v}} \overline{\lambda}(v) H(dv) = \int_{x=0}^{\infty} \overline{\lambda}(\widehat{w}(x)) g_N(x)\, dx = \overline{\Pi}.$$

PROOF: The first equality follows from the change of variables $v = \widehat{w}(x) = H^{-1}(G_N(x))$. For the second equality, it suffices to show that the middle integral in (18)

is equal to the ex ante expectation of the last integral, which using (20) and Tonelli's theorem is

$$\int_{x=0}^{\infty} \int_{y=0}^{x} g_N(y) \, dy \overline{\mu}(x) \widehat{w}(dx) = \int_{x=0}^{\infty} \overline{\mu}(x) G_N(x) \widehat{w}(dx).$$

Q.E.D.

LEMMA 7: *For every $m \neq 0$, $\overline{t}_i(m)$ is right-differentiable with respect to m_i, and*

$$\nabla \cdot \overline{t}(m) - \Sigma \overline{t}(m) = \overline{\overline{\Xi}}(\Sigma m).$$

PROOF: We first prove that $\overline{t}_i(m)$ is right-differentiable with respect to m_i at $m \neq 0$. By Lemma 5, it suffices to show that $\overline{T}$ is right-differentiable at every $x > 0$, and for that it suffices to show that $\overline{\overline{\Xi}}$ is right-continuous at every $x > 0$.

To that end, first note that $\overline{\mu}$ is right-continuous at every $x > 0$. This is clear from the definition of $\overline{\mu}$ in equation (17) when x is the left endpoint or in the interior of a graded interval, or when x is the left endpoint or in the interior of a nongraded interval. The only other case is when x is a limit point both of graded points and of nongraded points. In this case, $\overline{\mu}(x)$ is defined to be $(N-1)/x$ by (17), and for any sequence $\{x_n\}$ converging to x from the right, $\overline{\mu}(x_n)$ converges to $\overline{\mu}(x) = (N-1)/x$: The reason is that for any $\epsilon > 0$, there exists a nongraded point $x' \in (x, x + \epsilon)$. Hence, when n is sufficiently large, $x_n \in [x, x']$ so $\overline{\mu}(x_n) \in [(N-1)/(x+\epsilon), (N-1)/x]$, as $\overline{\mu}$ is decreasing from Lemma 3.

Next we prove right continuity of $\overline{\overline{\Xi}}$ at $x > 0$. Since $\overline{Q}(x)$ is right-continuous at all $x > 0$ by the proof of Lemma 5, the obstacle to right continuity of $\overline{\overline{\Xi}}(x)$ can only come from $\overline{\mu}(x)\widehat{w}(x)$ or $\overline{\lambda}(\widehat{w}(x))$. If $\widehat{w}$ is discontinuous at $x > 0$, then $\widehat{\Gamma}$ has a convex kink at x, so x must be in the interior of a graded interval $[a, b]$. Since $\overline{\mu}$ is continuous (in fact, constant) in (a, b), $\overline{\overline{\Xi}}$ is continuous at x, as the discontinuity in $\overline{\mu}(y)\widehat{w}(y)$ at $y = x$ is exactly canceled by the discontinuity in $\overline{\lambda}(\widehat{w}(y))$ at $y = x$ (recall that $\overline{\lambda}(\widehat{w}(y)) = C + \int_{z=0}^{y} \overline{\mu}(z) \, d\widehat{w}(z)$ for a constant C). On the other hand, if $\widehat{w}$ is continuous at $x > 0$, then $\overline{\overline{\Xi}}$ is right-continuous at x, because $\overline{\mu}$ is right-continuous there by the previous paragraph. We conclude that $\overline{\overline{\Xi}}$ is right-continuous and, hence, $\overline{T}$ is right-differentiable, at $x > 0$.

Finally, interpreting d/dx as the right derivative, we have $\frac{d}{dx}(g_N(x)\overline{T}(x)) = g_N(x)\overline{\overline{\Xi}}(x)$, so

$$\left(\frac{N-1}{x} - 1\right) \overline{T}(x) + \overline{T}'(x) = \overline{\overline{\Xi}}(x), \tag{24}$$

which proves the lemma, since by the product rule, $\nabla \cdot \overline{t}(m) = \frac{N-1}{\Sigma m} \overline{T}(\Sigma m) + \overline{T}'(\Sigma m)$.

Q.E.D.

LEMMA 8: *For any information structure $\mathcal{S}$ and equilibrium β of $(\overline{\mathcal{M}}, \mathcal{S})$,*

$$\int_{\mathcal{S}} \int_{\overline{M}} \left(w(s)\nabla \cdot \overline{q}(m) - \nabla \cdot \overline{t}(m)\right) \beta(dm|s)\pi(ds) \leq 0. \tag{25}$$

This result says that in any equilibrium, local upward deviations must not be attractive. If a bidder were to marginally increase all of the messages they send in equilibrium, the change in payoff would be

$$\int_{\mathcal{S}} \int_{\overline{M}} \left(w(s)\frac{\partial}{\partial m_i}\overline{q}_i(m) - \frac{\partial}{\partial m_i}\overline{t}_i(m)\right) \beta(dm|s)\pi(ds) \leq 0.$$

Summing across i gives (25). A technical complication is that the allocation sensitivity may blow up as the aggregate bid goes to zero, so that deriving this constraint as a limit of nonlocal deviations is not trivial. We resolve this technical complication by appealing to the second part of the left-tail assumption; a detailed proof can be found in Appendix A.

We now complete the proof of Proposition 3.

PROOF OF PROPOSITION 3: We have already argued in Lemma 4 that $\overline{\mathcal{M}}$ is well defined. To complete the proof, it suffices to show that profit in any equilibrium in any information structure is at least $\overline{\Pi}$. This is established in two steps.

Step 1. For any v and x,

$$
\begin{aligned}
\overline{\lambda}(v) &= \overline{\lambda}\big(\widehat{w}(x)\big) - \int_{\nu=v}^{\widehat{w}(x)} \overline{\mu}\big(G_N^{-1}\big(H(\nu)\big)\big)\,d\nu \\
&\leq \overline{\lambda}\big(\widehat{w}(x)\big) - \big(\widehat{w}(x) - v\big)\overline{\mu}(x) \\
&= v\,\overline{\mu}(x) - \overline{\Xi}(x) - c\,\overline{Q}(x),
\end{aligned}
$$

where the second line follows from the fact that $\overline{\mu}$ is decreasing (Lemma 3), and the third line follows from the definition of $\overline{\Xi}$.

Step 2. Fix an information structure $\mathcal{S}$. Profit in an equilibrium β of $(\overline{\mathcal{M}}, \mathcal{S})$ is

$$
\int_S \int_{\overline{M}} \big[\overline{T}(\Sigma m) - c\,\overline{Q}(\Sigma m)\big]\beta(dm|s)\pi(ds).
$$

By Lemma 8, this is at least

$$
\begin{aligned}
&\int_S \int_{\overline{M}} \big[w(s)\,\nabla \cdot \overline{q}(m) - \big(\nabla \cdot \overline{t}(m) - \Sigma \overline{t}(m)\big) - c\,\overline{Q}(\Sigma m)\big]\beta(dm|s)\pi(ds) \\
&= \int_S \int_{\overline{M}} \big[w(s)\,\overline{\mu}(\Sigma m) - \overline{\Xi}(\Sigma m) - c\,\overline{Q}(\Sigma m)\big]\beta(dm|s)\pi(ds) \\
&\geq \int_S \overline{\lambda}\big(w(s)\big)\pi(ds) \\
&\geq \int_V \overline{\lambda}(v)H(dv).
\end{aligned}
$$

The second line follows from Lemmas 5 and 7; the third line follows from Step 1; the last inequality uses concavity of $\overline{\lambda}$ (Lemma 3), the fact that H is a mean-preserving spread of the distribution of $w(s)$, and Jensen's inequality. The final integral is equal to $\overline{\Pi}$ by Lemma 6. _Q.E.D._

4.2.3. *Truth-Telling Equilibrium*

We now come to the last condition for $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ to be a strong maxmin solution.

PROPOSITION 4: *The truthful strategies $\overline{\beta}$ are an equilibrium of the game $(\overline{\mathcal{M}}, \overline{\mathcal{S}})$.*

PROOF: Let

$$
U_i\big(m_i, m_i'\big) = \int_{\overline{M}_{-i}} \big[\overline{w}(m_i + \Sigma m_{-i})\overline{q}_i\big(m_i', m_{-i}\big) - \overline{t}_i\big(m_i', m_{-i}\big)\big]\exp(-\Sigma m_{-i})\,dm_{-i}
$$

denote the interim expected utility from reporting m_i' when the true signal is m_i and others report truthfully. We will show that the difference $U_i(m_i, m_i) - U_i(m_i, m_i')$ is nonnegative for all i, m_i, and m_i'.

We first derive a convenient expression for the interim expected transfer,[30]

$$\int_{\overline{M}_{-i}} \overline{t}_i(m_i', m_{-i}) \exp(-\Sigma m_{-i}) \, dm_{-i}$$

$$= \int_{x=0}^{\infty} \frac{m_i'}{m_i' + x} \overline{T}(m_i' + x) g_{N-1}(x) \, dx$$

$$= -\int_{x=0}^{\infty} \left(\frac{N-1}{m_i' + x} g_N(x) - g_{N-1}(x) \right) \overline{T}(m_i' + x) \, dx$$

$$= -\int_{x=0}^{\infty} \left(\left(\frac{N-1}{m_i' + x} - 1 \right) \overline{T}(m_i' + x) + \overline{T}'(m_i' + x) \right) g_N(x) \, dx$$

$$= -\int_{x=0}^{\infty} \overline{\Xi}(m_i' + x) g_N(x) \, dx,$$

where the first equality is from the definition of $\overline{t}$ and the fact that $m_{-i} = 0$ occurs with zero probability; the second rearranges the formula for $(m_i' g_{N-1}(x))/(m_i' + x)$; the third is obtained by integrating $\int \overline{T}(m_i' + x) g_N'(x) \, dx$ by parts and the fact that $\overline{T}$ is bounded (Lemma 4); and the fourth substitutes using equation (24).

