

Compound Proportional Auctions*

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Abstract

We consider the design of optimal auctions when buyers may have asymmetric and/or interdependent values. All that is known to the seller is each buyer’s ex ante expected value and an upper bound on the values. We describe a new class of mechanisms which we term *compound proportional auctions*: Each buyer submits a bid, which is a non-negative real number. The auction then clears in a series of rounds. Within each round, a proportional auction (Brooks and Du, 2021b) is run allocate the remaining supply that is left over from previous rounds, among a set of active buyers. At the end of the round, those active buyers with the lowest expected value become inactive. Our main result is that compound proportional auctions maximize the *revenue guarantee*: minimum expected revenue across all information structures and Bayes Nash equilibria.

KEYWORDS: Optimal auctions, interdependent values, robustness, guarantee, potential.

JEL CLASSIFICATION: C72, D44, D82, D83.

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1 Introduction

A classic and fundamental problem in the theory of mechanism design concerns how to maximize revenue from the sale of goods. This question has been intensively studied for nearly fifty years. In certain specialized settings, economic theorists have arrived at compelling answers. If there is a single buyer for a single unit of a good, the optimal mechanism is simply a posted price (Riley and Zeckhauser, 1983). When there are multiple buyers with independent private values, then the class of optimal mechanisms has been concisely described by Myerson (1981). In the further special case where the buyers are symmetric and each buyer’s value distribution is log-concave, we have the celebrated result that both first- and second-price auctions with reserve prices are optimal mechanisms. Beyond those settings, the results are less satisfying. It has long been understood that relaxing independence in private information can lead to cases that are both very easy for the seller (meaning that they can get the same payoff as if the buyers’ information were not private) and moreover, the optimal mechanisms involve the buyers placing implausibly complicated and detailed bets about one another’s private information (Cr  mer and McLean, 1988). And if we move away from private values, then the precise form of interdependence in valuations matters a great deal, and the theory does not deliver clean and general insights about what form of auction is advantageous (Bulow and Klemperer, 1996, 2002; Bergemann et al., 2020).

This state of affairs reflects both the strengths and weaknesses of the Bayesian mechanism design framework: The designer’s understanding of the agents’ private information, expressed as an information structure, is absolutely correct and in complete agreement with the agents’ knowledge. This assumption is what permits the theory to deliver clean and precise conclusion about optimal mechanisms. It also means that those optimal mechanisms utilize all of the designer’s implausibly detailed understanding of the environment. But in reality, the designer may face large uncertainty about the nature of the agents’ private information. And even if the designer could fully articulate all of those possibilities, a mechanism that is equipped for all scenarios would be unwieldy and incomprehensible to the agents who are supposed to interact with it.¹

A recent literature has proposed to sidestep these issues by analyzing mechanisms that attain a favorable performance *guarantee*, that is, a uniform lower bound on performance that hold regardless of the information structure and equilibrium. Even though the guarantee applies to all information structures and equilibria, the guarantee maximizing mechanism is tailored to the worst-case informational environment and equilibrium, which has an especially

¹This perspective on the literature is discussed in greater detail in Brooks and Du (2025).

simple form (Brooks and Du, 2024a, 2025).²

Prior work has applied this methodology to optimal auctions, for the special case in which the buyers have a pure common value for the good being sold (Brooks and Du, 2021b). The main finding is that the revenue guarantee is maximized within a class of mechanisms termed *proportional auctions*: Each buyer submits a “bid,” a non-negative real number. The total amount of the good allocated, and the price per unit, are both determined as a function of the sum of the buyers’ bids. Finally, each buyer’s individual allocation is proportional to their bid. In benchmark cases, the total amount allocated increases linearly until the total supply is reached. As a result, bids can be interpreted as a demand for a quantity of the good, and if the total demand exceeds the supply, then the good is rationed proportionally.

In the present paper, we study an analogous problem, but where we substantially weaken assumptions about the distribution of buyers’ values. In particular, we suppose that the seller knows each buyer’s expected value, which may be different across buyers. There is also an upper bound on values, which is common across buyers. No other assumptions about the distribution of buyers’ values (and in particular the joint distribution) are made. In this setting, a mechanism’s revenue guarantee is its minimum expected revenue across all value distributions satisfying the mean constraints, and across information structures and equilibria. Our main result is that the revenue guarantee is maximized by a novel class of mechanisms, which we call *compound proportional auctions* (CPAs). This class includes proportional auctions as a special case.

A CPA is structured as follows: Each buyer submits a “bid,” which is a non-negative real number. Allocations and payments are then determined in a series of rounds. Within each round, a proportional auction is run among a subset of “active” buyers, for whatever residual supply is left over from the previous rounds. If the residual supply sells out, then there are no more rounds, and the allocation of the residual supply and payments are determined via the rules of the proportional auction. However, if the good does not sell out, then the process continues on to another round, with the active buyers with the lowest expected value becoming inactive. The buyers who are dropped from the active group receive allocations and payments in accordance with the rules of the proportional auction in their last round. But no allocations are made to buyers who remain active; their allocations and payments will be determined at a later round. The process terminates once the good sells out, or until no buyers remain.³ As with proportional auctions, we can interpret the bids as a demand

²A number of papers conduct a related but distinct exercise of holding fixed the information (typically assumed to be private values) and evaluating a mechanism by a worst case over all value distributions in some class, e.g., Che (2020), Suzdaltsev (2020), and He and Li (2022).

³We note that while we have offered a sequential interpretation of the auction, each buyer submits a single bid that is used in all rounds. We discuss possible dynamic interpretations later in the paper, in which

for a quantity of the good, with a weighted-proportional rationing rule when the good sells out.

In the special case where all of the buyers have the same expected value, the guarantee-maximizing CPA has only a single round, which is simply a proportional auction. Importantly, even though all of the buyers have the same *expected* value, their actual values may be distinct and may be correlated in an arbitrary manner. Thus, one contribution of our paper is to provide further foundations for proportional auctions as revenue-guarantee-maximizing mechanisms, beyond the case of pure common values.

Our characterization relies on the first-order methodology for informationally-robust mechanism design introduced in Brooks and Du (2021b, 2024b,a). In particular, the CPA is the auction that maximizes the expected lowest *strategic virtual objective*. In effect, it achieves the optimal balance across actions of revenue and local incentives for the buyers. This methodology is described in more detail in Sections 3 and 4.⁴

The rest of this paper is organized as follows: Section 2 describes our model. Section 3 presents a simple two buyer example that illustrates the logic behind the CPA. Section 4 describes the general version of the CPA, and proves a lower bound on its revenue guarantee. Section 5 proves that this lower bound is tight: There are information structures for which the CPA’s guarantee is equal to maximum revenue across all mechanisms and equilibria.

Not all mechanisms that maximize the guarantee are CPAs. In Section 6, we describe other guarantee maximizing mechanisms, and we discuss why the CPA is relatively simple compared to other known solutions.

Section 7 is a discussion. Omitted proofs are contained in Appendix A.

2 Model

There are buyers $i = 1, \dots, N$ and a seller. Each buyer i has a value $v_i \in [0, \bar{v}]$. Buyer i ’s ex ante expected value is known to be $\hat{v}_i$. Without loss, we order the buyers so that $\hat{v}_1 \geq \hat{v}_2 \geq \dots \geq \hat{v}_N$.

A *mechanism* is a tuple $M = (A, q, t)$ consisting of: a measurable set of actions A_i for each buyer, with $A = \prod_i A_i$; an allocation rule $q : A \rightarrow [0, 1]^N$ with $\sum_i q_i(a) \leq 1$ for all $a \in A$; and a transfer rule $t : A \rightarrow \mathbb{R}^N$. An action a_i is *secure* if $t_i(a_i, a_{-i}) \leq 0$ for all a_{-i} .⁵

the buyers can submit different bids for different rounds. However, any true dynamic implementation would entail some release of information about the progress of the auction, which could be used to support dynamic strategies that would alter the auction’s incentives.

⁴The present paper subsumes earlier work of Brooks and Du (2021a). That paper identified the same optimal guarantee, and showed that the proportional auction was a guarantee maximizer in the symmetric case, but did not identify the CPA as a guarantee-maximizing mechanism.

⁵This definition is equivalent to the one given in Brooks and Du (2024b).

A mechanism is *participation secure* if every buyer i has at least one secure action.

An *information structure* is a tuple $I = (S, \sigma)$ consisting of: a measurable set of signals S_i for each buyer, with $S = \prod_i S_i$; and a joint probability measure $\sigma \in \Delta(S \times [0, \bar{v}]^N)$, with the property that for each i , $\int_{s,v} v_i \sigma(ds, dv) = \widehat{v}_i$. In other words, each buyer i 's expected value under σ is $\widehat{v}_i$. The information structure has *common values* if $v_1 = \dots = v_N$ with probability one.

Given a pair (M, I) of a mechanism and an information structure, a strategy for buyer i is a probability transition kernel $b_i : S_i \rightarrow \Delta(A_i)$. Given a strategy profile $b = (b_1, \dots, b_N)$, we write $b(da|s) = \prod_i b_i(da_i|s_i)$ for the induced kernel from S to $\Delta(A)$. The ex ante expected utility of buyer i is

$$U_i(b; M, I) = \int_{s,v,a} (v_i q_i(a) - t_i(a)) b(da|s) \sigma(ds, dv).$$

Ex ante expected revenue is

$$R(b; M, I) = \int_{s,v,a} \left(\sum_i t_i(a) \right) b(da|s) \sigma(ds, dv).$$

The profile b is a (*Bayes Nash*) *equilibrium* of (M, I) if $U_i(b; M, I) \geq U_i(b'_i, b_{-i}; M, I)$ for every i and strategy b'_i . The set of equilibria is $E(M, I)$.

The *revenue guarantee* of a mechanism M is

$$G(M) = \inf_I \inf_{b \in E(M, I)} R(b; M, I).$$

The *revenue potential* of an information structure I is

$$P(I) = \sup_{M \text{ participation secure}} \sup_{b \in E(M, I)} R(b; M, I).$$

3 Example

Our goal in this paper is to understand which mechanisms provide favorable revenue guarantees. Our main contribution is to propose a new class of mechanisms, which we term *compound proportional auctions*. In this section, we will motivate these mechanisms by studying a simple example with two buyers. This discussion is informal; rigorous proofs of the results we describe will be given later in the paper.

3.1 Common-value auctions

To start, we will suppose that $\bar{v} = 1$ and $\hat{v}_1 = \hat{v}_2 = 1/2$. A variant of our model has been studied by Brooks and Du (2021b), where the further assumption was made that $v_1 = v_2$ with probability one. In other words, the buyers were assumed to have pure common values. A key result of that paper is that for a certain class of differentiable mechanisms, there is a lower bound on the revenue guarantee given by the *expected lowest strategic virtual objective*. In particular, suppose that $A_i = \mathbb{R}_+$ and that the partial derivatives $\partial q_i(a)/\partial a_i$ and $\partial t_i(a)/\partial a_i$ exist everywhere. Then $G(M)$ is at least

$$\inf_{\{\mu \in \Delta([0,1]) \mid \int_v v \mu(dv) = 1/2\}} \int_v \mu(dv) \inf_{a \in \mathbb{R}_+^2} \underbrace{\sum_i \left[t_i(a) + \frac{\partial}{\partial a_i} (v q_i(a) - t_i(a)) \right]}_{\equiv \lambda(v,a)}. \quad (1)$$

It is the term in square brackets that is referred to as the *strategic virtual objective* (SVO), which we denote by $\lambda(v, a)$. The expression (1) is the optimal value of a Lagrangian for the problem of minimizing expected revenue over all information structures and equilibria, but where we have dropped all equilibrium constraints except for those associated with “local” deviations in the standard order structure on $\mathbb{R}_+$, and we have attached a constant multiplier (equal to 1) on those constraints. This lower bound may seem quite arbitrary, but in fact it is known to be tight (Brooks and Du, 2024a,b).