We next compute the interim expected payoff from the allocation:

$$\int_{\overline{M}_{-i}} \overline{w}(m_i + \Sigma m_{-i}) \overline{q}_i(m_i', m_{-i}) \exp(-\Sigma m_{-i}) \, dm_{-i}$$

$$= \int_{x=0}^{\infty} \overline{w}(m_i + x) \frac{m_i'}{m_i' + x} \overline{Q}(m_i' + x) g_{N-1}(x) \, dx$$

---

[30]The final expression for the interim transfer substantiates the claim in Section 3 that boundedness of the transfers is a sufficient condition for incentive compatibility of $\overline{q}$ on $\overline{S}$. We can rewrite the transfer as

$$t_i(m) = \exp(m_i) \left(t_i(0, m_{-i}) + \int_{x=0}^{m_i} \xi_i(x, m_{-i}) \exp(-x) \, dx \right),$$

where $\xi_i(m) = \partial t_i(m)/\partial m_i - t_i(m)$ is bidder i's individual excess growth and $\Sigma \xi = \overline{\overline{\Xi}}$ (cf. (8)). Since $\overline{q}_i(0, m_{-i}) = 0$ for $m_{-i} \neq 0$, we restrict attention to $t_i(0, m_{-i}) = 0$. Boundedness of i's transfer then implies that

$$\int_{x=0}^{\infty} \xi_i(x, m_{-i}) \exp(-x) \, dx = 0 \tag{26}$$

for $m_{-i} \neq 0$, in which case we can rewrite

$$t_i(m) = -\int_{x=0}^{\infty} \xi_i(m_i + x, m_{-i}) \exp(-x) \, dx.$$

But if we take the expectation of $\xi_i(m_i + x, m_{-i})$ over m_{-i}, equation (26), combined with $\Sigma \xi = \overline{\overline{\Xi}}$, implies that the interim transfer is exactly as given. Moreover, when $N = 2$, this identity holds only if (26) holds as well. For more than two bidders, incentive compatibility of $\overline{q}$ on $\overline{S}$ is equivalent to, for all i and m_i, $\int_{\overline{M}_{-i}} (\xi_i(m_i, m_{-i}) - \overline{\Xi}(m_i, m_{-i})) \exp(-\Sigma m_{-i}) \, dm_{-i} = 0$.

$$= \int_{x=0}^{\infty} \overline{w}(m_i + x)\left(\overline{Q}(m_i' + x) - \frac{x}{m_i' + x}\overline{Q}(m_i' + x)\right)g_{N-1}(x)\,dx$$

$$= \int_{x=0}^{\infty} \overline{w}(m_i + x)\left(\overline{Q}(m_i' + x) - \frac{x}{N-1}\left(\overline{\mu}(m_i' + x) - \overline{Q}'(m_i' + x)\right)\right)g_{N-1}(x)\,dx$$

$$= \int_{x=0}^{\infty} \overline{w}(m_i + x)\left(\overline{Q}(m_i' + x)g_{N-1}(x) - \left(\overline{\mu}(m_i' + x) - \overline{Q}'(m_i' + x)\right)g_N(x)\right)dx$$

$$= \int_{x=0}^{\infty} \left(\overline{w}(m_i + x)\left(\overline{Q}(m_i' + x)g_{N-1}(x) - \overline{\mu}(m_i' + x)g_N(x)\right)dx\right.$$
$$\left. - \overline{Q}(m_i' + x)\,d\big(\overline{w}(m_i + x)g_N(x)\big)\right)$$

$$= \int_{x=0}^{\infty} \left[\overline{Q}(m_i' + x)g_N(x)\big(\overline{w}(m_i + x)\,dx - \overline{w}(m_i + dx)\big)\right.$$
$$\left. - \overline{\mu}(m_i' + x)\overline{w}(m_i + x)g_N(x)\,dx\right].$$

The first equality is the definition of $\overline{q}$ and the fact that $m_{-i} = 0$ has zero probability; the second rearranges the term $m_i'/(m_i' + x)$; the third substitutes in using (22); the fourth rearranges terms; the fifth integrates $\int \overline{Q}'(m_i' + x)\overline{w}(m_i + x)g_N(x)\,dx$ by parts; and the last equality applies the product rule (and the fact that $g_N' = g_{N-1} - g_N$) and rearranges terms.

We now use these expressions to compute the interim expected loss from deviating:

$$U_i(m_i, m_i) - U_i(m_i, m_i')$$
$$= \int_{x=0}^{\infty} \big(\overline{Q}(m_i + x) - \overline{Q}(m_i' + x)\big)g_N(x)\big(\overline{w}(m_i + x)\,dx - \overline{w}(m_i + dx)\big)$$
$$+ \int_{x=0}^{\infty} \left[\big(\overline{\mu}(m_i' + x) - \overline{\mu}(m_i + x)\big)\overline{w}(m_i + x) + \overline{\Xi}(m_i + x) - \overline{\Xi}(m_i' + x)\right]g_N(x)\,dx.$$

Observe that

$$\overline{\Xi}(m_i + x) - \overline{\Xi}(m_i' + x)$$
$$= \overline{\mu}(m_i + x)\widehat{w}(m_i + x) - \overline{\mu}(m_i' + x)\widehat{w}(m_i' + x) - \overline{\lambda}\big(\widehat{w}(m_i + x)\big) + \overline{\lambda}\big(\widehat{w}(m_i' + x)\big)$$
$$\quad - c\big(\overline{Q}(m_i + x) - \overline{Q}(m_i' + x)\big)$$

$$= \overline{\mu}(m_i + x)\widehat{w}(m_i + x) - \overline{\mu}(m_i' + x)\widehat{w}(m_i' + x) - \int_{y=m_i'+x}^{m_i+x} \overline{\mu}(y)\widehat{w}(dy)$$
$$\quad - c\big(\overline{Q}(m_i + x) - \overline{Q}(m_i' + x)\big)$$

$$= \int_{y=m_i'+x}^{m_i+x} \widehat{w}(y)\overline{\mu}(dy) - c\big(\overline{Q}(m_i + x) - \overline{Q}(m_i' + x)\big)$$

$$= \int_{y=m_i'+x}^{m_i+x} \overline{w}(y)\overline{\mu}(dy) - c\big(\overline{Q}(m_i + x) - \overline{Q}(m_i' + x)\big).$$

The first equality uses the definition of $\overline{\overline{\Xi}}$, equation (19); the second uses the definition of $\overline{\lambda}$, equation (18); the third is integration by parts; and the last equality uses the fact that $\overline{\mu}(y)$ is constant on the interior of graded intervals, so that the measure $\overline{\mu}(dy)$ assigns zero mass to the points where $\widehat{w} \neq \overline{w}$.

Substituting this last expression into the loss from deviating, we obtain

$$
\begin{aligned}
&U_i(m_i, m_i) - U_i(m_i, m_i') \\
&= \int_{x=0}^{\infty} \left(\overline{Q}(m_i + x) - \overline{Q}(m_i' + x)\right) g_N(x) \left(\left(\overline{w}(m_i + x) - c\right) dx - \overline{w}(m_i + dx)\right) \\
&\quad + \int_{x=0}^{\infty} \left(\left(\overline{\mu}(m_i' + x) - \overline{\mu}(m_i + x)\right) \overline{w}(m_i + x) + \int_{y=m_i'+x}^{m_i+x} \overline{w}(y)\overline{\mu}(dy)\right) g_N(x)\, dx.
\end{aligned}
$$

Lemma 2 implies that the measure

$$
\left(\overline{w}(m_i + x) - c\right) dx - \overline{w}(m_i + dx) = \overline{\gamma}(m_i + x)\, dx - \overline{\gamma}(m_i + dx)
$$

is nonnegative and its support consists of the nongraded intervals, on which $\overline{Q}(m_i + x) = 1 \geq \overline{Q}(m_i' + x)$, so that the first integral is nonnegative. The second integral is also nonnegative, because $\overline{w}$ is increasing, so

$$
\int_{y=m_i'+x}^{m_i+x} \overline{w}(y)\overline{\mu}(dy) \geq \overline{w}(m_i + x)\left(\overline{\mu}(m_i + x) - \overline{\mu}(m_i' + x)\right).
$$

We conclude that $U_i(m_i, m_i) \geq U_i(m_i, m_i')$, as desired.   *Q.E.D.*

Theorem 1 follows immediately from Propositions 2, 3, and 4.

4.3. *The Must-Sell Case*

We now discuss the variant of our model where the good must be sold. All of our tools still apply and almost immediately give us the solution.

A *must-sell mechanism* is a mechanism for which $\Sigma q(m) = 1$ for all m. A *must-sell strong maxmin solution* is a triple $(\mathcal{M}, \mathcal{S}, \beta)$ satisfying C1, C2, and C3 in Section 2.5, but where $\mathcal{M}$ is a must-sell mechanism and C1 only has to hold for $\mathcal{M}'$ that are must-sell mechanisms.

Let $\widehat{\mathcal{S}}$ be the information structure where signals are i.i.d. standard exponential and the value function is $\widehat{w}$. Also let $\widehat{\mathcal{M}}$ be the proportional auction with allocation

$$
\widehat{q}_i(m) = \begin{cases} \dfrac{1}{N} & \text{if } \Sigma m = 0, \\[2mm] \dfrac{m_i}{\Sigma m} & \text{if } \Sigma m > 0 \end{cases}
$$

and transfers

$$
\widehat{t}_i(m) = \begin{cases} \dfrac{1}{N}\widehat{T}(\Sigma m) & \text{if } \Sigma m = 0, \\[2mm] \dfrac{m_i}{\Sigma m}\widehat{T}(\Sigma m) & \text{if } \Sigma m > 0. \end{cases}
$$

We define $\widehat{\lambda}$, $\widehat{\Xi}$, and $\widehat{T}$ according to analogous formulae as those for $\overline{\lambda}$, $\overline{\Xi}$, and $\overline{T}$, using $\widehat{w}$, $\widehat{\mu}(x) = (N-1)/x$, and $\widehat{Q}(x) = 1$ in place of $\overline{w}$, $\overline{\mu}$, and $\overline{Q}$. Let

$$\widehat{\Pi} = \int_{x=0}^{\infty} \widehat{\gamma}(x) g_{N-1}(x)\, dx. \tag{27}$$

Finally, let $\widehat{\beta} = \overline{\beta}$.