It is important to note that the SVO is linear in v , so that we can replace the outer infimum over μ with an expectation over $v \in \{0, 1\}$, both equally likely, which must weakly decrease the expression (1). Thus, if we define

$$\bar{\lambda}(v) = \inf_a \lambda(v, a),$$

then (1) reduces to

$$\frac{1}{2} \bar{\lambda}(0) + \frac{1}{2} \bar{\lambda}(1).$$

In fact, as Brooks and Du (2021b) show, not only is this a lower bound; there are guarantee-maximizing auctions for which $G(M)$ is equal to (1).⁶

Brooks and Du (2021b) identify a particular class of guarantee maximizers that they term *proportional auctions*: For some scalar $\eta > 0$ and for a differentiable function $T : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

⁶The result that the guarantee is at least the expected lowest strategic virtual objective was generalized to a broad class of mechanism design problems by Brooks and Du (2024b). Moreover, as shown in Brooks and Du (2024a), for any mechanism M , there is another mechanism M' for which (1) is at least $G(M)$. In that sense, the lower bound is always tight.

the mechanism is given by

$$q_i(a) = \min \left\{ \eta a_i, \frac{a_i}{a_1 + a_2} \right\};$$

$$t_i(a) = \left(\frac{a_i}{a_1 + a_2} \right) \cdot T(a_1 + a_2).$$

In other words, as long as $\eta(a_1 + a_2) \leq 1$, each buyer's allocation is ηa_i , and above that threshold, the good is allocated proportionally. The parameter η controls the sensitivity of the allocation to buyers' actions. The buyers together pay an aggregate transfer $T(a_1 + a_2)$ that only depends on the sum of their actions. Each buyer's share of this aggregate transfer is proportional to their action.

To see why this structure is effective for maximizing (1), note that the strategic virtual objective simplifies to

$$\lambda(v, a) = \begin{cases} T(a_1 + a_2) + 2\eta v - \frac{1}{a_1 + a_2}(T(a_1 + a_2)) - T'(a_1 + a_2) & \text{if } \eta(a_1 + a_2) < 1. \\ T(a_1 + a_2) + \frac{1}{a_1 + a_2}(v - T(a_1 + a_2)) - T'(a_1 + a_2) & \text{if } \eta(a_1 + a_2) \geq 1. \end{cases}$$

In other words, $\lambda(v, a)$ only depends on $x = a_1 + a_2$. This limits the scope for information to coordinate equilibrium behavior to depress the SVO, since all that matters is the sum of the buyers' actions. This reflects a common feature of informationally robust mechanisms, that they tend to equalize the SVO across action profiles (see also Brooks and Du, 2023, 2024b).

Within the class of proportional auctions, and given a value for the sensitivity η , the guarantee is maximized by a particular aggregate transfer rule T , chosen so that $\lambda(1, a)$ is constant everywhere, $\lambda(0, a)$ is constant when $\eta \Sigma a \leq 1$, and $\lambda(0, 1/\eta) < \lambda(0, a)$ for $\Sigma a > 1/\eta$.⁷ This further equalizes and “balances” the SVO across action profiles and limits the scope for adversarial information and equilibrium to depress revenue. On the region $\eta x \leq 1$, an aggregate transfer rule with these properties solves the ordinary differential equation

$$T'(x) + \left(\frac{1}{x} - 1 \right) T(x) = -\bar{\lambda}(1) + 2\eta$$

⁷This pattern for SVO minimizers satisfies complementary slackness with a potential-minimizing information structure, in which interim values are in $[0, 1]$ when $a_1 + a_2 \leq 1/\eta$, and the interim value is 1 when $a_1 + a_2 > 1/\eta$. The complementarity is that both $v = 0$ and $v = 1$ are SVO-minimizers when $a_1 + a_2 \leq 1/\eta$, and $v = 1$ is the unique minimizer when $a_1 + a_2 > 1/\eta$. See Brooks and Du (2024b) for a further discussion of complementary slackness in informationally-robust mechanism design.

with the solution (subject to $T(0) = 0$) being

$$T(x) = \frac{1}{x \exp(-x)} \left[\int_{y=0}^x y \exp(-y) [-\bar{\lambda}(1) + 2\eta] dy \right].$$

For $\eta x > 1$, the aggregate transfer solves

$$\begin{aligned} T'(x) + \left(\frac{1}{x} - 1 \right) T(x) &= -\bar{\lambda}(1) + \frac{1}{x} \\ \implies T(x) &= \frac{1}{x \exp(-x)} \left[(1/\eta) \exp(-1/\eta) T(1/\eta) + \int_{y=1/\eta}^x y \exp(-y) \left[-\bar{\lambda}(1) + \frac{1}{y} \right] dy \right]. \end{aligned}$$

Because $\lambda(0, a)$ is minimized when $\sum a < 1$, this construction implies that $\bar{\lambda}(0) = \bar{\lambda}(1) - 2\eta$. All else equal, the seller would prefer a higher $\bar{\lambda}(1)$. However, if $\bar{\lambda}(1)$ is too high, in particular larger than

$$\int_{y=0}^{1/\eta} y \exp(-y) \cdot 2\eta dy + \int_{y=1/\eta}^{\infty} y \exp(-y) \cdot \frac{1}{y} dy, \quad (2)$$

then $\lim_{x \rightarrow \infty} T(x) \rightarrow -\infty$, and the resulting mechanism would have no equilibria: Given a candidate equilibrium in which a_i is a best reply, a buyer could always deviate to a large number $a'_i > a_i$, win the good with weakly higher probability, and guarantee themselves a large payment from the mechanism close to $-T(a'_i)$. Thus, the optimal solution is to set $\bar{\lambda}(1)$ equal to (2).⁸

Let us denote the proportional auction just derived by $\bar{M}(\eta)$, with associated SVO $\lambda(v, a; \eta)$, optimal SVO $\bar{\lambda}(v; \eta)$, and so on. The associated lower bound on the guarantee is

$$G(\bar{M}(\eta)) = (1 - \hat{v})\bar{\lambda}(0; \eta) + \hat{v}\bar{\lambda}(1; \eta) = 2\eta\hat{v} + \bar{\lambda}(0; \eta).$$

This formula reveals a key tradeoff: The more sensitive the allocation is to buyers' actions (meaning the higher is η), the larger is the spread $\bar{\lambda}(1; \eta) - \bar{\lambda}(0; \eta) = 2\eta$ between the SVOs for high and low values. At the same time, $\bar{\lambda}(0; \eta)$ is decreasing in η . This tradeoff is depicted graphically in Figure 1. The left panel shows the curves $\bar{\lambda}(v; \eta)$, as well as the frontier of $(\bar{\lambda}(0; \eta), \bar{\lambda}(1; \eta))$ pairs that can be induced in a proportional auction for some η . The optimal η therefore depends on $\hat{v}$, and the higher is the expected value, the higher is the allocation sensitivity that maximizes the guarantee. We denote the optimal sensitivity when $\hat{v} = 1/2$ by $\bar{\eta} \approx 0.375$, with associated optimal guarantee of approximately 0.253.

⁸Brooks and Du (2021b) address the possibility of equilibrium non-existence, and that the guarantee may be vacuous, by only considering *strong maxmin solutions*. We revisit this approach in Theorem 2 below.

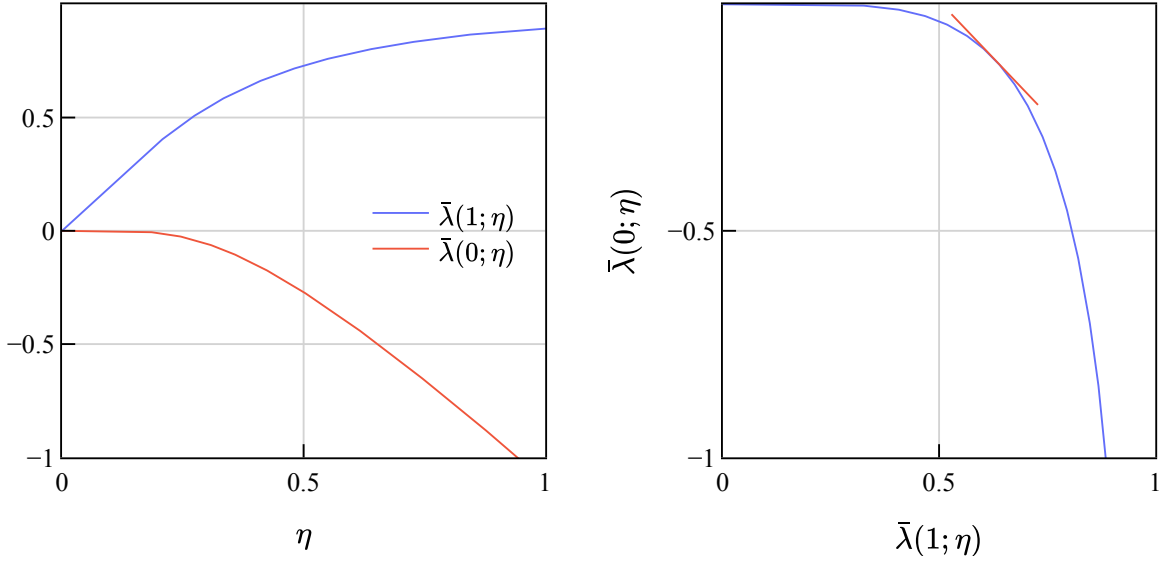


Figure 1: Left: $\bar{\lambda}(0; \eta)$. Right: The locus of $(\bar{\lambda}(0; \eta), \bar{\lambda}(1; \eta))$ pairs.

3.2 Adding interdependence

We now return to the more general version of the problem studied in this paper, where each buyer's expected value is $1/2$, but the correlation structure is unknown. It is still possible that the values are perfectly correlated, as in the pure common value case, but other distributions are also possible, e.g., where the value profiles $(1, 0)$ and $(0, 1)$ are both equally likely. Because there are now more information structures over which the guarantee has to hold, the guarantee must be weakly lower than when values are known to be common.

In this more general model, there is an analogous lower bound to (1), but where the buyers' values may be different:

$$\inf_{\{\mu \in \Delta([0,1]^2) \mid \int_v v_i \mu(dv) = 1/2 \forall i=1,2\}} \int_v \mu(dv) \inf_{a \in \mathbb{R}_+^2} \sum_i \underbrace{\left[t_i(a) + \frac{\partial}{\partial a_i} (v_i q_i(a) - t_i(a)) \right]}_{\equiv \lambda_i(v_i, a)}. \quad (3)$$

Again, the SVO is linear in v , so we may without loss rewrite the problem as minimization over all μ that are supported on $\{0, 1\}^2$, and for which the marginals are uniform. Thus, the expression (3) reduces to

$$\inf_{\{\mu \in \Delta(\{0,1\}^2) \mid \int_v v_i \mu(dv) = 1/2 \forall i=1,2\}} \int_v \mu(dv) \inf_a [\lambda_1(v_1, a) + \lambda_2(v_2, a)].$$

This expression is similar to what we had before, but with two important differences: First,

the values need not be the same, and second, the strategic virtual objective is now additively separable across players and has the functional form $\lambda_1(v_1, a) + \lambda_2(v_2, a)$.

Now, suppose the mechanism is the same proportional auction that maximized the revenue guarantee in the common value setting. We will argue that even though the set of information structures is larger, in fact, the guarantee has not changed. To conclude this, we first calculate the SVO induced by $\overline{M}(\overline{\eta})$:

$$\sum_i \lambda_i(v_i, a) = \begin{cases} T(a_1 + a_2) + \overline{\eta}(v_1 + v_2) - \frac{1}{a_1 + a_2}(T(a_1 + a_2)) - T'(a_1 + a_2) & \text{if } \overline{\eta}(a_1 + a_2) < 1. \\ T(a_1 + a_2) + \frac{a_2 v_1 + a_1 v_2}{(a_1 + a_2)^2} - \frac{1}{a_1 + a_2}(T(a_1 + a_2)) - T'(a_1 + a_2) & \text{if } \overline{\eta}(a_1 + a_2) \geq 1. \end{cases}$$

Clearly, this expression reduces to the common value SVO $\lambda(v, a; \overline{\eta})$ when $v = (0, 0)$ and when $v = (1, 1)$. Moreover, when $\overline{\eta}(a_1 + a_2) \leq 1$, we have that

$$\lambda_1(0, a) + \lambda_2(1, a) = \lambda_1(1, a) + \lambda_2(0, a) = \lambda(0, a; \overline{\eta}) + \overline{\eta} = \frac{\lambda(0, a; \overline{\eta}) + \lambda(1, a; \overline{\eta})}{2}.$$

Thus, for low aggregate actions, the SVO from the value profile $(0, 1)$ is the same as that from an equiprobable mixture over $(0, 0)$ and $(1, 1)$. In addition, on the high region where $\overline{\eta}(a_1 + a_2) > 1$, we have that

$$\lambda_1(0, a) + \lambda_2(1, a) = \frac{a_1}{a_1 + a_2} \lambda(0, a; \overline{\eta}) + \frac{a_2}{a_1 + a_2} \lambda(1, a; \overline{\eta}). \quad (4)$$

Recall that $\lambda(1, a; \overline{\eta})$ is constant in a , and $\lambda(0, a; \overline{\eta})$ is constant in a when $\overline{\eta}(a_1 + a_2) \leq 1$ and is higher when $\overline{\eta}(a_1 + a_2) \geq 1$. Hence, the expression (4) is minimized when $\overline{\eta}(a_1 + a_2) \leq 1$, so that for value profiles $(1, 0)$ and $(0, 1)$, the SVO minimizers are the action profiles in the low region. This, combined with the fact that the marginals are symmetric, implies that the expected lowest SVO is achieved by value distributions concentrated on the value profiles $\{(0, 0), (1, 1)\}$. In other words, common values is the worst case for revenue.