THEOREM 2: $(\widehat{\mathcal{M}}, \widehat{\mathcal{S}}, \widehat{\beta})$ *is a must-sell strong minmax solution with a profit guarantee of* $\widehat{\Pi}$.

PROOF: The proof of Proposition 2 goes through with $\widehat{\gamma}$ in place of $\overline{\gamma}$, except that we do not need to invoke Lemma 2 (which does not hold for the fully-revealing gains function) to conclude that the profit upper bound is maximized by setting $Q(x) = 1$. Instead, the conclusion follows directly from the must-sell assumption, so that (27) is an upper bound on profit in $\widehat{\mathcal{S}}$.

The proof of Proposition 3 remains valid with $\widehat{\gamma}$ in place of $\overline{\gamma}$. (In the proof of Lemma 7, $\widehat{\mu}$ is continuous everywhere, in particular where $\widehat{w}$ is discontinuous).

Proposition 4 also goes through, with the only modification being that the last term in the deviation payoff involving $\overline{Q}$ disappears, so that again we do not need to invoke Lemma 2. Thus, the mechanism $\widehat{\mathcal{M}}$ guarantees the seller at least $\widehat{\Pi}$ in any equilibrium, and $\widehat{\beta}$ is an equilibrium of the game $(\widehat{\mathcal{M}}, \widehat{\mathcal{S}})$. *Q.E.D.*

It is sometimes possible to use the formulae for $\widehat{\mu}$ and $\widehat{Q}$ to simplify the must-sell aggregate transfer. For example, Appendix D.1 shows that when v is standard uniform,

$$\widehat{T}(x) = G_{N-1}(x) + \frac{\dbinom{2N-3}{N-1}}{2^{2N-3}} \frac{G_N(x) - G_{2N-2}(2x)}{g_N(x)}.$$

When $N = 2$, this further simplifies to $\widehat{T}(x) = (1 - (1 - e^{-x})/x)/2$. We shall describe the many-bidder limit of this formula in Section 6.

4.4. *Single-Crossing Distributions*

We now discuss a class of distributions for which the maxmin aggregate allocation is relatively simple. The distribution H is *single crossing* if there is a cutoff $\overline{x}$ such that $\widehat{\Gamma} \circ E^{-1}$ is convex on $[0, \overline{x}]$ and concave on $[\overline{x}, \infty)$. When the gains function is differentiable, this is equivalent to saying that $\widehat{\gamma}(x) - \widehat{\gamma}'(x)$ is single crossing from below at $x = E^{-1}(\overline{x})$. This is in a sense a counterpart to the regular case of Myerson (1981), under which the seller only has an incentive to ration the good when signals are a below a cutoff.

If H is single crossing, then there is a single graded interval, denoted $[0, x^*]$, on which $\overline{Q}(x) = xC(0, x^*)/N = x/x^*$ (since $D(0, x^*) = 0$). The maxmin allocation is therefore $\overline{q}_i(m_i, m_{-i}) = m_i / \max\{x^*, \Sigma m\}$. We can interpret m_i as bidder i's demand for the good in units where the aggregate supply is x^*. The allocation rule simply says that the bidders get their demands if the aggregate demand is feasible, and the good is rationed proportionally otherwise.

The uniform distribution is single crossing for all N. To see this, observe that the fully-revealing gains function is $\widehat{\gamma}(x) = G_N(x) - c$, so $\widehat{\gamma}(x) - \widehat{\gamma}(x) = G_N(x) - c - g_N(x)$. This is $-c$ when $x = 0$, and its derivative is

$$2g_N(x) - g_{N-1}(x) = \left(\frac{2x}{N-1} - 1 \right) g_{N-1}(x),$$

so that $\widehat{\gamma}(x) - \widehat{\gamma}(x)$ is decreasing for $x < (N-1)/2$ and increasing otherwise, which implies that it crosses zero once, from below. Thus, $\overline{\gamma} = \overline{\gamma}(0) \exp(x)$ on $[0, x^*]$ and it is fully revealing above x^*. For these segments to meet continuously, it must be that $\overline{\gamma}(0) = \exp(-x^*)(G_N(x^*) - c)$. This implies that x^* is the unique positive solution to[31]

$$\int_{x=0}^{x^*} (G_N(x) - c) g_N(x)\, dx = \exp(-x^*)(G_N(x^*) - c) \int_{x=0}^{x^*} \exp(x) g_N(x)\, dx. \qquad (28)$$

Maxmin profit is

$$\overline{\Pi} = \int_{x=0}^{x^*} \overline{\gamma}(0) \exp(x) g_{N-1}(x)\, dx + \int_{x=x^*}^{\infty} (G_N(x) - c) g_{N-1}(x)\, dx,$$

while maxmin profit among must-sell mechanisms is only

$$\widehat{\Pi} = \int_{x=0}^{\infty} (G_N(x) - c) g_{N-1}(x)\, dx.$$

This example is continued in Section 6. We also give an explicit formula for $\overline{T}$ in Appendix D.2 when $c = 0$.

5. UNIQUENESS AND FINITE APPROXIMABILITY

We have constructed a particular strong maxmin solution. We know for a fact that there are others (and we will comment further on this shortly). However, as we now show, any sufficiently well behaved solution must have the same profit guarantee of $\overline{\Pi}$. As part of developing this result, we will show that the mechanism $\overline{\mathcal{M}}$ and information structure $\overline{\mathcal{S}}$ can both be approximated with finite objects that guarantee profit arbitrarily close to $\overline{\Pi}$. A further implication of this result is that $\overline{\mathcal{M}}$ and $\overline{\mathcal{S}}$ are limits of ϵ equilibria of standard zero-sum games, where the seller and Nature choose finite mechanisms and finite information structures, respectively. These results provide additional foundations for the strong maxmin solution that we construct.

Let us say that a mechanism is *finite* if M_i is finite for all i. Finite message sets can be identified with subsets of $\mathbb{N}$, so that the set of finite mechanisms exists and is denoted by $\mathbf{M}^F$. Similarly, an information structure is *finite* if S_i is finite for all i, and $\mathbf{S}^F$ is the set of finite information structures.

A strong maxmin solution $(\mathcal{M}, \mathcal{S}, \beta)$ with profit guarantee Π is *finitely approximable* if for any $\epsilon > 0$, the following statements hold:

---

[31]Both sides of (28) are zero at $x^* = 0$, and the difference between the left and right has a derivative with respect to x^* that has the same sign as $G_N(x^*) - c - g_N(x^*)$, which, as we have argued, is single crossing from below.

(i) There exists $\mathcal{M}^F \in \mathbf{M}^F$ such that for any information structure $\mathcal{S}'$ and equilibrium β' of $(\mathcal{M}^F, \mathcal{S}')$, $\Pi(\mathcal{M}^F, \mathcal{S}', \beta') \geq \Pi - \epsilon$.

(ii) There exists $\mathcal{S}^F \in \mathbf{S}^F$ such that for any mechanism $\mathcal{M}'$ and equilibrium β' of $(\mathcal{M}', \mathcal{S}^F)$, $\Pi(\mathcal{M}', \mathcal{S}^F, \beta') \leq \Pi + \epsilon$.

In other words, a solution is finitely approximable if there are finite mechanisms and finite information structures that guarantee profit close to Π for the seller and Nature, respectively.

THEOREM 3: *The solution $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ is finitely approximable.*

The theorem follows from two propositions, whose proofs are given in Appendix B. Given a nonnegative real number $\underline{m}$ and a positive integer K, let $\overline{\mathcal{M}}(\underline{m}, K)$ be the mechanism where each bidder's message space is $\{\underline{m} + l/K \,|\, l = 0, \ldots, K^2\}$, the allocations are the restriction of $\overline{q}$ to this message space, and transfers are given by $t_i(m) = \overline{t}_i(m) - L_p \underline{m}$, where L_p is a Lipschitz constant for the premium aggregate transfer $\overline{T} - \underline{v}\overline{Q}$.[32]

The purpose of the discount $L_p \underline{m}$ is to satisfy participation security, since

$$t_i(\underline{m}, m_{-i}) = \frac{\underline{m}}{\underline{m} + \Sigma m_{-i}} \overline{T}(\underline{m} + \Sigma m_{-i}) - L_p \underline{m}$$

$$\leq \frac{\underline{m}}{\underline{m} + \Sigma m_{-i}} \left(\underline{v}\overline{Q}(\underline{m} + \Sigma m_{-i}) + L_p(\underline{m} + \Sigma m_{-i}) \right) - L_p \underline{m}$$

$$\leq \frac{\underline{m}}{\underline{m} + \Sigma m_{-i}} \underline{v}\overline{Q}(\underline{m} + \Sigma m_{-i}).$$

As a result, $m_i = \underline{m}$ guarantees a nonnegative ex post payoff. An interpretation is that $\overline{\mathcal{M}}(\underline{m}, K)$ is a *discrete proportional auction*, where allocation and transfers are proportional, and, in addition, every bidder receives a small subsidy of $L_p \underline{m}$.

PROPOSITION 5: *For all $\epsilon > 0$, there exist $\underline{m}$ and K such that for any information structure $\mathcal{S}$ and equilibrium β of $(\overline{\mathcal{M}}(\underline{m}, K), \mathcal{S})$, $\Pi(\overline{\mathcal{M}}(\underline{m}, K), \mathcal{S}, \beta) \geq \overline{\Pi} - \epsilon$.*

Next, given a positive integer K, let $\overline{S}(K)$ be the information structure derived from $\overline{S}$ by coarsening each bidder's information, so that instead of observing s_i, bidder i only observes the element of the partition

$$\left\{ [0, K^{-1}), [K^{-1}, 2K^{-1}), \ldots, [K - K^{-1}, K), [K, \infty) \right\}$$

that contains s_i.

PROPOSITION 6: *For all $\epsilon > 0$, there exists a K such that for every mechanism $\mathcal{M}$ and equilibrium β of $(\mathcal{M}, \overline{S}(K))$, $\Pi(\mathcal{M}, \overline{S}(K), \beta) \leq \overline{\Pi} + \epsilon$.*

Thus, the solution we constructed is finitely approximable, and the finite approximations are natural discrete analogues of their counterparts in the strong maxmin solution. The proofs of these propositions follow the arguments in Section 4, adapted to the discrete setting. We note that analogous statements of Theorem 3 and other results of this section hold for the must-sell case. This is discussed further in Appendix B.

We now present our uniqueness result.

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[32]The Lipschitz continuity of $\overline{T}^p = \overline{T} - \underline{v}\overline{Q}$ is established in the proof of Lemma 14.

THEOREM 4: *Every finitely approximable strong maxmin solution has a profit guarantee of $\overline{\Pi}$.*

PROOF: If a strong maxmin solution has value Π and is finitely approximable, then for any $\epsilon > 0$, there exists $\mathcal{M}^F \in \mathbf{M}^F$ such that expected profit in any information structure and equilibrium is at least $\Pi - \epsilon$. In particular, for any K, there exists an equilibrium of $(\mathcal{M}^F, \overline{\mathcal{S}}(K))$ (since both are finite) in which profit is at least $\Pi - \epsilon$. By Proposition 6, there is a K such that profit in this equilibrium is at most $\overline{\Pi} + \epsilon$. This shows that $\Pi - \epsilon \leq \overline{\Pi} + \epsilon$. As ϵ was arbitrary, we conclude that $\Pi \leq \overline{\Pi}$. The reverse inequality follows from an analogous argument, using a finite approximation $\mathcal{S}^F$ and $\overline{\mathcal{M}}(\underline{m}, K)$.   Q.E.D.

An additional implication of Theorem 3 is that the finite approximations of $\overline{\mathcal{M}}$ and $\overline{\mathcal{S}}$ are approximate solutions of a large family of maxmin mechanism design and minmax information design problems, parametrized by the equilibrium selection rule.

COROLLARY 1: *Fix an arbitrary selection $\beta^*(\mathcal{M}, \mathcal{S})$ from the (nonempty) equilibrium correspondence B on $\mathbf{M}^F \times \mathbf{S}^F$. Then*

$$\sup_{\mathcal{M} \in \mathbf{M}^F} \inf_{\mathcal{S} \in \mathbf{S}^F} \Pi\big(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S})\big) = \inf_{\mathcal{S} \in \mathbf{S}^F} \sup_{\mathcal{M} \in \mathbf{M}^F} \Pi\big(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S})\big) = \overline{\Pi}.$$

Moreover, the sup-inf is attained by the discrete proportional auctions $\overline{\mathcal{M}}(\underline{m}, K)$, and the inf-sup is attained by the information structures $\overline{\mathcal{S}}(K)$.

This result provides an additional foundation for proportional auctions, as limits of ϵ equilibria of zero-sum games where we fix the equilibrium selection rule.

Corollary 1 can be strengthened in the following manner. The reason for appealing to finite approximations is to ensure the existence of an equilibrium for a suitably large class of alternative mechanisms or information structures. Let us say that an information structure $\mathcal{S}$ is *regular* if for all $\mathcal{M} \in \mathbf{M}^F$, the game $(\mathcal{M}, \mathcal{S})$ has an equilibrium. A mechanism $\mathcal{M}$ is *regular* if for all $\mathcal{S} \in \mathbf{S}^F$, the game $(\mathcal{M}, \mathcal{S})$ has an equilibrium.

Due to the noncompactness of the signal space, we do not know whether $\overline{\mathcal{S}}$ is regular. However, it is easy to regularize $\overline{\mathcal{S}}$ by adding an infinite signal and defining $\overline{w}(\infty) = \overline{v}$, so that $\overline{w}$ is continuous at infinity. The extended information structure $\overline{\mathcal{S}}^*$ is regular, since for any finite mechanism, the associated Bayesian game satisfies the sufficient conditions in Milgrom and Weber (1985). One can also extend the arguments of Section 4.2 to show that equilibrium profit is at most $\overline{\Pi}$. As a result, if the domain of the infimum in Corollary 1 is a set of regular information structures that contain $\overline{\mathcal{S}}^*$, then the inf-sup is attained by $\overline{\mathcal{S}}^*$.

It is an open question whether or not $\overline{\mathcal{M}}$ can be regularized by a similar technique.[33] The difficulty is how to define the allocation and transfer at infinity. There are, however, other mechanisms that are regular and have the same profit lower bound. In particular, an

---

[33]Proposition 5 shows that it is possible to approximate $\overline{\mathcal{M}}$ with finite mechanisms, which are necessarily regular. In fact, it can be shown that the variant of $\overline{\mathcal{M}}$ where bids are capped at some $\overline{m} > 0$ is regular, and similar steps as in the proof of Proposition 5 can be used to show that it has a profit lower bound that converges to $\overline{\Pi}$ as $\overline{m} \to \infty$.

earlier version of this paper constructed a maxmin mechanism with the same proportional allocation and the transfer

$$\bar{t}_i(m) = \underline{v}\overline{q}_i(m) + \frac{1}{N!} \sum_{\zeta \in Z} \int_{x=0}^{\infty} \big(\overline{\Xi}^p(\Sigma m_{\zeta < \zeta(i)} + x) - \overline{\Xi}^p(\Sigma m_{\zeta \le \zeta(i)} + x)\big) g_{N - \zeta(i) + 1}(x) \, dx,$$

where $\overline{\Xi}^p(x) = \overline{\Xi}(x) - \underline{v}(\overline{\mu}(x) - \overline{Q}(x))$, Z is the set of permutations of $\{1, \ldots, N\}$, $m_{\zeta \le k}$ is the subvector of messages $m_{\{j|\zeta(j) \le k\}}$, and $m_{\zeta < k}$ is the subvector $m_{\{j|\zeta(j) < k\}}$. This transfer rule is continuous at infinity. If the allocation is extended so that the good is equally shared between bidders who submit infinite bids and transfers are extended via continuity, then the resulting mechanism is regular. Indeed, with a finite information structure, the resulting Bayesian game is payoff secure and upper semicontinuous, so that the existence of an equilibrium follows from Reny (1999). More details can be found in our working paper (Brooks and Du (2019)).

In addition to the solutions we have described, there may be other solutions to the allocation sensitivity and excess growth equations. Indeed, when the support of H is $\{0, 1\}$, distinct allocation and transfer rules are constructed by Bergemann, Brooks, and Morris (2016). As discussed in Section 3, however, the allocation rule is unique in the must-sell case when $N = 2$ and the support of H is convex, and when we restrict attention to continuous allocation rules and one-dimensional bids. There may also be more exotic solutions, such as mechanisms that explicitly elicit belief hierarchies. The characterization of the set of strong maxmin solutions is an interesting topic for future work.

6. MAXMIN AUCTIONS IN THE MANY-BIDDER LIMIT

6.1. *Profit Comparison*

In this section, we further explore the properties of the maxmin proportional auctions and the optimal profit guarantee. We begin with a comparison of mechanisms for the standard uniform distribution with $c = 0$, for which the optimal profit guarantee was computed in Section 4.4. In Figure 4, we have plotted the optimal guarantee for N ranging from 1 to 30.[34] The can-keep and must-sell guarantees are denoted by the dots and circles, respectively.

For comparison, the gray dots denote the profit guarantee of the first-price auction, as computed by Bergemann, Brooks, and Morris (2017), which is $(N - 1)/(4N - 2)$. Also, the solid black line denotes the best guarantee from a posted price mechanism, which is $1/8$ and is obtained with a price of $1/4$.[35]

A striking feature of this picture is that the optimal profit guarantee increases in N and appears to be converging toward 0.5. The latter is the ex ante expected value, which is obviously an upper bound on profit in any mechanism. In fact, as N goes to infinity, the optimal profit guarantee converges to the expected surplus. This remarkable fact is implied by the earlier result of Du (2018), who constructed a particular sequence of mechanisms and profit guarantees (the diamonds) which converge to the expected value. A fortiori, the optimal profit guarantee must converge as well.

The rest of this section explores and extends this result. We generalize the bound to positive production costs, in which case the correct limit profit guarantee is the ex ante

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[34]A similar figure previously appeared in Du (2018).

[35]The worst-case information is a public signal indicating whether the value is above or below $1/2$.
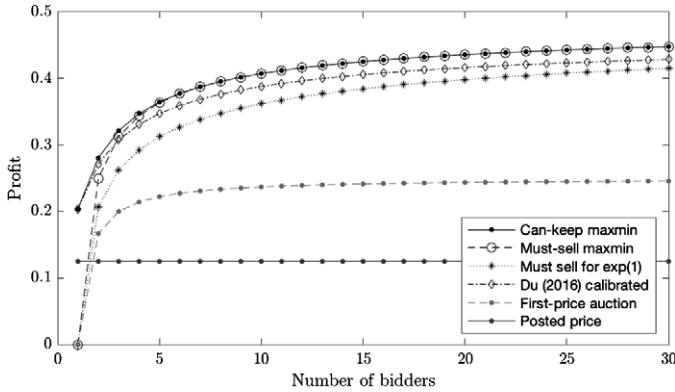
FIGURE 4.—Comparing maxmin mechanisms to other mechanisms.

gains from trade. The optimal rate of convergence is characterized. We also show that the limit is attained even with must-sell mechanisms, and perhaps most surprisingly, we argue that the same limit holds even if the distribution of the value is incorrectly specified. As an illustration, the triangles in Figure 4 denote a profit guarantee for the maxmin auction that is calibrated to an exponentially distributed value, but when the value is actually standard uniform.[36] Finally, we describe the limiting maxmin aggregate allocation and transfer.