In a way, this is not so surprising: the common value distribution minimizes the expected highest value, subject to the marginal constraints. Thus, common values minimizes the total amount of surplus available to both buyers and seller. Of course, if values are not common, then there could be more private information, and resulting rents for the buyers, which would also cut into revenue. But in the event, it seems that the lower surplus associated with common values is the dominant factor.

The bottom line is that if the seller uses the mechanism $\overline{M}(\overline{\eta})$ in the symmetric interdependent value model, the resulting guarantee is the same as when we had assumed pure common values. Therefore, proportional auctions maximize the interdependent-values revenue guarantee, when buyers are symmetric.

3.3 Asymmetric buyers

We now enrich the problem one step further. Previously, both buyers expected values were equal to $1/2$. But now suppose that buyer 1's expected value is $3/5$, while buyer 2's value is $2/5$. Thus, we have preserved the average expected value, which is still $1/2$.

An important observation is that the seller can continue to use the same proportional auction $\bar{M}(\bar{\eta})$ and achieve at least the same guarantee $G(\bar{M}(\bar{\eta}))$. To see why, suppose there is some information structure I and equilibrium b of $\bar{M}(\bar{\eta})$ for which revenue is $R < G(\bar{M}(\bar{\eta}))$. Then in the symmetric interdependent values setting of the previous section, when both expected values were $1/2$, we could construct an information structure where, with equal probability, it becomes public information that one buyer's expected value is $3/5$ and the other's is $2/5$. Clearly, this satisfies the constraint that each buyer's ex ante expected value is $1/2$. But conditional on knowing which buyer has the higher value, the buyers receive further information and play the equilibrium that induces expected revenue R (exchanging the identities of the buyers in I and b when it is buyer 2 who has the higher expected value), so that ex ante expected revenue is also R .

Our point is that asymmetry in the buyers' expected values can only help the seller, relative to the symmetric model with the same average expected value across buyers. It is analogous to a lower bound on the buyers' information that induces some interim asymmetry. This effectively shrinks the set of information structures over which the guarantee has to hold, and therefore raises the guarantee.

The natural next question is: can the seller exploit this asymmetry to achieve a strictly higher guarantee? The answer turns out to be yes. Recall the fundamental tradeoff across proportional auctions in the common value setting: The seller can adjust the sensitivity of the allocation, and the higher is the sensitivity, the lower is the SVO when $v = 0$ but the higher is the SVO when $v = 1$, so that all else equal, higher sensitivity is desirable when expected values are higher. But now, buyer 1's expected value is relatively high and buyer 2's expected value is relatively low. We might want to modify the auction so that we increase the allocation sensitivity of buyer 1 and decrease that of buyer 2.

Here is a simple way to do so: We start with the proportional auction, as before, but with a lower sensitivity, denoted by $\eta_2 < \bar{\eta}$. We run this auction. If the good sells out (meaning that $\eta_2(a_1 + a_2) \geq 1$), then we implement the outcome of the proportional auction. If the good does not sell out (meaning that $\eta_2(a_1 + a_2) < 1$), buyer 2 still receives their allocation and payment, as per the rules of the proportional auction. Thus, player 2's allocation is $\bar{q}_2(a; \eta_2)$. However, in the case where $\eta_2(a_1 + a_2) < 1$, for buyer 1 we do something different. In particular, we implement a second stage of the auction in which we allocate some or all of

the *residual supply* of the good to buyer 1, according to an additional sensitivity of $\eta_1 - \eta_2$, for some $\eta_1 > \bar{\eta}$ (so that also $\eta_1 > \eta_2$). As we will see, this rule “tops up” buyer 1’s total sensitivity to η_1 . This two-stage rule is an example of what we refer to more generally as a *compound proportional auction*.

In particular, buyer 1’s allocation in the second stage, when this stage occurs, is

$$\tilde{q}_1(a_1; a_1 + a_2) = \min\{(\eta_1 - \eta_2)a_1, 1 - \eta_2(a_1 + a_2)\}$$

and their overall allocation is

$$q_1(a_1, a_2) = \begin{cases} \min\{\eta_1 a_1, 1 - \eta_2 a_2\} & \text{if } \eta_2(a_1 + a_2) < 1 \\ \frac{a_1}{a_1 + a_2} & \text{if } \eta_2(a_1 + a_2) \geq 1. \end{cases}$$

Thus, on the region where $\eta_1 a_1 + \eta_2 a_2 < 1$, buyer 1’s sensitivity is indeed η_1 .

We modify both buyers’ payments as well, to preserve the “balance” in the SVO, meaning the equalization of the SVO across action profiles. This is done while preserving the structure that payments per unit only depend on the aggregate action in a given round. In particular, we look for a pair of aggregate transfer rules for the two stages of this auction that, together, equalize the total SVO $\lambda_1(1, a) + \lambda_2(1, a) =: \Lambda(1)$ everywhere. The component of this value that is derived from the second stage is a *continuation strategic virtual objective* that depends on the allocations in the first stage, which we denote by $\tilde{\lambda}(1; a_1 + a_2)$. Because the total SVO, $\Lambda(1)$, must be constant at all action profiles, the proportional first stage must have an SVO of $\Lambda(1) - \tilde{\lambda}(1; a_1 + a_2)$.

Given this structure on incentives, we can solve for the corresponding aggregate transfer rules in the same manner as before to get:

$$T(x) = \begin{cases} \frac{1}{x \exp(-x)} \int_0^x y \exp(-y) \left[2\eta_2 - (\Lambda(1) - \tilde{\lambda}(1; y)) \right] dy & \text{if } \eta_2 x \leq 1; \\ \frac{1}{y \exp(-y)} \left[(1/\eta_2) \exp(-1/\eta_2) T(1/\eta_2) \right. \\ \quad \left. + \int_{1/\eta_2}^x y \exp(-y) \left[\frac{1}{y} - (\Lambda(1) - \tilde{\lambda}(1; y)) \right] dy \right] & \text{if } \eta_2 x > 1, \end{cases}$$

$$\tilde{T}(a_1; a_2) = \frac{1}{\exp(-a_1)} \int_0^{\min(a_1, \frac{1-\eta_2 a_2}{\eta_1})} \exp(-z) \left[(\eta_1 - \eta_2) - \tilde{\lambda}(1; z + a_2) \right] dz.$$

Also as before, the guarantee is increasing in $\Lambda(1)$ and $\tilde{\lambda}(1; a_1 + a_2)$, but if these constants are too large, then transfers will diverge to $-\infty$. This constraint fully determines the remaining

constants:

$$\tilde{\lambda}(1; a_1 + a_2) = \begin{cases} (\eta_1 - \eta_2) \left(\frac{1 - \exp\left(\frac{1 - \eta_2(a_1 + a_2)}{\eta_1}\right)}{\eta_1} \right) & \text{if } \eta_2(a_1 + a_2) < 1 \\ 0 & \text{if } \eta_2(a_1 + a_2) \geq 1 \end{cases}$$

$$\Lambda(1) = \bar{\lambda}(1; \eta_2) + \int_0^{\frac{1}{\eta_2}} y \exp(-y) [\tilde{\lambda}(1; y)] dy$$

Note that the optimal constants Λ and $\tilde{\lambda}$ implicitly depend on the sensitivities (η_1, η_2) .

With transfer rules pinned down, we can study how this compound proportional auction improves upon the one-round proportional auction. We denote the compound proportional auction with these transfer rules as $\bar{M}(\eta_1, \eta_2)$.⁹ We calculate the SVO induced by $\bar{M}(\eta_1, \eta_2)$ in a similar fashion as before:

$$\sum_i \lambda_i(v_i, a) = \begin{cases} T(a_1 + a_2) + \eta_2(v_1 + v_2) - \frac{1}{a_1 + a_2}(T(a_1 + a_2)) - T'(a_1 + a_2) \\ \quad + \tilde{T}(a_1; a_2) + (\eta_1 - \eta_2)v_1 - \tilde{T}'(a_1; a_2) & \text{if } \eta_1 a_1 + \eta_2 a_2 < 1. \\ T(a_1 + a_2) + \eta_2(v_1 + v_2) - \frac{1}{a_1 + a_2}(T(a_1 + a_2)) - T'(a_1 + a_2) \\ \quad + \tilde{T}(a_1; a_2) - \tilde{T}'(a_1; a_2) & \text{if } \eta_1 a_1 + \eta_2 a_2 \geq 1, \\ & \eta_2(a_1 + a_2) < 1. \\ T(a_1 + a_2) + \frac{a_2 v_1 + a_1 v_2}{(a_1 + a_2)^2} - \frac{1}{a_1 + a_2}(T(a_1 + a_2)) - T'(a_1 + a_2) & \text{if } \eta_2(a_1 + a_2) \geq 1. \end{cases}$$

Because $\sum_i \lambda_i(1, a)$ is equalized at all action profiles by the constructed transfer rules, we can use the equations above to determine when $\sum_i \lambda_i(v_i, a)$ is minimized for other value profiles. As before, when $v = (0, 0)$, the SVO is minimized when the total supply of the good is not fully allocated, or when $\eta_1 a_1 + \eta_2 a_2 < 1$. When $v = (1, 0)$, the SVO is minimized when the total supply of the good is not fully allocated in the first stage, or when $\eta_2(a_1 + a_2) < 1$. Together, these equations imply that

$$\bar{\lambda}_1(0) = \bar{\lambda}_1(1) - \eta_1 \text{ and } \bar{\lambda}_2(0) = \bar{\lambda}_2(1) - \eta_2.$$

As before, the allocation sensitivities relate to the spread of the optimal SVO's. Using the fact that $\sum_i \bar{\lambda}_i(1) = \Lambda(1)$, we can write the lower bound on the guarantee of the mechanism

⁹Note that we are overloading our notation from the previous section.

is

$$\begin{aligned} G(\overline{M}(\eta_1, \eta_2)) &= \sum_{i \in \{1,2\}} \widehat{v}_i \cdot \bar{\lambda}_i(1) + (1 - \widehat{v}_i) \cdot \bar{\lambda}_i(0) \\ &= \Lambda(1) - (1 - \widehat{v}_1)\eta_1 - (1 - \widehat{v}_2)\eta_2. \end{aligned}$$

To focus on the benefits of exploiting the asymmetry between the buyers, we simply vary the difference between allocation sensitivities while keeping the average allocation sensitivity the same as that from the optimal proportional auction when $\widehat{v}_1 = \widehat{v}_2 = 1/2$. There will always be a strict increase in the lower bound of the guarantee. For an example of this, consider setting $\eta_1 = \bar{\eta} + \epsilon$ and $\eta_2 = \bar{\eta} - \epsilon$. For the case of $\widehat{v}_1 = 3/5$ and $\widehat{v}_2 = 2/5$, we have plotted in Figure 2 how the guarantee varies with ϵ . The optimal value of ϵ is approximately 0.170, which gives a guarantee of 0.270, strictly greater than the value of 0.253 when $\epsilon = 0$.

Setting the sensitivities equidistant around $\bar{\eta}$ is not optimal; it is a simple perturbation that illustrates how assortatively matching higher allocation sensitivities with higher expected values can raise the guarantee. We return to this example and describe the globally optimal sensitivities in Section 5.