6.2. *Information and Welfare in the Many-Bidder Limit*

We now proceed formally. As a preliminary step, we address the left-tail assumption on the value distribution introduced in Section 2, which were only assumed for a single N, whereas we now take N to infinity. It turns out, however, that no additional assumption is needed.

LEMMA 9: *If the left-tail assumption holds for N, it also holds for any $N' > N$.*

Proofs for all results of this section are provided in Appendix C.

We now denote the optimal profit guarantees for the can-keep and must-sell models by $\overline{\Pi}_N(H)$ and $\widehat{\Pi}_N(H)$, respectively, emphasizing their dependence on the number of bidders and the distribution of the value. The production cost is held fixed when we vary N and H.

A simple upper bound on the profit guarantee that holds for all N is the ex ante gains from trade. For if the bidders have no information about the value, the best the seller can do is make a take-it-or-leave-it offer at a price equal to the ex ante expected value. We now show that this upper bound is tight.

PROPOSITION 7: *In the limit as N goes to infinity, the profit guarantees $\overline{\Pi}_N(H)$ and $\widehat{\Pi}_N(H)$ converge to the ex ante gains from trade at a rate of $1/\sqrt{N}$.*

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[36]In Section 2, we assumed that the support of H is bounded, which is violated by the exponential distribution. In this calculation, we have taken the limit of the formulae for bounded distributions. We suspect that our results extend to unbounded distributions as long as the right tail is not too heavy.

Here is a sketch of the argument. Recall that under the minmax information, the aggregate signal is a sufficient statistic for the value. Let us change the units of each bidder's signal according to[37] $s_i^C = (s_i - (N-1)/N)/\sqrt{N-1}$, where the "$C$" denotes a central limit normalization. The centered aggregate signal $x = \Sigma s^C = (\Sigma s - (N-1))/\sqrt{N-1}$ has cumulative distribution $G_N^C(x) = G_N(\sqrt{N-1}x + N - 1)$ and density $g_N^C(x) = \sqrt{N-1}g_N(\sqrt{N-1}x + N - 1)$. We can correspondingly center the value function as $\overline{w}_N^C(x) = \overline{w}_N(\sqrt{N-1}x + N - 1)$, where we now emphasize the dependence on N. The centered gains function $\overline{\gamma}_N^C$ is defined analogously.

The Lindeberg–Lévy theorem implies that the distribution of the centered aggregate signal converges to a standard Normal with distribution Φ and density ϕ. We argue in Appendix C that the normalized fully-revealing gains function converges almost surely to $\widehat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x)) - c$, which is just a change of units from $\widehat{\gamma}_N$, and the graded gains function converges almost surely to

$$\overline{\gamma}_\infty^C(x) = \begin{cases} 0 & \text{if } x < x^*, \\ H^{-1}(\Phi(x)) - c & \text{if } x \geq x^*, \end{cases}$$

where x^* is the largest x such that

$$0 = \int_{y=-\infty}^{x} \widehat{\gamma}_\infty^C(y)\phi(y)\,dy. \tag{29}$$

(Note that x^* is $-\infty$ if $v - c > 0$ with probability 1.) Thus, in the limit, there is only grading at the bottom, and then only if the gains from trade may be negative.

With this normalization, the hazard rate of each bidder's signal becomes $\sqrt{N-1}$, so that each bidder's virtual value is

$$\overline{\gamma}_N^C(x) - \frac{1}{\sqrt{N-1}}\frac{d}{dx}\overline{\gamma}_N^C(x).$$

Since $\overline{\gamma}_N^C$ is bounded, information rents go to zero at a rate of $1/\sqrt{N-1}$. Note that only the bidder who is allocated the good gets an information rent. Thus, total bidder surplus goes to zero at a rate of $1/\sqrt{N-1}$. At the same time, it is always weakly optimal for the seller to allocate the good, so that profit converges to the ex ante gains from trade.

This sketch glosses over significant technical complications. The convergence of the gains function is only almost everywhere, and along the sequence of minmax information structures, the hazard rate and the graded gains function are both changing. The formal proof deals with these issues by working directly with the integral for the difference between ex ante gains from trade and profit, scaled up by $\sqrt{N-1}$. This sequence converges to a positive constant, thus establishing the proposition.

6.3. *Robustness to the Prior*

We have assumed that the seller does not know the information structure, but does know the value distribution exactly. There is clear tension here. It turns out, however, that our results are robust to misspecification of the prior, as we now explain.

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[37]This central limit normalization is obviously asymptotically equivalent to the more conventional normalization, $(s_i - 1)/\sqrt{N}$. The normalization with $\sqrt{N-1}$ turns out to be much more analytically convenient, for example, in the proof of Lemma 12.

Suppose that the seller runs the maxmin proportional auction for the prior H, denoted $\overline{\mathcal{M}}_N(H)$. Let $\overline{\lambda}_N(v; H)$ denote the associated optimal multipliers given by (18). The proof of Proposition 3 establishes that a lower bound on profit is the expectation of $\overline{\lambda}_N(v; H)$. In that argument, the prior H only appears at the last step as a mean-preserving spread of the distribution of $w(s)$. As a result, even if the prior is some $H' \neq H$, we still obtain a lower bound on profit, which is the expectation of $\overline{\lambda}_N(v; H)$ under H'. Since $\overline{\lambda}_N(v; H)$ is bounded and continuous, the change in the profit guarantee is small as long as H is close to H' in the weak-∗ topology.

PROPOSITION 8: *Fix a distribution H' with support contained in $[\underline{v}, \overline{v}]$. In any equilibrium of $\overline{\mathcal{M}}_N(H)$ for any information structure where the value distribution is H', expected profit is at least*

$$\overline{\Pi}_N(H, H') = \int_{v=\underline{v}}^{\overline{v}} \overline{\lambda}_N(v; H) H'(dv),$$

which is a linear and weak-∗ continuous function of H'.

Thus, when the prior is only slightly misspecified, the loss in the profit guarantee is small. If the prior is badly misspecified, the loss may be substantial, but when the number of bidders is large, the loss from misspecification vanishes.

PROPOSITION 9: *Fix a distribution H' with support contained in $[\underline{v}, \overline{v}]$. As N goes to infinity, $\overline{\Pi}_N(H, H')$ converges to the ex ante gains from trade under H'.*

Note that this limiting profit guarantee need not be positive, in which case the optimal profit guarantee is zero and it is better to shut down production.

When $c \leq \underline{v}$, this result follows from Propositions 7 and 8. To see why, consider what would happen if the seller ran $\overline{\mathcal{M}}_N(H)$, but the true prior puts probability 1 on a particular value $v \in [\underline{v}, \overline{v}]$. Proposition 8 says that profit must be at least $\overline{\lambda}_N(v; H)$. At the same time, profit in this counterfactual cannot be greater than $v - c$, which is the efficient surplus, so $\overline{\lambda}_N(v; H) \leq v - c$. But Proposition 7 says that expected profit guarantee under H converges to the ex ante gains from trade, which is only possible if $\overline{\lambda}_N(v; H)$ converges to $v - c$ for H-almost all v.

This argument establishes Proposition 9 if H' is absolutely continuous with respect to H and there is common knowledge of (ex post) gains from trade. The result is much stronger. In Appendix C, we show that $\overline{\lambda}_N(v)$ converges pointwise to $v - c$ for all $v \in [\underline{v}, \overline{v}]$, even when there is not common knowledge of gains from trade.

Analogues of Propositions 8 and 9 also hold for the must-sell model. The necessary modifications to the proof are minor, as explained in Appendix C.

6.4. *Limiting Allocation and Transfer*

As a last topic, we present two descriptions of the limiting maxmin allocation and transfer. For this section, we focus on a relatively simple case when the value distribution is asymptotically single crossing. In particular, we assume there exists a $C > 0$ such that for all $a, b \in [\underline{v}, \overline{v}]$ such that $a \leq b$, $H(b) - H(a) \geq C(b - a)$.[38] In addition, we assume that $\underline{v} \neq c$.

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[38]This is equivalent to assuming that the absolutely continuous part of H has a density that is bounded away from zero on $[\underline{v}, \overline{v}]$.

LEMMA 10: *Suppose that the above stated conditions hold. Then there exists an $\widehat{N}$ such that for all $N > \widehat{N}$, if $\underline{v} > c$, there are no graded intervals, and if $\underline{v} < c$, $\overline{\gamma}_N^C$ has a single graded interval of the form $[-\sqrt{N-1}, x_N]$.*

Let us next define

$$\overline{Q}_N^C(x) = \overline{Q}_N(x\sqrt{N-1} + N - 1), \qquad \overline{T}_N^C(x) = \overline{T}_N(x\sqrt{N-1} + N - 1).$$

PROPOSITION 10: *Under the conditions preceding Lemma 10, for all $x \in \mathbb{R}$, $\lim_{N\to\infty} \overline{Q}_N^C(x) = 1$ and*

$$\lim_{N\to\infty} \overline{T}_N^C(x) = \frac{1}{\phi(x)} \int_{y=-\infty}^{x} F(y)\phi(y)\, dy,$$

where

$$F(x) = \begin{cases} -cx + \widehat{\gamma}_\infty^C(x^*)(1 - \Phi(x^*)) + \int_{y=x^*}^{\infty} \widehat{\gamma}_\infty^C(y)(1 - \Phi(y))\, dy \\ \quad \text{if } x < x^*, \\ -cx - \widehat{\gamma}_\infty^C(x^*)\Phi(x^*) - \int_{y=x^*}^{x} \widehat{\gamma}_\infty^C(y)\Phi(y)\, dy + \int_{y=x}^{\infty} \widehat{\gamma}_\infty^C(y)(1 - \Phi(y))\, dy \\ \quad \text{if } x > x^*, \end{cases}$$

where $\widehat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x)) - c$ and x^* is the largest solution to equation (29).