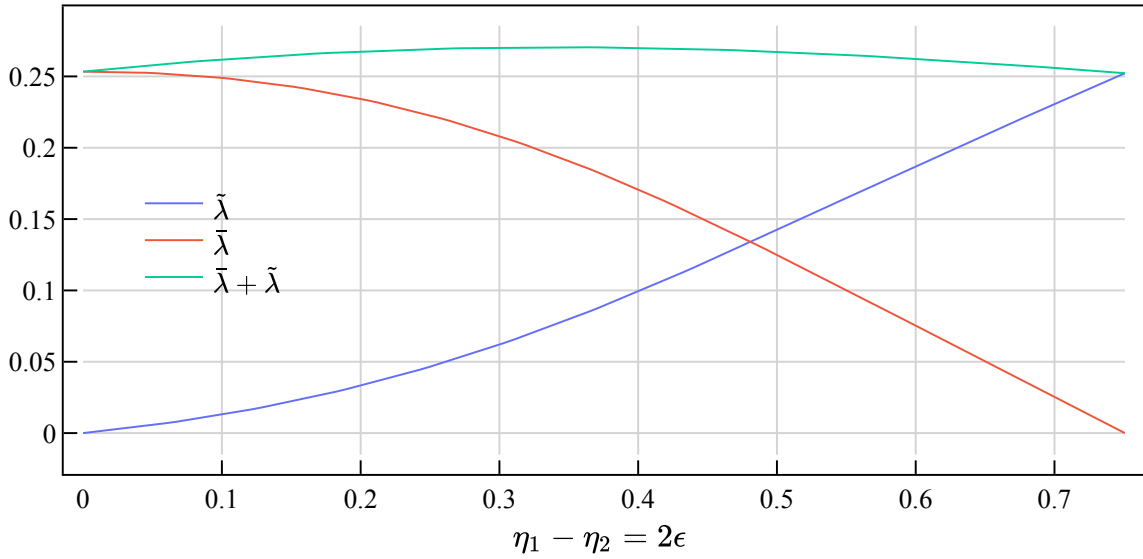


Figure 2: A lower bound on the revenue guarantee of the two-round CPA as a function of $\eta_1 - \eta_2$, with $\widehat{v}_1 = 3/5$ and $\widehat{v}_2 = 2/5$. In red is the contribution to the guarantee from the first round, and in blue is the contribution from the second round, with the total guarantee in cyan.

The takeaways from the example are that the proportional auction remains optimal even if the buyers have symmetric interdependent values. If the buyers are asymmetric, then it is

possible to achieve a higher revenue guarantee by adding an extra “round” to the auction. In our main results, we generalize this idea to construct compound proportional auctions for many buyers. Our main result is that these mechanisms, with appropriately chosen sensitivities, maximize the revenue guarantee.

4 Compound Proportional Auctions

4.1 Definition and Interpretation

We now formally define and characterize the CPA.

Definition 1. *The compound proportional auction (CPA) with allocation sensitivity $\eta \in \mathbb{R}_+^N$ (with $\eta_1 \geq \dots \geq \eta_N \geq \eta_{N+1} \equiv 0$) and aggregate transfer rules $T^i : \mathbb{R}^2 \rightarrow \mathbb{R}$ for $i = 1, \dots, N$ is defined as follows: At the action profile a , let*

$$C^n(a) \equiv \sum_{i=n+1}^N \eta_i a_i;$$

$$S^n(a) \equiv 1 - C^n(a) - \sum_{i=1}^n \eta_{n+1} a_i.$$

The allocation and transfer rules of the CPA are:

$$q_i(a) = \sum_{n=1}^N q_i^n(a), \quad t_i(a) = \sum_{n=1}^N t_i^n(a),$$

where for all $n \leq N$,

$$q_i^n(a) \equiv \mathbb{I}(i \leq n, S^n(a) \geq 0) \cdot \begin{cases} (\eta_n - \eta_{n+1})a_i & \text{if } \sum_{j=1}^n (\eta_n - \eta_{n+1})a_j < S^n(a); \\ S^n(a) \cdot \frac{a_i}{\sum_{j=1}^n a_j} & \text{if } \sum_{j=1}^n (\eta_n - \eta_{n+1})a_j \geq S^n(a), \end{cases}$$

$$t_i^n(a) \equiv \mathbb{I}(i \leq n, S^n(a) \geq 0) \cdot \frac{a_i}{\sum_{j=1}^n a_j} T^n \left(\sum_{j=1}^n a_j, S^n(a) \right).$$

The CPA can be understood as a series of proportional auctions conducted over N rounds. After each round, the weakest buyer (meaning the one with the lowest expected value and also the lowest sensitivity) drops out, while the rest of the buyers proceed to the following round. Rounds are indexed by the number of buyers participating. Thus, the auction starts with round N and ends with round 1.

We call buyer i *active* in round n if he participates in round n ($i \leq n$) and *inactive* otherwise ($i > n$). At round n , $C^n(a)$ is the “committed” supply allocated to the inactive buyers, $\sum_{i=1}^n \eta_{n+1} a_i$ is the supply allocated to the active buyers in the previous rounds, and $S^n(a)$ is the “residual” supply which is still to be allocated to the remaining active buyers

At round n , we attempt to allocate $(\eta_n - \eta_{n+1})a_i$ of the residual supply to each active buyer in $\{1, 2, \dots, n\}$. If we cannot do this without running out of the good (meaning that $(\eta_n - \eta_{n+1}) \sum_{j=1}^n a_j > S^n(a)$), we allocate the residual supply $S^n(a)$ proportionally between the n active buyers and the auction ends. But if the good does not run out, and $n > 1$, we proceed to round $n - 1$ with one fewer active buyers, and new residual supply $S^{n-1}(a) = S^n(a) - (\eta_n - \eta_{n+1}) \sum_{j=1}^n a_j$. After round 1, the auction ends, even if the good is not fully allocated. Regardless of the round, the active buyers make proportional payments $t_i^n(a)$ for the good they are allocated in round n .

In a CPA, each buyer’s action can be viewed as a (rescaled) demand for a quantity of the good, where buyer i ’s action a_i corresponds to a total demand of $\eta_i a_i$. But within a given round, the good is allocated among the buyers using the proportional auction as if the active buyers have submitted demands $(\eta_n - \eta_{n+1})a_i$. For example, in the first round N , a maximum of $\eta_N a_i$ is allocated to each buyer. For buyers with $i < N$, this may not satisfy all of their demand, and so they are permitted to participate in the next round. To account for the fact that buyer i ’s demand has been partially satisfied, they enter the next round with a residual demand of $(\eta_i - \eta_N)a_i$. The auction continues in this fashion, treating actions symmetrically in each round and allowing buyers who still have residual demand to progress to further rounds.¹⁰

Note that the class of CPAs includes proportional auctions as a special case: If all buyers have the same allocation sensitivity, then after round N , no buyers will have any residual demand so the subsequent rounds will not allocate any proportion of the good. On the other hand, if $\eta_1 = \eta_2 = \dots = \eta_n > \eta_{n+1} = \eta_{n+2} = \dots = \eta_N$, the CPA has only two rounds: round N where all buyers participate, and round n where only buyers $1, \dots, n$ participate. It is also possible for some buyers to have a sensitivity of 0, in which case the first rounds of the auction are degenerate, no good is allocated, and no payments are made. This is equivalent to excluding the zero-sensitivity buyers from the auction altogether.

We emphasize that while the auction clears according to a sequential algorithm, the mechanism itself is static: The buyers simultaneously submit bids, and the same bids are used in all rounds. Any true sequentiality, in which the buyers can submit different bids in

¹⁰An equivalent implementation of the CPA would allow each buyer to directly submit a demand $d_i \in \mathbb{R}_+$ that is in units of the good, and the rules of the auction would be the same as if buyer i played the action $a_i = d_i/\eta_i$. Equivalently, buyer i ’s demand is split across rounds, with a fraction $(\eta_n - \eta_{n+1})/\eta_i$ of their demand being used in round n .

different rounds or get feedback about the progress of the algorithm, would give the buyers a richer space of strategies that they could exploit to manipulate the outcome and could change the auction's guarantee.

4.2 CPA with Balanced Transfers

Given an allocation sensitivity vector $\eta \in \mathbb{R}_+^N$ where $\eta_1 \geq \dots \geq \eta_N \geq \eta_{N+1} \equiv 0$, we next define a particular set of aggregate transfer rules that equalize the SVO across all action profiles when $v_i = \bar{v}$, and across all action profiles with $\eta \cdot a \leq 1$ when $v_i = 0$. We refer to a CPA for which the SVO has this property as being *balanced*.

To derive the balanced CPA, first note that the divergence of the allocation q^n in round n , $\nabla \cdot q^n(a) \equiv \sum_{i=1}^N \frac{\partial q_i^n}{\partial a_i}(a)$, only depends on the total action of the active buyers $\sum_{i=1}^n a_i$ and the supply committed to the inactive buyers $C^n(a)$:

$$\nabla \cdot q^n \left(\sum_{i=1}^n a_i, C^n(a) \right) = \begin{cases} 0 & \text{if } S^n(a) \leq 0; \\ n(\eta_n - \eta_{n+1}) & \text{if } (\eta_n - \eta_{n+1}) \sum_{i=1}^n a_i < S^n(a); \\ \frac{n-1}{\sum_{i=1}^n a_i} S^n(a) - \eta_{n+1} & \text{if } (\eta_n - \eta_{n+1}) \sum_{i=1}^n a_i \geq S^n(a) > 0, \end{cases}$$

where $S^n(a) = 1 - C^n(a) - \eta_{n+1} \sum_{i=1}^n a_i$ is the residual supply at round n . Our convention is that $\nabla \cdot q^n(\tau, C) = 0$ if $C \neq C^n(a)$ for every a such that $\sum_{i=1}^n a_i = \tau$.

We define the expected continuation guarantee $\Lambda^n : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\Lambda^n(C) = \bar{v} \int_{a \in \mathbb{R}_+^n} \sum_{r=1}^n \nabla \cdot q^r \left(\sum_{i=1}^r a_i, C + \sum_{i=r+1}^n \eta_i a_i \right) \exp \left(- \sum_{i=1}^n a_i \right) da. \quad (5)$$

Also define $\lambda^n : (-\infty, 1] \rightarrow \mathbb{R}$ by

$$\lambda^n(S) = \begin{cases} \Lambda^N(0) & n = N; \\ (\eta_{n+1} \cdot \frac{d}{dS} + \mathbf{1})^n (\Lambda^n(1 - S)) & 0 < n < N; \\ 0 & n = 0, \end{cases}$$

where $\frac{d}{dS}$ is the derivative operator and $\mathbf{1}$ is the identity operator. Next, we define the round-based *aggregate excess growth* $\Xi^n : \mathbb{R}_+ \times (-\infty, 1] \rightarrow \mathbb{R}$ as

$$\Xi^n(\tau, S) = \begin{cases} \bar{v} \nabla \cdot q^n(\tau, 1 - S - \eta_{n+1}\tau) - (\lambda^n(S) - \lambda^{n-1}(S - (\eta_n - \eta_{n+1})\tau)) & \text{if } S \geq 0; \\ 0 & \text{if } S < 0. \end{cases}$$

With these expressions, we can write the balanced transfer rule for round n as:

$$T^n(x_n, S) = \frac{1}{g_n(x_n)} \int_0^{x_n} \Xi^n(\tau, S + \eta_{n+1}(x_n - \tau)) g_n(\tau) d\tau, \quad (6)$$

where

$$g_n(\tau) = \frac{x^{n-1} e^{-\tau}}{n!}$$

is the probability density function for the Erlang distribution (i.e., the density of the sum of n i.i.d. unit exponential random variables).

By construction, $T^n(\sum_{i=1}^n a_i, S^n(a))$ is a bounded and continuous function of a , and $T^n(\sum_{i=1}^n a_i, S^n(a)) = 0$ whenever $S^n(a) \leq 0$, i.e., whenever round n does not occur¹¹. Moreover, as in a standard proportional auction, T^n is designed for the transfer to satisfy

$$\sum_{i=1}^n \left(\frac{\partial t_i^n}{\partial a_i}(a) - t_i^n(a) \right) = \Xi^n \left(\sum_{i=1}^n a_i, S^n(a) \right).$$

Thus, the SVO of a balanced CPA is

$$\begin{aligned} & \sum_i \lambda_i(v_i, a) \\ &= \sum_n \sum_i \left(v_i \frac{\partial q_i^n}{\partial a_i}(a) + t_i^n(a) - \frac{\partial t_i^n}{\partial a_i}(a) \right) \\ &= \sum_n \left(\sum_i v_i \frac{\partial q_i^n}{\partial a_i}(a) - \bar{v} \nabla \cdot q^n(x_n, 1 - S^n(a) - \eta_{n+1}x_n) + \lambda^n(S^n(a)) - \lambda^{n-1}(S^{n-1}(a)) \right) \\ &= \sum_i (v_i - \bar{v}) \frac{\partial q_i}{\partial a_i}(a) + \Lambda^N(0), \end{aligned} \quad (7)$$

where $x_n = \sum_{i=1}^n a_i$.

It is easy to check that

Lemma 1. *For the compound proportional auction, we have*

$$\frac{\partial q_i}{\partial a_i}(a) \begin{cases} = \eta_i & \eta \cdot a < 1, \\ \leq \eta_i & \eta \cdot a \geq 1. \end{cases}$$

Lemma 1 implies that the SVO is balanced as discussed in the beginning of the subsection,

¹¹See Lemmas 3,4, and 5 in the Appendix for details

and the lowest SVO is:

$$\min_{a \in \mathbb{R}_+^N} \sum_i \lambda_i(v_i, a) = \Lambda^N(0) - \sum_i \eta_i(\bar{v} - v_i).$$

Let us write $\Lambda^N(0; \eta) \equiv \Lambda^N(0)$ to emphasize its dependence on the parameter η . As in Section 3, we denote the balanced CPA with aggregate transfer defined by (6) as $\bar{M}(\eta)$.¹² Using the fact that the revenue guarantee is at least the expected lowest SVO (formally established as Lemma 6 in the Appendix), we then have:

Theorem 1. *For $\eta \in \mathbb{R}_+^N$, with $\eta_1 \geq \eta_2 \geq \dots \geq \eta_N$, the revenue guarantee $G(\bar{M}(\eta))$ of the balanced CPA $\bar{M}(\eta)$ is at least*

$$\Lambda^N(0; \eta) - \sum_{i=1}^N \eta_i(\bar{v} - \hat{v}_i). \quad (8)$$

Theorem 1 allows us to calculate a lower bound on the guarantee for $\bar{M}(\eta)$.¹³ However, it does not tell us the optimal choice of the sensitivities η , which depends on the expected values. In the next section, we will determine the optimal η . It should be noted that while we have established a lower bound on the guarantee for a balanced CPA $\bar{M}(\eta)$, we have not shown this bound is tight nor that this guarantee is optimal. Both will be shown in the next section.

5 Optimality of the CPA

We now show that the revenue guarantee of the CPA in Theorem 1 is unimprovable, by identifying an information structure $\bar{I}$ and sensitivities $\bar{\eta}$ for which the potential is equal to (8). In this information structure, the signal space of each buyer i is $S_i = \mathbb{R}_+$. In addition, the marginal distribution of the signals is an independent product of standard exponential distributions (i.e., the probability density function for a signal profile $s \in \mathbb{R}_+^N$ is $\exp(-\Sigma s)$). The interim expected value of buyer i given s is also parametrized by a vector $\eta \in \mathbb{R}_+^N$, and

¹²Whereas before η was a scalar, now it is an N -vector of sensitivities. In the case where all the entries of η are the same, the definition reduces to the proportional auction with sensitivity equal to that of common value.

¹³Theorem 1 does not use the assumption of $\hat{v}_1 \geq \hat{v}_2 \geq \dots \geq \hat{v}_N$ and works for any expected values $\hat{v}_1, \hat{v}_2, \dots, \hat{v}_N$.

is of the form

$$w_i(s; \eta) = \begin{cases} \bar{v} \min\{(\exp(\eta \cdot s - 1))^{1/\eta_i}, 1\} & \text{if } \eta_i > 0; \\ 0 & \text{if } \eta_i = 0 \text{ and } \eta \cdot s < 1; \\ \frac{\hat{v}_i}{\int_{\{s \in \mathbb{R}_+^N | \eta \cdot s \geq 1\}} \exp(-\Sigma s) ds} & \text{if } \eta_i = 0 \text{ and } \eta \cdot s \geq 1. \end{cases} \quad (9)$$

The parameters $\eta \in \mathbb{R}_+^N$ are chosen so that each interim expected value has the correct ex-ante expectation:

$$\int_{\mathbb{R}_+^N} w_i(s; \eta) \exp(-\Sigma s) ds = \hat{v}_i. \quad (10)$$

We can equivalently interpret $w_i(s; \eta)/\bar{v}$ as the probability that $v_i = \bar{v}$ conditional on s , with the complementary probability on $v_i = 0$.

The following Lemma asserts that such sensitivities exist:

Lemma 2. *For any $\hat{v} \in [0, 1]^N$, there exists an $\eta \in \mathbb{R}_+^N$ such that the mean constraints (10) are satisfied for every i . If $\hat{v}_1 = \hat{v}_2 = \dots = \hat{v}_N$, then we can take $\eta_1 = \eta_2 = \dots = \eta_N$.*

We let $\bar{\eta}$ denote a solution to the system (10), which we hold fixed throughout the rest of the analysis, and we denote by $\bar{I}$ the information structure with iid standard exponential signals and interim value functions $\bar{w}_i(s; \bar{\eta})$. Clearly, because of the assumption $\hat{v}_1 \geq \hat{v}_2 \geq \dots \geq \hat{v}_N$, we have $\bar{\eta}_1 \geq \bar{\eta}_2 \geq \dots \geq \bar{\eta}_N$. Moreover, in the symmetric case where all buyers have the same expected value ex ante, the information structure $\bar{I}$ features pure common values. Also note that some of the sensitivities may be zero, in which case those buyers are excluded from the auction.

We now proceed to characterize the potential of $\bar{I}$.

Proposition 1. *The revenue potential of the information structure $\bar{I}$ is at most*

$$\bar{R} \equiv \bar{v} \int_{\{s \in \mathbb{R}_+^N | \bar{\eta} \cdot s \geq 1\}} \exp(-\Sigma s) ds. \quad (11)$$

The proof of Proposition 1 in the Appendix proves the even stronger result that $\bar{R}$ is an upper bound on revenue across all incentive compatible and interim individually rational mechanisms. Indeed, $\bar{R}$ is the expected highest virtual value of the buyers. Following Myerson (1981) and Bulow and Klemperer (1996), the virtual value of buyer i is

$$\bar{w}_i(s) - \frac{1 - F_i(s_i)}{f_i(s_i)} \frac{\partial \bar{w}_i}{\partial s_i}(s)$$

where F_i and f_i are respectively the marginal cumulative distribution and density of buyer i 's signal. Plugging in the functional forms for $\bar{w}_i$ in (9), and $F_i(s_i) = 1 - \exp(-s_i)$ and $f_i(s_i) = \exp(-s_i)$ for the exponential distribution, we conclude that this expression is exactly zero when $\bar{\eta} \cdot s < 1$, and is exactly $\bar{v}$ if $\bar{\eta} \cdot s \geq 1$. Thus, the expected highest virtual value is simply $\bar{v}$ times the probability that $\bar{\eta} \cdot s \geq 1$.

Clearly, the potential of $\bar{I}$ (or of any information structure) is an upper bound on the guarantee of any mechanism M for which an equilibrium exists of the game $(M, \bar{I})$. Remarkably, the balanced CPA $\bar{M}(\bar{\eta})$ attains this upper bound:

Proposition 2. *The guarantee of $\bar{M}(\bar{\eta})$ is at least*

$$\Lambda^N(0; \bar{\eta}) - \sum_{i=1}^N \bar{\eta}_i \cdot (\bar{v} - \hat{v}_i) = \bar{v} \int_{\{a \in \mathbb{R}_+^N | \bar{\eta} \cdot a \geq 1\}} \exp(-\Sigma a) da = \bar{R}.$$

From Propositions 1 and 2 we can nearly conclude that the revenue guarantee of the CPA $\bar{M}(\bar{\eta})$ is unimprovable, and $\bar{I}$ is a worst-case information structure for $\bar{M}(\bar{\eta})$. However, for both of these conclusions, we would have to know that an equilibrium exists at $(\bar{M}(\bar{\eta}), \bar{I})$, so that the guarantee of $\bar{M}(\bar{\eta})$ is at most $\bar{R}$. This is what we now establish.

Following Brooks and Du (2021b), a *strong maxmin solution* is a tuple (M, I, b) , consisting of a mechanism M , and information structure I , and a strategy profile b in (M, I) , such that

- (i) b is an equilibrium of (M, I) ;
- (ii) The revenue guarantee of M is at least $R(M, I, b)$;
- (iii) The revenue potential of I is at most $R(M, I, b)$.

Clearly, at a strong maxmin solution (M, I, b) , the revenue guarantee of M and the revenue potential of I are both equal to $R(M, I, b)$, and this is true for any equilibrium b of (M, I) . Moreover, M maximizes the guarantee among all mechanisms that have an equilibrium in I , and I minimizes the potential among all information structures with an equilibrium for M .

Theorem 2. *Let $\bar{I}$ be the information structure from Proposition 1, $\bar{M}(\bar{\eta})$ the compound proportional auction from Proposition 2, and $\bar{b}$ the profile of truthful strategies in $(\bar{M}, \bar{I})$ for which $\bar{b}_i(\{s_i\} | s_i) = 1$ for all i and s_i . Then $(\bar{M}(\bar{\eta}), \bar{I}, \bar{b})$ is a strong maxmin solution. The revenue guarantee/potential of this solution is (11).*

Thus, the strong maxmin solution $(\bar{M}(\bar{\eta}), \bar{I}, \bar{b})$ exhibits a kind of “double revelation principle”: $\bar{M}(\bar{\eta})$ is a revenue-maximizing, direct revelation mechanism on $\bar{I}$ (subject to incentive compatibility and individual rationality constraints), while $\bar{I}$ is a revenue-minimizing Bayes correlated equilibrium on $\bar{M}(\bar{\eta})$ (Bergemann and Morris, 2016).

Finally, we have the following straightforward corollary of Theorem 2, that in the symmetric case, the proportional auction maximizes the revenue guarantee:

Corollary 1. *If the buyers are symmetric, meaning $\hat{v}_1 = \hat{v}_2 = \dots = \hat{v}_N$, then in the strong maxmin solution described in Theorem 2, the buyers have common values and the mechanism is a proportional auction.*

Let us now return to the two buyer example where $\hat{v}_1 + \hat{v}_2 = 1$ and $\bar{v} = 1$. When $\hat{v}_1 = 3/5$ and $\hat{v}_2 = 2/5$, we showed in Section 3 that it was possible to improve on the proportional auction with a CPA in which the buyers' sensitivities were perturbed by equal amounts. This achieved a guarantee of 0.270, with sensitivities of $\eta_1 = 0.546$ and $\eta_2 = 0.205$. The globally optimal sensitivities are $\bar{\eta}_1 = 0.572$ and $\bar{\eta}_2 = 0.211$, which gives an optimal guarantee of 0.271.

In the top panel of Figure 3, we have plotted the optimal sensitivities $\bar{\eta}$ as a function of the difference in values $\hat{v}_1 - \hat{v}_2$. As we can see, $\bar{\eta}_1$ is increasing and $\bar{\eta}_2$ is decreasing. Once the difference in values exceeds a threshold that is approximately 0.36, $\bar{\eta}_2$ is equal to zero, meaning that buyer 2 is excluded from the auction and the problem collapses to one with a single buyer. Once this happens, the sensitivity of buyer 1 continues to rise, and in fact diverges to infinity as $\hat{v}_1$ approaches 1. In the bottom panel of Figure 3, we have plotted the corresponding optimal guarantee, which approaches 1 as $\hat{v}_1 - \hat{v}_2$ approaches 1, since in this case $\hat{v}_1 \approx \bar{v} = 1$, and there is no uncertainty about buyer 1's value.