The limits for the must-sell mechanism are the same as those in Proposition 10, substituting $x^* = -\infty$.

Thus, under the central limit normalization, the good is asymptotically always allocated. One might have guessed that in this limit the aggregate transfer would be equal to the expected value under the minmax information, but Proposition 10 shows that this is not the case.

In some cases, the formula for the limit transfer simplifies substantially. The running uniform example does not satisfy the hypotheses of Lemma 10, since $\underline{v} = c = 0$. Nonetheless, in Appendix D.2, it is shown that there is no grading in the limit, and

$$\lim_{N\to\infty} \overline{T}_N^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}\phi(x)}.$$

We get a somewhat different perspective on the limit when we use a law of large numbers normalization, in which signals are i.i.d. exponential with arrival rate N. Let us define

$$\overline{Q}_N^L(x) = \overline{Q}_N(Nx), \qquad \overline{T}_N^L(x) = \overline{T}_N(Nx).$$

PROPOSITION 11: *Under the conditions preceding Lemma 10, for all $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \overline{Q}_N^L(x) = \begin{cases} 1 & \text{if } x^* = -\infty, \\ \min\{x, 1\} & \text{otherwise} \end{cases}$$

and

$$\lim_{N\to\infty} \overline{T}_N^L(x) = \begin{cases} \max\{\underline{v}, c\} & \text{if } x < 1, \\ \overline{v} & \text{if } x > 1, \end{cases}$$

PROOF: The limit of the transfer follows from Proposition 10 and computations using l'Hôpital's rule for $\lim_{x\to-\infty}\lim_{N\to\infty}\overline{T}_N^C(x)$ and $\lim_{x\to\infty}\lim_{N\to\infty}\overline{T}_N^C(x)$.

For the allocation, if $\underline{v} < c$, then there is a finite $\widehat{N}$ such that $\overline{Q}_N^L = \min\{Nx/(x_N\sqrt{N-1} + N - 1), 1\}$ for all $N > \widehat{N}$. Since $x_N \to x^*$, we conclude that $\overline{Q}_N^L$ converges to $\min\{x, 1\}$. If $\underline{v} > c$, then there is a finite $\widehat{N}$ such that $\overline{Q}_N^L(x) = 1$ for all $N > \widehat{N}$. *Q.E.D.*

Thus, rationing persists in the limit under the law of large numbers normalization when the gains from trade might be negative. However, the aggregate transfer pushes the aggregate bid to 1, so that in equilibrium, the good will almost always be allocated.

7. CONCLUSION

This paper has studied the canonical auction design problem when values are common. The novelty is to use an informationally robust criterion for measuring auction performance. The spirit of the exercise is to identify mechanisms that are less vulnerable to misspecification of information and behavior, and are therefore more viable in a practical setting, where a designer may be unwilling or unable to commit to a specific description of information.

The literature to which we contribute has previously shown that it is possible to obtain nontrivial profit guarantees across all information structures and equilibria, even with simple mechanisms like the first-price auction (Bergemann, Brooks, and Morris (2017)). It has also shown that there are mechanisms whose profit guarantees are unimprovable when the number of bidders is large (Du (2018)). Our marginal contribution is to establish, in a rich class of environments, the precise limit of what can be attained. We have also developed new methodology for the characterization of maxmin mechanisms, namely the double revelation principle and the critical conditions on the aggregate allocation sensitivity and the aggregate excess growth. Finally, we have shown that the optimal guarantee can be attained with the relatively simple class of proportional auctions, which are parametrized by just the aggregate allocation and aggregate transfer as functions of the aggregate bid. The analysis also indicates that simple aggregate allocation rules that increase linearly until the available supply is exhausted can perform well.

To our knowledge, proportional auctions are new to the literature, and we are unaware of instances where these auctions have been used in practice. We therefore view our contribution as normative. Our model stays within the Bayesian mechanism design framework, broadly defined, but also allows us to remedy conceptual and practical limitations associated with having to commit to a specific information structure. To be sure, this approach introduces new conceptual issues: Why should the bidders have common knowledge of the information structure, while the seller does not? Why does the seller not simply induce the bidders to reveal the information structure and then run the optimal mechanism for the environment they report? This is clearly a theoretical possibility, but it runs contrary to our primary motivation, which is to identify mechanisms with desirable welfare properties that remain feasible when we respect the designer and the agents' limited ability to articulate higher-order beliefs. We have not imposed such constraints

explicitly. To us, the value of the model is not just in its assumptions, but also in the form of the results: mechanisms that have desirable welfare properties but also feature a simple bidding interface and rule set, so that they remain feasible in the face of additional practical constraints. As for the common prior among the agents, this is obviously a strong assumption, but we find it relatively palatable as an as-if description of agents' behavior, as long as it does not need to be explicitly communicated by the bidders and it is not a necessary input to compute the optimal mechanism.

Nonetheless, it is true that in distancing ourselves from the untenable assumptions of the standard model, we have taken an equally extreme and implausible position, which is that the designer puts no restrictions on information except for the value distribution and the existence of a common prior. Verily, the truth must lie somewhere in between. Auction designers may be willing to rule out some models without committing themselves to a single description of the world. We expect the theory to become even more useful as we explore the middle ground between these two extremes by incorporating reasonable restrictions on beliefs into the informationally robust mechanism design problem.

APPENDIX A: Omitted Proofs for Section 4

PROOF OF PROPOSITION 1: (i) $\Longrightarrow$ (ii). Suppose $(\mathcal{M}, \mathcal{S}, \beta)$ is a strong maxmin solution with profit guarantee Π. Let X be the disjoint union of $\{M_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$. Clearly, $\mathcal{M} \in \mathbf{M}(X)$ and $\mathcal{S} \in \mathbf{S}(X)$. We will argue that $(\mathcal{M}, \mathcal{S})$ is a nontrivial Nash equilibrium of $\mathcal{G}(X, \beta^*)$ for every $\beta^* \in \mathcal{B}(X)$. Fix a selection β^*. By C3, $B(\mathcal{M}, \mathcal{S})$ is nonempty, so that the seller and Nature's payoffs from $(\mathcal{M}, \mathcal{S})$ are finite. Moreover, C1 and C2 imply that all equilibria of $(\mathcal{M}, \mathcal{S})$ have the same profit, which is Π. The payoffs from $(\mathcal{M}, \mathcal{S})$ are therefore $(\Pi, -\Pi)$, Clearly, neither party can profit by deviating so that the resulting game has no bidder equilibria, so we restrict attention to deviations that result in an equilibrium. For any $\mathcal{M}' \in \mathbf{M}(X)$ such that $B(\mathcal{M}', \mathcal{S}) \neq \emptyset$, C1 implies that $\Pi(\mathcal{M}', \mathcal{S}, \beta^*(\mathcal{M}', \mathcal{S})) \leq \Pi$, and for any $\mathcal{S}' \in \mathbf{S}$ with $B(\mathcal{M}, \mathcal{S}') \neq \emptyset$, C2 implies that $\Pi(\mathcal{M}, \mathcal{S}', \beta^*(\mathcal{M}, \mathcal{S}')) \geq \Pi$. Thus, $(\mathcal{M}, \mathcal{S})$ is a nontrivial Nash equilibrium.

(ii) $\Longrightarrow$ (i). Suppose that $(\mathcal{M}, \mathcal{S}) \in \mathbf{M}(X) \times \mathbf{S}(X)$ is a nontrivial Nash equilibrium of $\mathcal{G}(X, \beta^*)$ for all $\beta^* \in \mathcal{B}(X)$. Fix $\beta \in B(\mathcal{M}, \mathcal{S})$. We claim that $(\mathcal{M}, \mathcal{S}, \beta)$ is a strong maxmin solution with profit guarantee $\Pi = \Pi(\mathcal{M}, \mathcal{S}, \beta)$. C3 is immediate. We now show C1. Fix any $\mathcal{M}'$ and equilibrium β' of $(\mathcal{M}', \mathcal{S})$. Then by the revelation principle, there is an incentive compatible direct mechanism on $\mathcal{S}$ for which truth-telling is an equilibrium. Since $\mathcal{M}'$ is participation secure, bidders' payoffs must all be nonnegative, so we can extend the direct mechanism by adding a participation-secure message, so that if any bidder sends the participation-secure message, the seller keeps the good and all transfers are zero. Call this mechanism $\mathcal{M}''$, and observe that $\mathcal{M}'' \in \mathbf{M}(X)$ (since by construction $\mathbf{M}(X)$ contains all participation mechanisms with message spaces S_i plus an additional message). Since a bidder's payoff from this participation-secure message is zero, the truthful strategies β'' are an equilibrium. Now fix a selection β^* for which the truthful equilibrium β'' is selected on $(\mathcal{M}'', \mathcal{S})$ and β is selected on $(\mathcal{M}, \mathcal{S})$. Then since $(\mathcal{M}, \mathcal{S})$ is a Nash equilibrium of $\mathcal{G}(X, \beta^*)$, we conclude that

$$\Pi(\mathcal{M}', \mathcal{S}, \beta') = \Pi(\mathcal{M}'', \mathcal{S}, \beta'')$$
$$= \Pi(\mathcal{M}'', \mathcal{S}, \beta^*(\mathcal{M}'', \mathcal{S}))$$
$$\leq \Pi(\mathcal{M}, \mathcal{S}, \beta^*(\mathcal{M}, \mathcal{S}))$$
$$= \Pi(\mathcal{M}, \mathcal{S}, \beta) = \Pi.$$