6 Other Guarantee-Maximizing Mechanisms

In this section, we describe a more general class of guarantee-maximizing mechanisms, which includes the CPA. For a $v_i \in [0, \bar{v}]$, let us define

$$\bar{\lambda}_i(v_i) \equiv \frac{\bar{R}}{N} + (v_i - \hat{v}_i)\bar{\eta}_i. \quad (12)$$

Theorem 3. *Suppose that $M = (\mathbb{R}_+^N, q, t)$ satisfies the following conditions:*

1. $q_i(0, a_{-i}) = 0$ for all i and a_{-i} , $\sum_i q_i(a) = 1$ if $\bar{\eta} \cdot a \geq 1$, $q_i(a)$ is continuously right-differentiable in a_i , $\frac{\partial q_i}{\partial a_i}(a) = \bar{\eta}_i$ if $\bar{\eta} \cdot a < 1$, and $0 \leq \frac{\partial q}{\partial a_i}(a) \leq \bar{\eta}_i$ if $\bar{\eta} \cdot a \geq 1$.
2. $t_i(a)$ is right-differentiable in a_i , $t_i(a)$ and $\frac{\partial t_i}{\partial a_i}(a)$ are bounded, $t_i(0, a_{-i}) = 0$, and

$$\nabla \cdot t(a) - \sum_i t_i(a) = \nabla \cdot q(a) - \sum_i \bar{\lambda}_i(\bar{v}) \text{ for all } a \in \mathbb{R}_+^N. \quad (13)$$

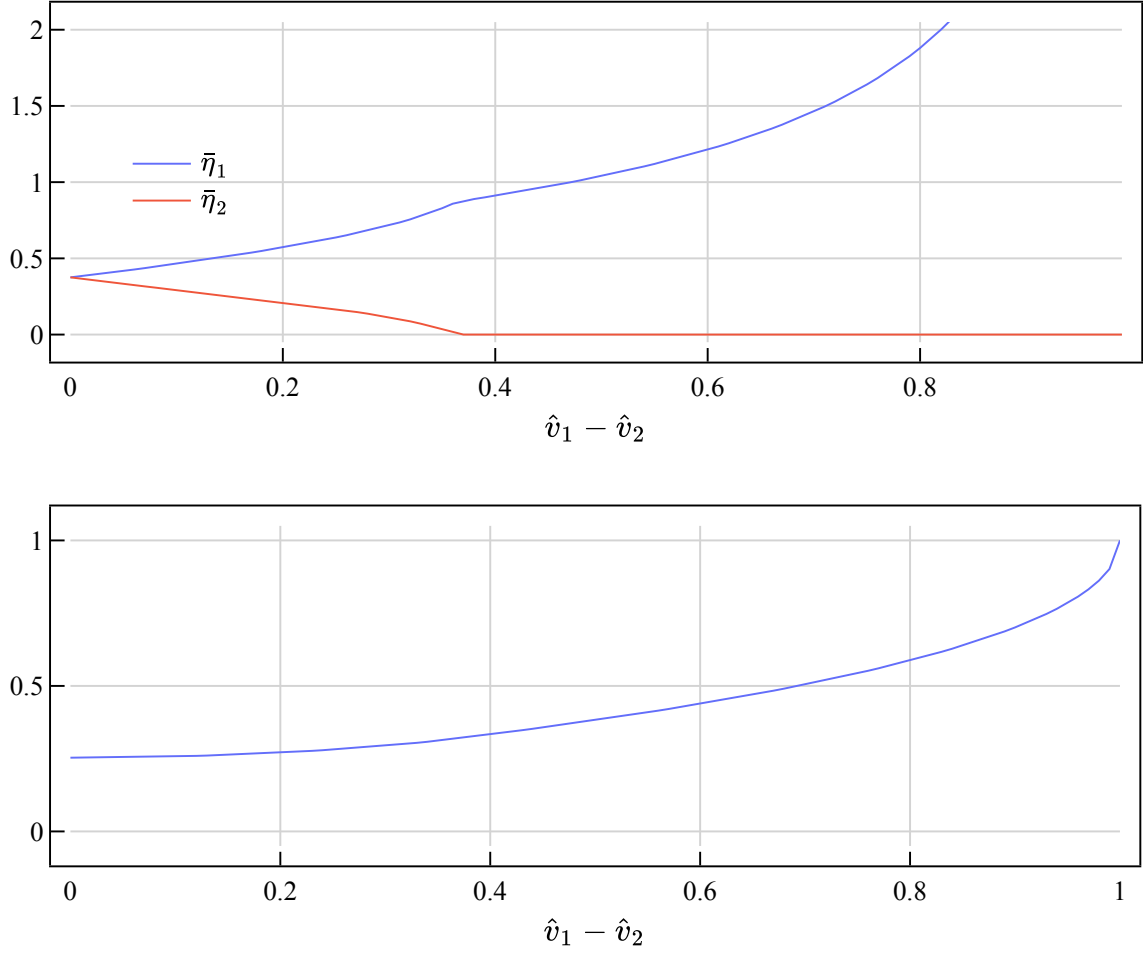


Figure 3: Top: Optimal sensitivities $\bar{\eta}$ as a function of $\hat{v}_1 - \hat{v}_2$, with the restriction that $\hat{v}_1 + \hat{v}_2 = 1$. Bottom: Optimal guarantee.

Let $\bar{I}$ be the information structure from Proposition 1 and $\bar{b}$ the truthful strategy profile. Then $(M, \bar{I}, \bar{b})$ is a strong maximin solution, and the revenue guarantee/potential of this solution is (11).

Moreover, for any allocation rule q satisfying Condition 1, there exists a transfer rule t satisfying Condition 2.

A leading example of a mechanism satisfying the hypotheses of Theorem 3 is the balanced CPA defined by Equation (6): Condition 1 clearly holds by Lemma 1 and the fact that if $\bar{\eta} \cdot a \geq 1$, then the CPA must terminate in a round where all residual supply is exhausted. Condition 2 follows from Equation (7) and Proposition 2. Thus, Theorem 2 follows from Theorem 3.

There are other allocation rules that satisfy Condition 1 of Theorem 3 and hence are part

of a guarantee-maximizing mechanism. One example is

$$q_i(a) = \frac{\bar{\eta}_i a_i}{\max\{1, \bar{\eta} \cdot a\}},$$

which reduces to the proportional allocation rule when $\bar{\eta}_i$'s are all equal. It is straightforward to verify that $\frac{\partial q_i}{\partial a_i}(a) \leq \bar{\eta}_i$, with an equality when $\bar{\eta} \cdot a < 1$.

Another example is the following *Shapley rule*¹⁴: Each buyer submits a message a_i . Buyers are then randomly ordered, with all orders being equally likely. Let us denote by i_k the k th buyer in the realized order. Then buyer i_k 's allocation is equal to

$$q_i(a) = \min \left\{ \bar{\eta}_i a_i, \max \left\{ 1 - \sum_{k' < k} \bar{\eta}_{i_{k'}} a_{i_{k'}}, 0 \right\} \right\}.$$

In words, each buyer i demands $\bar{\eta}_i a_i$ units of the good, buyers are “served” in order, and on their turn a buyer receives the lesser of their demand and the remaining amount of the good. Clearly, if $\bar{\eta} \cdot a < 1$, then all buyers demands are met, regardless of the order, and $\frac{\partial q_i}{\partial a_i}(a) = \bar{\eta}_i$. If $\bar{\eta} \cdot a \geq 1$, then under every order, some buyer will not receive their demanded amount. When this happens, the buyer's allocation is insensitive to their action. Hence, for every action profile, $\frac{\partial q_i}{\partial a_i}(a) \leq \bar{\eta}_i$.

Yet another example is the “consistent” rule of Aumann and Maschler (1985), which reduces to the Shapley rule when $N = 2$ but differs for $N > 2$. In particular, if we let $f_i(d_1, \dots, d_N)$ denote the share of buyer i under the consistent rule when there is a unit surplus to be divided among the N buyers, and each buyer i demands d_i . As shown by Aumann and Maschler, $\partial f_i(d)/\partial d_i \in \{0, 1/2, 1\}$. Thus, if we define the allocation rule $q_i(a) = f_i(\bar{\eta}_1 a_1, \dots, \bar{\eta}_N a_N)$, then $\frac{\partial q_i}{\partial a_i}(a) \in \{0, \bar{\eta}_i/2, \bar{\eta}_i\}$, as required by Theorem 3.

In equations (22)–(24) in the proof of Theorem 3, we construct the transfers for these allocation rules to satisfy Condition 2. Unfortunately, the construction is rather complicated, and the resulting transfer does not reflect the structure of the corresponding allocation rule. Moreover, as the asymmetry in $\hat{v}_i$ vanishes, the resulting mechanisms do not converge to the proportional auction. The CPA on the other hand has a more transparent relationship between the allocation and the transfer, where the latter can be implemented as a constant price-per-unit that depends only on the residual supply and the aggregate action among the remaining buyers. Thus, the CPA is distinguished by admitting transfers that depend on

¹⁴Bergemann, Brooks, and Morris (2016) show that the Shapley rule with the symmetric $\bar{\eta}_i$ is part of a guarantee-maximizing mechanism in the common value model when there are two buyers; this is generalized to many buyers in an early working paper version of Brooks and Du (2021b) (available from the authors upon request). It is also an immediate corollary of Theorem 3, since the information structure $\bar{I}$ in Theorem 3 is a common value information structure when all $\bar{\eta}_i$'s are the same.

relatively low dimensional features of the action profile. It remains an open question whether there exist similarly appealing transfer rules for other guarantee-maximizing allocation allocation rules.

7 Discussion

The main contribution of our paper is to describe and characterize compound proportional auctions. Not only can these mechanisms be rationalized as optimal, but they also maximize the guarantee for revenue under the basic assumption that each buyer’s expected value is known. They are therefore a strong candidate for practical implementation, when the seller is uncertain about the precise form of private information. Moreover, the theoretical arguments underlying the performance of the CPA only rely on the fact that in equilibrium, buyers must be playing *local* best responses. In that sense, our arguments do not use the full force of equilibrium.

In a CPA, each buyer will in general be allocated only a portion of the good. This seems relatively easy to implement in a setting where the good is divisible, e.g., a commodity like wheat or corn, shares of common stock, or dollars of government debt. The CPA may be less compelling in settings where the good is indivisible and the allocation is interpreted as a probability of being allocated the entire unit.

We regard it as a strength of our theory that the CPA has so few parameters, which are just the sensitivities and the aggregate transfer rules. Moreover, when restricting attention to balanced CPAs, the space of parameters is reduced to just the vector sensitivities. To determine the optimal sensitivities, one only needs to know the expected value of each buyer and an upper bound on buyers’ values. Thus, the degree of understanding of the environment that is required in order to calibrate the CPA is relatively modest, especially compared to the standard model in which the optimal auction depends on the entire structure of higher-order beliefs. For future work, we think an important direction is to consider further restrictions on fundamentals, such as specifying either the full marginals of each buyer’s value or the full joint distribution.

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A Omitted Proofs

A.1 Section 4

Proof of Lemma 1. Fix $\eta_1 \geq \eta_2 \geq \dots \geq \eta_N \geq \eta_{N+1} \equiv 0$ for a CPA.

If $\eta \cdot a < 1$, then $q_i(a) = \sum_{r=i}^N q_i^r(a) = \eta_i a_i$.

Now suppose $\eta \cdot a \geq 1$.

If the CPA ends in round $n < i$, then we also have $q_i(a) = \sum_{r=i}^N q_i^r(a) = \eta_i a_i$.

If the CPA ends in round $n \geq i$, where $S^n(a) > 0$ but $S^{n-1}(a) \leq 0$, then

$$q_i(a) = \eta_{n+1} a_i + \frac{a_i}{\sum_{j=1}^n a_j} \underbrace{\left(1 - \sum_{r=n+1}^N \eta_r a_r - \eta_{n+1} \sum_{j=1}^n a_j \right)}_{S^n(a)}.$$

Let j sum over $\{1, \dots, n\}$, we have

$$\begin{aligned} \frac{\partial q_i}{\partial a_i}(a) &= \eta_{n+1} + \frac{\sum_{j \neq i} a_j}{\left(\sum_j a_j \right)^2} \left(1 - \sum_{r=n+1}^N \eta_r a_r - \eta_{n+1} \sum_j a_j \right) - \eta_{n+1} \frac{a_i}{\sum_j a_j} \\ &= \frac{\sum_{j \neq i} a_j}{\left(\sum_j a_j \right)^2} \left(1 - \sum_{r=n+1}^N \eta_r a_r \right) \\ &\leq \eta_n, \end{aligned}$$

since $S^{n-1}(a) = 1 - \sum_{r=n+1}^N \eta_r a_r - \eta_n \sum_j a_j \leq 0$. Thus we have $\frac{\partial q_i}{\partial a_i}(a) \leq \eta_i$ since $i \leq n$. $\square$

Lemma 3. For $n < N$ we have $\Lambda^n(1) = 0$. Moreover, if $n \geq 2$, then $\frac{d^m \Lambda^n}{dC^m}(1) = 0$ for every m .

Proof of Lemma 3. If $\eta_{n+1} \sum_{i=1}^n a_i + C \geq 1$, then round n of CPA will not occur, and thus $\nabla \cdot q^n(\sum_{i=1}^n a_i, C) = 0$. Thus the first part of the lemma follows. The second part of the lemma follows from $g_n(0) = 0$ if $n \geq 2$. $\square$

Lemma 4. For $n < N$ and $\tilde{S} \geq 0$, we have

$$\int_0^{\tilde{S}/\eta_{n+1}} \lambda^n(\tilde{S} - \eta_{n+1}\tau) g_n(\tau) d\tau = \Lambda^n(1 - \tilde{S}).$$

Proof of Lemma 4. Denote $F^m(\tilde{S}) = (\eta_{n+1} \frac{d}{d\tilde{S}} + \mathbf{1})^m (\Lambda^n(1 - \tilde{S}))$, so $F^n(\tilde{S}) = \lambda^n(\tilde{S})$. We

claim

$$\int_0^{\tilde{S}/\eta_{n+1}} F^m(\tilde{S} - \eta_{n+1}\tau) g_m(\tau) d\tau = \Lambda^n(1 - \tilde{S}),$$

for every $m \leq n$. We proceed by induction on m .

If $m = 1$, by integration by parts we have

$$\begin{aligned} & \int_0^{\tilde{S}/\eta_{n+1}} F^1(\tilde{S} - \eta_{n+1}\tau) g_1(\tau) d\tau \\ &= \int_0^{\tilde{S}/\eta_{n+1}} \left(\eta_{n+1} \frac{d}{d\tilde{S}} \Lambda^n(1 - (\tilde{S} - \eta_{n+1}\tau)) + \Lambda^n(1 - (\tilde{S} - \eta_{n+1}\tau)) \right) g_1(\tau) d\tau \\ &= -\Lambda^n(1 - (\tilde{S} - \eta_{n+1}\tau)) g_1(\tau) \Big|_0^{\tilde{S}/\eta_{n+1}} \\ &= \Lambda^n(1 - \tilde{S}), \end{aligned}$$

since $\Lambda^n(1) = 0$ by Lemma 3.

Now suppose the claims are true for $m - 1$, where $m \geq 2$ and $n \geq 2$. We have

$$\begin{aligned} & \int_0^{\tilde{S}/\eta_{n+1}} F^m(\tilde{S} - \eta_{n+1}\tau) g_m(\tau) d\tau \\ &= \int_0^{\tilde{S}/\eta_{n+1}} \left(\eta_{n+1} \frac{d}{d\tilde{S}} (F^{m-1}(\tilde{S} - \eta_{n+1}\tau)) + F^{m-1}(\tilde{S} - \eta_{n+1}\tau) \right) g_m(\tau) d\tau \\ &= -F^{m-1}(\tilde{S} - \eta_{n+1}\tau) g_m(\tau) \Big|_0^{\tilde{S}/\eta_{n+1}} \\ &\quad + \int_0^{\tilde{S}/\eta_{n+1}} (F^{m-1}(\tilde{S} - \eta_{n+1}\tau) g'_m(\tau) + F^{m-1}(\tilde{S} - \eta_{n+1}\tau) g_m(\tau)) d\tau \\ &= \int_0^{\tilde{S}/\eta_{n+1}} F^{m-1}(\tilde{S} - \eta_{n+1}\tau) g_{m-1}(\tau) d\tau \\ &= \Lambda^n(\tilde{S}) \end{aligned}$$

where we used the facts that $F^{m-1}(0) = 0$ for $n \geq 2$ (by Lemma 3), $g'_m(x) + g_m(x) = g_{m-1}(x)$, and applied the inductive hypothesis. □

Lemma 5. For $n < N$ and $\tilde{S} \in [0, 1]$, we have

$$\int_0^{\tilde{S}/\eta_{n+1}} \Xi^n(\tau, \tilde{S} - \eta_{n+1}\tau) g_n(\tau) d\tau = 0.$$

When $n = N$, we have

$$\int_0^\infty \Xi^n(\tau, 1) g_n(\tau) d\tau = 0.$$

Proof of Lemma 5. For $n < N$, we show that

$$\int_0^{\tilde{S}/\eta_{n+1}} \left(\nabla \cdot q^n(\tau, 1 - \tilde{S}) - \lambda^n(\tilde{S} - \eta_{n+1}\tau) + \lambda^{n-1}(\tilde{S} - \eta_n\tau) \right) g_n(\tau) d\tau = 0.$$

By Lemma 4, the second term above integrates to $-\Lambda^n(1 - \tilde{S})$. From the same lemma, we can express the third term as

$$\int_0^{\tilde{S}/\eta_n} \left(\int_0^{(\tilde{S} - \eta_n a_n)/\eta_n} \lambda^{n-1}(\tilde{S} - \eta_n a_n - \eta_n \tau') g_{n-1}(\tau') d\tau' \right) e^{-a_n} da_n = \int_0^{\tilde{S}/\eta_n} \Lambda^{n-1}(1 - \tilde{S} + \eta_n a_n) e^{-a_n} da_n$$

From the definition of Λ^n , the difference between these terms is

$$\begin{aligned} & \Lambda^n(1 - \tilde{S}) - \int_0^{\tilde{S}/\eta_n} \Lambda^{n-1}(1 - \tilde{S} + \eta_n a_n) e^{-a_n} da_n \\ &= \int_{a \in \mathbb{R}_+^n} \sum_{r=1}^n \nabla \cdot q^r \left(\sum_{i=1}^r a_i, 1 - \tilde{S} + \sum_{i=r+1}^n \eta_i a_i \right) \exp \left(- \sum_{i=1}^n a_i \right) da \\ & \quad - \int_0^{\tilde{S}/\eta_n} \left(\int_{a \in \mathbb{R}_+^{n-1}} \sum_{r=1}^{n-1} \nabla \cdot q^r \left(\sum_{i=1}^r a_i, 1 - \tilde{S} + \eta_n a_n + \sum_{i=r+1}^{n-1} \eta_i a_i \right) \exp \left(- \sum_{i=1}^{n-1} a_i \right) da \right) e^{-a_n} da_n \\ &= \int_0^{\tilde{S}/\eta_{n+1}} \nabla \cdot q^n(\tau, 1 - \tilde{S}) g_n(\tau) d\tau, \end{aligned}$$

which proves the first part of the lemma.

When $n = N$, since $\lambda^N(1) = \Lambda^N(0)$ by definition, we need to show that

$$\int_0^\infty \left(\nabla \cdot q^N(\tau, 1) - \Lambda^N(0) + \lambda^{N-1}(1 - \eta_N \tau) \right) g_N(\tau) d\tau = 0,$$

where the third term integrates to

$$\int_0^{1/\eta_N} \left(\int_0^{(1 - \eta_N a_N)/\eta_N} \lambda^{N-1}(1 - \eta_N a_N - \eta_N \tau') g_{N-1}(\tau') d\tau' \right) e^{-a_N} da_N = \int_0^{1/\eta_N} \Lambda^{N-1}(\eta_N a_N) e^{-a_N} da_N.$$

Thus, the second part of the lemma follows from the proof of the first part when $n = N$ and $\tilde{S} = 1$. $\square$

Lemma 6. Fix a mechanism $M = (\mathbb{R}_+^N, q, t)$ whose allocation and transfer functions are right-differentiable and have bounded right-derivatives. Define the strategic virtual objective:

$$\lambda_i(v_i, a) = t_i(a) + v_i \frac{\partial q_i}{\partial a_i}(a) - \frac{\partial t_i}{\partial a_i}(a).$$

Suppose $\inf_{a \in \mathbb{R}_+^N} \sum_i \lambda_i(v_i, a)$ is an affine function of v . Then the revenue guarantee of M is at least

$$\inf_{a \in \mathbb{R}_+^N} \sum_i \lambda_i(\hat{v}_i, a).$$

Proof of Lemma 6. Fix an information structure I and equilibrium b of (M, I) . For the information structure I , let $\pi(ds)$ be the marginal distribution of signals s and $w_i(s)$ a version of the conditional expectation of v_i given s .

For all $\Delta > 0$, the fact that b is an equilibrium implies that

$$\int_S \int_{\mathbb{R}_+^N} \sum_{i=1}^N [w_i(s)(q_i(a_i + \Delta, a_{-i}) - q_i(a)) - (t_i(a_i + \Delta) - t_i(a))] b(da|s) \pi(ds) \leq 0.$$

Since $\frac{\partial q_i}{\partial a_i}$ and $\frac{\partial t_i}{\partial a_i}$ are bounded, we conclude that the integrand in left-hand side of the preceding inequality is bounded by $K\Delta b(da|s)\pi(ds)$ for some constant K . We can divide through by Δ . The limit of the left-hand side as $\Delta \rightarrow 0$ must be non-positive as well. Finally, the Dominated Convergence Theorem implies

$$\int_S \int_{\mathbb{R}_+^N} \sum_i \left[w_i(s) \frac{\partial q_i}{\partial a_i}(a) - \frac{\partial t_i}{\partial a_i}(a) \right] b(da|s) \pi(ds) \leq 0.$$

Therefore

$$\begin{aligned} \int_S \int_{\mathbb{R}_+^N} \sum_i t_i(a) b(da|s) \pi(ds) &\geq \int_S \int_{\mathbb{R}_+^N} \sum_i \left[t_i(a) + w_i(s) \frac{\partial q_i}{\partial a_i}(a) - \frac{\partial t_i}{\partial a_i}(a) \right] b(da|s) \pi(ds) \\ &= \int_S \int_{\mathbb{R}_+^N} \sum_i \lambda_i(w_i(s), a) b(da|s) \pi(ds) \\ &\geq \int_S \inf_a \sum_i \lambda_i(w_i(s), a) \pi(ds) \\ &\geq \inf_a \sum_i \lambda_i(a), \end{aligned}$$

where in the last line we use the assumption that $\inf_a \sum_i \lambda_i(v_i, a)$ is an affine function of v . $\square$

Proof of Theorem 1. It is easy to check that the allocation and transfer functions in $\overline{M}(\eta)$

are right differentiable with bounded right derivatives. The proof of Theorem 1 then follows from the discussion on page 18 and Lemma 6. $\square$

A.2 Section 5

In the remaining sections let us introduce some notations to streamline the exposition. For a function $f : \mathbb{R}_+^N \rightarrow \mathbb{R}^N$. Let us denote $\nabla_i f_i(x) = \frac{\partial f_i}{\partial x_i}(x)$ for $x \in \mathbb{R}_+^N$, $\nabla f(x) = (\nabla_1 f_1(x), \dots, \nabla_N f_N(x))$, and $\nabla \cdot f(x) = \sum_i \nabla_i f_i(x)$. Moreover, for $x \in \mathbb{R}_+^N$, let $\Sigma x = \sum_i x_i$.

Proof of Lemma 2. Let $\hat{\eta} \in \mathbb{R}_+$ such that

$$\bar{v} \int_{s_1=0}^{\infty} \min\{(\exp(\hat{\eta}s_1 - 1))^{1/\hat{\eta}}, 1\} \exp(-s_1) ds_1 \geq \hat{v}_1.$$

Such a $\hat{\eta}$ exists because as $\hat{\eta} \rightarrow \infty$, the integrand converges monotonically pointwise to 1, so by the Dominated Convergence Theorem the integral converges monotonically to $\bar{v}$, which is strictly greater than the right-hand side.

Now, let us define the mapping $G_i : [0, \hat{\eta}]^N \rightarrow \mathbb{R}$ according to

$$G_i(\eta) = \begin{cases} \bar{v} \int_{s \in \mathbb{R}_+^N} \min\left\{(\exp(\eta \cdot s - 1))^{1/\eta_i}, 1\right\} \exp(-\Sigma s) ds & \text{if } \eta_i > 0; \\ \bar{v} \int_{\{s \in \mathbb{R}_+^N \mid \eta_{-i} \cdot s_{-i} \geq 1\}} \exp(-\Sigma s_{-i}) ds_{-i} & \text{if } \eta_i = 0. \end{cases}$$

Note that G_i is continuous and strictly increasing in η_i for $\eta_i > 0$. Moreover, the Dominated Convergence Theorem implies that

$$\lim_{\eta_i \rightarrow 0} G_i(\eta_i, \eta_{-i}) = G_i(0, \eta_{-i}),$$

so that G_i is continuous at $\eta_i = 0$.