This proves C1. C2 follows by an analogous argument, where we use the well known fact that for any $\mathcal{S}'$ and equilibrium of $(\mathcal{M}, \mathcal{S}')$, there is a BCE $\mathcal{S}''$ that, together with the obedient strategies, induces the same profit. Moreover, this BCE, using the message space of $\mathcal{M}$ as the signal space, is necessarily in $\mathbf{S}(X)$.   *Q.E.D.*

LEMMA 11: *The ex ante expectation of $\overline{\overline{\Xi}}$ is zero:*

$$\int_{x=0}^{\infty} \overline{\overline{\Xi}}(x) g_N(x) \, dx = 0.$$

PROOF: Using the formula for $\overline{\overline{\Xi}}$ in equation (19) and Lemma 6, it suffices to show that

$$\overline{\Pi} = \int_{x=0}^{\infty} \left[\overline{\mu}(x) \widehat{w}(x) - c \overline{Q}(x) \right] g_N(x) \, dx. \tag{30}$$

From (22), note that

$$\int_{x=0}^{\infty} \left(\overline{\mu}(x) - \overline{Q}(x) \right) g_N(x) \, dx = \int_{x=0}^{\infty} \left(\overline{Q}'(x) + \left(\frac{N-1}{x} - 1 \right) \overline{Q}(x) \right) g_N(x) \, dx$$

$$= \int_{x=0}^{\infty} \left(\overline{Q}'(x) g_N(x) + \left(g_{N-1}(x) - g_N(x) \right) \overline{Q}(x) \right) dx$$

$$= \int_{x=0}^{\infty} \frac{d}{dx} \left(g_N(x) \overline{Q}(x) \right) dx = 0, \tag{31}$$

since $g_N(0) \overline{Q}(0) = 0$ and $g_N(x) \overline{Q}(x)$ goes to zero as $x \to \infty$. Thus, (30) is equivalent to

$$\overline{\Pi} = \int_{x=0}^{\infty} \left(\widehat{w}(x) - c \right) \overline{\mu}(x) g_N(x) \, dx.$$

Using $\widehat{\gamma}(x) = \widehat{w}(x) - c$ and the formula for $\overline{\Pi}$ in (3), we see that the above equation is equivalent to

$$\int_{x=0}^{\infty} \overline{\gamma}(x) g_{N-1}(x) \, dx = \int_{x=0}^{\infty} \widehat{\gamma}(x) \overline{\mu}(x) g_N(x) \, dx. \tag{32}$$

When $\overline{\Gamma}(x) = \widehat{\Gamma}(x)$, we have $\overline{\gamma}(x) = \widehat{\gamma}(x)$ and $\overline{\mu}(x) = (N-1)/x$, so $\overline{\mu}(x) g_N(x) = g_{N-1}(x)$ and the two integrands are exactly equal. On the other hand, over a graded interval $[a, b]$,

$$\int_{x=a}^{b} \widehat{\gamma}(x) \overline{\mu}(x) g_N(x) \, dx = C(a, b) \left(\widehat{\Gamma}(b) - \widehat{\Gamma}(a) \right)$$

$$= C(a, b) \left(\overline{\Gamma}(b) - \overline{\Gamma}(a) \right)$$

$$= C(a, b) \int_{x=a}^{b} \overline{\gamma}(x) g_N(x) \, dx$$

$$= \int_{x=a}^{b} \overline{\gamma}(x) g_{N-1}(x) \, dx,$$

where the third equality follows from $\overline{\gamma}(x) = \overline{\gamma}(a)\exp(x - a)$, and the fourth equality follows from the fact that for $n \geq 1$,

$$\int_{x=a}^{b} \overline{\gamma}(x)g_n(x)\,dx = \overline{\gamma}(a)\exp(-a)\frac{x^n}{n!}\bigg|_{x=a}^{b} = \overline{\gamma}(a)\exp(-a)\frac{b^n - a^n}{n!}. \qquad Q.E.D.$$

LEMMA 12: *For all $x > 0$, $|\overline{Q}'(x)| \leq (N-1)/x$, and if $\Sigma m = x$, then*

$$\left|\frac{1}{\Delta}\big(\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)\big)\right| \leq \frac{N+1}{x}.$$

PROOF: Note that on a graded interval,

$$\overline{Q}'(x) = \frac{C(a,b)}{N} - \frac{(N-1)D(a,b)}{x^N}.$$

This is at most $1/x$ when we replace $C(a,b)$ with the bound in equation (21) and set $D(a,b) = 0$. Moreover, since $C(a,b) \geq 0$ and

$$D(a,b) = \frac{b^N - ab^{N-1}}{b^N - a^N}a^{N-1} \leq a^{N-1} \leq x^{N-1}, \qquad (33)$$

we conclude that $\overline{Q}'(x)$ is at least $-(N-1)/x$, so $|\overline{Q}'(x)| \leq (N-1)/x$.

Next observe that

$$\left|\frac{1}{\Delta}\big(\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)\big)\right|$$

$$= \left|\frac{1}{\Delta}\left(\frac{m_i + \Delta}{\Sigma m + \Delta}\overline{Q}(\Sigma m + \Delta) - \frac{m_i}{\Sigma m}\overline{Q}(m)\right)\right|$$

$$= \left|\frac{1}{\Delta}\left(\frac{m_i}{\Sigma m + \Delta}\big(\overline{Q}(\Sigma m + \Delta) - \overline{Q}(\Sigma m)\big) + \frac{\Delta}{\Sigma m + \Delta}\overline{Q}(\Sigma m + \Delta)\right.\right.$$

$$\left.\left. + \left(\frac{m_i}{\Sigma m + \Delta} - \frac{m_i}{\Sigma m}\right)\overline{Q}(m)\right)\right|$$

$$\leq \left|\frac{m_i}{\Delta}\frac{\overline{Q}(\Sigma m + \Delta) - \overline{Q}(m)}{\Sigma m + \Delta}\right| + \left|\frac{\overline{Q}(\Sigma m + \Delta)}{\Sigma m + \Delta}\right| + \left|\frac{m_i}{\Sigma m(\Sigma m + \Delta)}\overline{Q}(\Sigma m)\right|$$

$$\leq \frac{m_i}{\Delta}\frac{\Delta\frac{N-1}{x}}{\Sigma m + \Delta} + \frac{2}{\Sigma m} \leq \frac{N+1}{x},$$

where the last line follows from the facts that $\overline{Q}'(x) \leq (N-1)/x$ and $\overline{Q}(x) \leq 1$. $\qquad Q.E.D.$

LEMMA 13: *As $x \to \infty$, $\overline{Q}(x) \to 1$.*

PROOF: For each $x \in \mathbb{R}_+$, either x is not graded, in which case $\overline{Q}(x) = 1$, or x is in a graded interval $[a, b]$, in which case $|\overline{Q}(x) - 1| \leq \frac{N-1}{a}(b - a)$ since $\overline{Q}(a) = 1$ and

$|\overline{Q}'(x)| \leq \frac{N-1}{a}$ by Lemma 12. But the length of a graded interval $[a, b]$ is bounded above by a constant $C = \log(\overline{v}) - \log(\overline{\gamma}(0))$ since

$$\overline{v} \geq \overline{\gamma}(b) = \overline{\gamma}(a) \exp(b - a) \geq \overline{\gamma}(0) \exp(b - a).$$

So $|\overline{Q}(x) - 1| \leq \frac{N-1}{x-C} C$ when x is in a graded interval $[a, b]$. We conclude that $\lim_{x \to \infty} \overline{Q}(x) = 1$.   *Q.E.D.*

For proving Lemma 8, it is convenient to decompose $\overline{t}_i(m)$ into a "base" component of $\underline{v}\overline{q}_i(m)$ (whose derivative may be unbounded around 0) and a "premium" component of $\overline{t}_i^p(m) = \overline{t}_i(m) - \underline{v}\overline{q}_i(m)$ (whose derivative is bounded; see Lemma 14 below). Let $\overline{\overline{\Xi}}^p(x) = \overline{\overline{\Xi}}(x) - \underline{v}(\overline{\mu}(x) - \overline{Q}(x))$ and

$$\overline{T}^p(x) = \overline{T}(x) - \underline{v}\overline{Q}(x) = \frac{1}{g_N(x)}\left(\int_{y=0}^{x} \overline{\overline{\Xi}}(y)g_N(y)\,dy - \underline{v}\overline{Q}(x)g_N(x)\right)$$

$$= \frac{1}{g_N(x)}\int_{y=0}^{x} \overline{\overline{\Xi}}^p(y)g_N(y)\,dy,$$

where the last equality follows from $\frac{d}{dx}(\overline{Q}(x)g_N(x)) = (\overline{\mu}(x) - \overline{Q}(x))g_N(x)$ by the reasoning leading to (31). Since both $\overline{Q}$ and $\overline{T}$ are right-differentiable (Lemmas 5 and 7), so is $\overline{T}^p$, and it is easy to verify the differential equation

$$\left(\frac{N-1}{x} - 1\right)\overline{T}^p(x) + \frac{d}{dx}\overline{T}^p(x) = \overline{\overline{\Xi}}^p(x). \tag{34}$$

LEMMA 14: *The ratio*

$$\frac{\overline{t}_i^p(m_i + \Delta, m_{-i}) - \overline{t}_i^p(m)}{\Delta}$$

is bounded over all $m \in \overline{M}$ and $\Delta > 0$.

PROOF: First, we show that $\overline{\overline{\Xi}}^p$ is bounded. Given that $\widehat{w}(x)$ is bounded above and $\overline{\mu}(x)$ is decreasing from Lemma 3, it suffices to show that $\limsup_{x \to 0}(\widehat{w}(x) - \underline{v})/x < \infty$. This is a direct implication of the first part of the left-tail assumption, since $x^\varphi < x$ for x sufficiently small.

Next suppose $\Sigma m > 0$. Since $\overline{t}_i^p(m)$ is right-continuous at m, we have

$$\left|\frac{\partial \overline{t}_i^p(m)}{\partial m_i}\right| = \left|\frac{\Sigma m_{-i}}{(\Sigma m)^2}\overline{T}^p(\Sigma m) + \frac{m_i}{\Sigma m}\frac{d\overline{T}^p}{dx}(\Sigma m)\right| \leq \left|\frac{\overline{T}^p(\Sigma m)}{\Sigma m}\right| + \left|\frac{d\overline{T}^p}{dx}(\Sigma m)\right|.$$

We argue that the right-hand side of the above equation is bounded over all $m \neq 0$, which implies the boundedness of $(\overline{t}_i^p(m_i + \Delta, m_{-i}) - \overline{t}_i^p(m))/\Delta$. Since $\overline{T}^p$ satisfies equation (34), and $\overline{\overline{\Xi}}^p$ and $\overline{T}^p$ are bounded, it suffices to show that $\overline{T}^p(x)/x$ is bounded. This follows from l'Hôpital's rule:

$$\lim_{x \to 0}\frac{\overline{T}^p(x)}{x} = \lim_{x \to 0}\frac{\displaystyle\int_{y=0}^{x} \overline{\overline{\Xi}}^p(y)g_N(y)\,dy}{xg_N(x)}$$

$$= \lim_{x \to 0} \frac{\overline{\Xi}^p(x) g_N(x)}{g_N(x) + x(g_{N-1}(x) - g_N(x))}$$

$$= \lim_{x \to 0} \frac{\overline{\Xi}^p(x)}{1 + x((N-1)/x - 1)}$$

$$= \frac{-\overline{\lambda}(\underline{v}) + (\underline{v} - c)\overline{Q}(0)}{N}. \tag{35}$$

Finally, for $m = 0$, we note $(\bar{t}_i^p(\Delta, 0) - \bar{t}_i^p(0))/\Delta = \overline{T}^p(\Delta)/\Delta$ and again appeal to the boundedness of $\overline{T}^p(x)/x$. $\qquad$ Q.E.D.

LEMMA 15: *There exists $b > 0$ such that either $[0, b]$ is a graded interval or $[0, b]$ is a nongraded interval.*

PROOF: Case 1: $\underline{v} \le c$. If there were no graded interval at zero, then we would have $\overline{\gamma}(0) = \widehat{\gamma}(0) = \underline{v} - c \le 0$. Lemma 2 then implies that for all $x \ge 0$, $\overline{\gamma}(x) \le \overline{\gamma}(0) \exp(x) \le 0$, so that $\overline{\gamma}$ has a nonpositive expectation. This contradicts Lemma 1, which implies that the expectation of $\overline{\gamma}$ equals the expectation of $v - c$ under H, and the hypothesis that the ex ante gains from trade is strictly positive.

Case 2: $\underline{v} > c$. The second part of the left-tail assumption says there exists an $\epsilon > 0$ such that

$$\frac{\widehat{\gamma}(x')}{\exp(x')} \ge \frac{\widehat{\gamma}(x)}{\exp(x)}$$

for all $0 \le x' \le x \le \epsilon$. Thus, $\widehat{\Gamma} \circ E^{-1}$ is concave on the interval $[0, E(\epsilon)]$, so that if a subset $[a, b]$ of $[0, E(\epsilon)]$ is not graded, then $[0, a]$ is not graded as well. The claim of the lemma follows since either $[0, E(\epsilon)]$ is a subset of a graded interval or there exists a nongraded interval that starts at zero. $\qquad$ Q.E.D.

PROOF OF LEMMA 8: Fix an information structure $\mathcal{S}$ and an equilibrium β of $(\overline{\mathcal{M}}, \mathcal{S})$. Let us rewrite equation (25) as

$$\int_S \int_{\overline{M}} \left[(w(s) - \underline{v}) \nabla \cdot \overline{q}(m) - \nabla \cdot \overline{t}^p(m) \right] \beta(dm|s) \pi(ds) \le 0.$$

Equation (35) implies that $\nabla \cdot \overline{t}^p(0) = -\overline{\lambda}(\underline{v}) + (\underline{v} - c)\overline{Q}(0)$.

Since $(\bar{t}_i^p(m_i + \Delta, m_{-i}) - \bar{t}_i^p(m))/\Delta$ is bounded over $\Delta \ge 0$ (Lemma 14), the dominated convergence theorem then implies that

$$\lim_{\Delta \searrow 0} \int_S \int_M \frac{\bar{t}_i^p(m_i + \Delta, m_{-i}) - \bar{t}_i^p(m)}{\Delta} \beta(dm|s) \pi(ds) = \int_S \int_M \frac{\partial \bar{t}_i^p}{\partial m_i}(m) \beta(dm|s) \pi(ds). \tag{36}$$

We will argue that

$$\liminf_{\Delta \searrow 0} \int_S \int_M (w(s) - \underline{v}) \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} \beta(dm|s) \pi(ds)$$

$$\ge \int_S \int_M (w(s) - \underline{v}) \frac{\partial \overline{q}_i}{\partial m_i}(m) \beta(dm|s) \pi(ds) \tag{37}$$

by appealing to Fatou's lemma: notice that $\frac{\partial \overline{q}_i}{\partial m_i}(m) = \lim_{\Delta \searrow 0} (\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m))/\Delta$ is well defined when $m \neq 0$ by Lemma 5; for $m = 0$, the limit is $\frac{\partial \overline{q}_i}{\partial m_i}(m) = \frac{1}{b}$ if $[0, b]$ is a graded interval, and $\frac{\partial \overline{q}_i}{\partial m_i}(m) = \infty$ otherwise.

We consider two cases.

Case 1. If there is no grading anywhere (so $\overline{q}_i(m) = m_i/\Sigma m$) or if $[0, b]$ is the only graded interval for some $b > 0$ (so $\overline{q}_i(m) = m_i/\max(\Sigma m, b)$), then Fatou's lemma applies to (37), since $\overline{q}_i(m_i + \Delta, m_{-i}) \geq \overline{q}_i(m)$ for all m and all $\Delta > 0$.

Case 2. Let $\underline{a}$ be the infimum of left endpoints of graded intervals that are strictly positive (of which there must be at least one if we are not in Case 1); Lemma 15 implies that $\underline{a} > 0$. We claim

$$\frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} \geq -\frac{(N+1)}{\underline{a}} \tag{38}$$

holds for all m and all $\Delta > 0$, so Fatou's lemma also applies to (37) in this case.

If $\Sigma m \geq \underline{a}$, then (38) follows from Lemma 12. If $\Sigma m + \Delta \leq \underline{a}$, then $\overline{q}_i(m_i + \Delta, m_{-i}) \geq \overline{q}_i(m)$ by examining the functional form of $\overline{q}$ as in Case 1, so (38) again holds.

Now suppose $\Sigma m < \underline{a}$ and $\Sigma m + \Delta > \underline{a}$. Let $m_i' = \underline{a} - \Sigma m_{-i}$. Clearly, we have

$$\frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} = \delta \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m_i', m_{-i})}{m_i + \Delta - m_i'} + (1 - \delta) \frac{\overline{q}_i(m_i', m_{-i}) - \overline{q}_i(m)}{m_i' - m_i}$$

for $\delta = (m_i + \Delta - m_i')/\Delta \in (0, 1)$. By applying Lemma 12 to the term following δ and applying $\overline{q}_i(m_i', m_{-i}) \geq \overline{q}_i(m)$ to the term following $1 - \delta$, we conclude that (38) again holds.

Since β is an equilibrium, for any bidder i and any fixed $\Delta > 0$, we have

$$\int_S \int_M \left((w(s) - \underline{v}) \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} - \frac{\overline{t}_i^p(m_i + \Delta, m_{-i}) - \overline{t}_i^p(m)}{\Delta} \right) \beta(dm|s) \pi(ds)$$
$$\leq 0.$$

Therefore, equations (36) and (37) imply (25) if we sum across i.   Q.E.D.

LEMMA 16: *There exists a $\widehat{\Delta}$ such that for all $\Delta < \widehat{\Delta}$ and $x \in \mathbb{R}_+$,*

$$\overline{\gamma}(x) \left(g_N(x) + \frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} \right) \leq \begin{cases} \overline{\gamma}(x)(g_N(x) + 2) & \text{if } x < 1, \\ \overline{\gamma}(x)[g_N(x) + 2g_{N-1}(x)] & \text{if } x \geq 1. \end{cases}$$

This bounding function is integrable.

PROOF: If $x \geq \Delta$, then

$$\frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} = \frac{\exp(-x)}{(N-1)!} \frac{x^{N-1} - (x - \Delta)^{N-1} \exp(\Delta)}{\exp(\Delta) - 1}$$
$$\leq \frac{\exp(-x)}{(N-1)!} \frac{x^{N-1} - (x - \Delta)^{N-1}}{\exp(\Delta) - 1}$$
$$\leq \frac{\exp(-x)}{(N-1)!} (N-1) x^{N-2} \frac{\Delta}{\exp(\Delta) - 1} = g_{N-1}(x) \frac{\Delta}{\exp(\Delta) - 1},$$

where the second-to-last line follows from convexity of x^{N-1}, so $x^{N-1} - (x - \Delta)^{N-1} \leq (N-1)x^{N-2}\Delta$. Since $\Delta/(\exp(\Delta) - 1) \to 1$ as $\Delta \to 0$, we can take $\widehat{\Delta} \leq 1$ small enough so that for $\Delta < \widehat{\Delta}$, the ratio is less than 2. Also, as long as $\Delta < N - 1$, g_N is increasing for $x \in [0, \Delta]$, so that for x in this range,

$$\frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} \leq \frac{g_N(\Delta)}{\exp(\Delta) - 1}. \tag{39}$$

This expression converges to zero pointwise for $N > 2$ and converges to 1 for $N = 2$. Thus, we can take Δ small so that for $x < \Delta$, the right-hand side of (39) is less than 2.

Integrability follows from the fact that $\overline{\gamma}(x)$ is bounded by $\overline{v}$. *Q.E.D.*

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