Define the mapping $F : [0, \hat{\eta}]^N \rightarrow [0, \hat{\eta}]^N$ as follows: For fixed $\eta \in [0, \hat{\eta}]^N$, we define $F_i(\eta)$ as the solution $\eta'_i \in [0, \hat{\eta}]$ to

$$G_i(\eta'_i, \eta_{-i}) = \max\{\hat{v}_i, G_i(0, \eta_{-i})\}. \quad (14)$$

Note $G_i(\eta)$ is strictly increasing in η_i , so if a solution to (14) exists, it is unique. Moreover, G_i is increasing in η_{-i} , so from how we have defined $\hat{\eta}$, there exists a $\eta'_i > 0$ that satisfies (14) as an equality if and only if $G_i(0, \eta_{-i})$ is weakly less than $\hat{v}_i$. Otherwise, the unique solution is $\eta'_i = 0$.

Since the left-hand side of (14) is strictly increasing in η'_i , the Implicit Function Theorem in Kumagai (1980) implies that $F_i(\eta)$ is continuous. The Brouwer Fixed-Point Theorem then

implies that F has a fixed point, which necessarily solves the system (14).

We next claim that for any η that is a fixed point of F , $\eta_i = 0$ if and only if

$$\bar{v} \int_{\{s \in \mathbb{R}_+^N | \eta \cdot s \geq 1\}} \exp(-\Sigma s) ds \geq \hat{v}_i. \quad (15)$$

For if this condition is satisfied and $\eta_i > 0$, then $G_i(\eta)$ is strictly greater than the left-hand side of (15), which is in turn weakly greater than $G_i(0, \eta_{-i})$. Thus, $G_i(\eta)$ is strictly greater than both terms on the right-hand side of (14), which contradicts the hypothesis that η satisfies (14). (Note that $G_i(0) = 0$, so there must be at least one i for which $\eta_i > 0$.)

Finally, if η is a fixed point of F , then $w_i(\cdot | \eta)$ satisfies (10) for all i such that $\eta_i > 0$. And since (15) is satisfied for any i such that $\eta_i = 0$, $\bar{w}_i(s; \eta) \in [0, \bar{v}]$ for all s , and also satisfies (10). $\square$

Proof of Proposition 1. Fix an incentive compatible and individually rational direct mechanism (q, t) and define

$$U_i(s_i, s'_i) \equiv \int_{\bar{S}_{-i}} (\bar{w}_i(s_i, s_{-i}) q_i(s'_i, s_{-i}) - t_i(s'_i, s_{-i})) \exp(-\Sigma s_{-i}) ds_{-i},$$

and $U_i(s_i) \equiv U_i(s_i, s_i)$. Incentive compatibility says that for all i , s_i , and s'_i ,

$$U_i(s_i) \geq U_i(s_i, s'_i) = U_i(s'_i) + \int_{\bar{S}_{-i}} (\bar{w}_i(s_i, s_{-i}) - \bar{w}_i(s'_i, s_{-i})) q_i(s'_i, s_{-i}) \exp(-\Sigma s_{-i}) ds_{-i}.$$

and individual rationality says that $U_i(s_i) \geq 0$. Thus, for all $\Delta \geq 0$,

$$\begin{aligned} U_i &\equiv \int_{\bar{S}_i} U_i(s_i) \exp(-s_i) ds_i \\ &\geq \int_{\{s \in \bar{S} | s_i \geq \Delta\}} [U_i(s_i - \Delta) + (\bar{w}_i(s_i, s_{-i}) - \bar{w}_i(s_i - \Delta, s_{-i})) q_i(s_i - \Delta, s_{-i})] \exp(-\Sigma s) ds \\ &= \exp(-\Delta) \left(U_i + \int_{\bar{S}} (\bar{w}_i(s_i + \Delta, s_{-i}) - \bar{w}_i(s)) q_i(s) \exp(-\Sigma s) ds \right). \end{aligned}$$

Rearranging, we have

$$U_i \geq \frac{1}{\exp(\Delta) - 1} \int_{\bar{S}} (\bar{w}_i(s_i + \Delta, s_{-i}) - \bar{w}_i(s)) q_i(s) \exp(-\Sigma s) ds.$$

Since total surplus is

$$\sum_{i=1}^N \int_{\bar{S}} \bar{w}_i(s) q_i(s) \exp(-\Sigma s) ds,$$

we conclude that an upper bound on profit is

$$\sum_{i=1}^N \int_{\bar{S}} \left[\bar{w}_i(s) - \frac{1}{\exp(\Delta) - 1} (\bar{w}_i(s_i + \Delta, s_{-i}) - \bar{w}_i(s)) \right] q_i(s) \exp(-\Sigma s) ds.$$

To apply the Dominated Convergence Theorem and take $\Delta \rightarrow 0$, we just need to show that the discrete derivative is bounded:

$$\begin{aligned} & \max_{s \in \bar{S}} \frac{1}{\exp(\Delta) - 1} (\bar{w}_i(s_i + \Delta, s_{-i}) - \bar{w}_i(s)) \\ & \leq \bar{v} \max_{\{s \in \bar{S} | \bar{\eta} \cdot s \leq 1\}} \frac{1}{\exp(\Delta) - 1} (\exp((\bar{\eta} \cdot s)/\bar{\eta}_i + \Delta) - \exp((\bar{\eta} \cdot s)/\bar{\eta}_i)) \\ & = \bar{v} \max_{\{s \in \bar{S} | \bar{\eta} \cdot s \leq 1\}} \exp((\bar{\eta} \cdot s)/\bar{\eta}_i) \\ & = \bar{v} \exp(1/\bar{\eta}_i). \end{aligned}$$

Thus, the limit of the profit upper bound as $\Delta \rightarrow 0$ is

$$\begin{aligned} & \sum_{i=1}^N \int_{\bar{S}} [\bar{w}_i(s) - \nabla_i \bar{w}_i(s)] q_i(s) \exp(-\Sigma s) ds \\ & = \bar{v} \int_{\{s \in \bar{S} | \bar{\eta} \cdot s \geq 1\}} \sum_{i=1}^N q_i(s) \exp(-\Sigma s) ds \\ & \leq \bar{v} \int_{\{s \in \bar{S} | \bar{\eta} \cdot s \geq 1\}} \exp(-\Sigma s) ds = \bar{R}. \end{aligned}$$

□

Proof of Proposition 2. Since

$$\Lambda^N(0; \bar{\eta}) = \bar{v} \int_{\mathbb{R}_+^N} \nabla \cdot q(a) \exp(-\Sigma a) da,$$

where q is the allocation rule for the CPA $\bar{M}(\bar{\eta})$, we need to show that

$$\bar{v} \int_{\mathbb{R}_+^N} \nabla \cdot q(a) \exp(-\Sigma a) da - \sum_{i=1}^N \eta_i \cdot (\bar{v} - \hat{v}_i) = \bar{v} \int_{\{a \in \mathbb{R}_+^N | \eta \cdot a \geq 1\}} \exp(-\Sigma a) da = \bar{R}.$$

This is implied by Lemma 7, which states that (cf. equations (12) and (16))

$$\bar{v} \int_{\mathbb{R}_+^N} \nabla \cdot q(a) \exp(-\Sigma a) da = \bar{R} + \sum_i (\bar{v}_i - \hat{v}_i) \bar{\eta}_i.$$

□

Proof of Theorem 2. The proof follows from Theorem 3. □

A.3 Section 6

Given an allocation rule q that satisfies Condition 1 of Theorem 3, we define its associated *aggregate excess growth function* $\Xi : \mathbb{R}_+^N \rightarrow \mathbb{R}$ to be the right-hand side of (13), i.e.,

$$\Xi(a) \equiv \bar{v} \nabla \cdot q(a) - \sum_i \bar{\lambda}_i(\bar{v}). \quad (16)$$

Also, given an information structure $I = (S, \sigma)$, we denote by $\pi(ds)$ the marginal of σ on S , and we let $w_i(s)$ be a version of the conditional expectation of v_i given s .

Proposition 3. *Suppose that M satisfies the hypotheses of Theorem 3. Then for any information structure I and equilibrium b of (M, I) , $R(M, I, b) \geq \bar{R}$.*

Proof of Proposition 3. By Condition 2 of Theorem 3, the SVO of M is

$$\sum_i \lambda_i(v_i, a) = v \cdot \nabla q(a) - \Xi(a) = \sum_i (v_i - \bar{v}) \nabla_i q(a) + \bar{\lambda}_i(\bar{v}).$$

Since $\nabla_i q_i(a) \leq \bar{\eta}_i$, with an equality if $\bar{\eta} \cdot a < 1$, we have

$$\inf_a \sum_i \lambda_i(v_i, a) = \sum_i (v_i - \bar{v}) \bar{\eta}_i + \bar{\lambda}_i(\bar{v}) = \sum_i \bar{\lambda}_i(v_i). \quad (17)$$

Then Lemma 6 implies that $R(M, I, b) \geq \sum_i \bar{\lambda}_i(\hat{v}_i) = \bar{R}$.

□

Lemma 7. *Suppose that q satisfies Condition 1 of Theorem 3. Then*

$$\int_{\mathbb{R}_+^N} \Xi(a) \exp(-\Sigma a) da = 0.$$

Proof of Lemma 7. Equation (17) implies that

$$\sum_{i=1}^N \bar{\lambda}_i(\bar{w}_i(a)) = \bar{w}(a) \cdot \nabla q(a) - \Xi(a)$$

for all $a \in \mathbb{R}_+^N$, since $\bar{w}_i(a) = \bar{v}$ whenever $\bar{\eta} \cdot a \geq 1$.

The ex ante expectation of Ξ is therefore the sum over i of the integrals

$$\int_{\mathbb{R}_+^N} \bar{w}_i(a) \nabla_i q_i(a) \exp(-\Sigma a) da - \bar{\lambda}_i(\hat{v}_i).$$

Integrating by parts and using the fact that $q_i(0, a_{-i}) = 0$ and the definition of $\bar{\lambda}_i$, this is

$$\int_{\mathbb{R}_+^{N-1}} \left(\int_{\mathbb{R}_+} (\bar{w}_i(a) - \nabla_i \bar{w}(a)) q_i(a) \exp(-a_i) da_i \right) \exp(-\Sigma a_{-i}) da_{-i} - \frac{\bar{R}}{N}.$$

Summing across i , we get

$$\begin{aligned} & \int_{\mathbb{R}_+^N} \sum_{i=1}^N (\bar{w}_i(a) - \nabla_i \bar{w}(a)) q_i(a) \exp(-\Sigma a) da - \bar{R} \\ &= \int_{\{a \in \mathbb{R}_+^N \mid \bar{\eta} \cdot a \geq 1\}} \bar{v} \sum_{i=1}^N q_i(a) \exp(-\Sigma a) da - \bar{R} = 0, \end{aligned}$$

since $\bar{w}_i(a) - \nabla_i \bar{w}(a) = 0$ when $\bar{\eta} \cdot a < 1$, and $\bar{w}_i(a) = \bar{v}$ and $\Sigma q(a) = 1$ when $\bar{\eta} \cdot a \geq 1$. $\square$

Proposition 4. *Suppose that M satisfies the hypotheses of Theorem 3. Then the truthful strategy profile $\bar{b}$ is an equilibrium of $(M, \bar{I})$.*

Proof of Proposition 4. We first derive an expression for the interim expected transfer in terms of the allocation (equation (20)). Define the individual excess growth as

$$\xi_j(a) = \nabla_j t(a) - t_j(a).$$

With the assumption that $t_j(0, a_{-j}) = 0$, the above equation is equivalent to

$$t_j(a) = \exp(a_j) \int_{s_j=0}^{a_j} \xi_j(s_j, a_{-j}) \exp(-s_j) ds_j. \quad (18)$$

Therefore, we can write the interim expected transfer of buyer i in $(M, \bar{I})$ as

$$\begin{aligned} t_i(a_i) &= \int_{\mathbb{R}_+^{N-1}} t_i(a_i, s_{-i}) \exp(-\Sigma s_{-i}) ds_{-i} \\ &= \int_{\mathbb{R}_+^{N-1}} \exp(a_i) \int_{s_i=0}^{a_i} \xi_i(s_i, s_{-i}) \exp(-s_i) ds_i \exp(-\Sigma s_{-i}) ds_{-i}. \end{aligned}$$

Since t_j is bounded in equation (18), it must be that

$$\int_{s_j=0}^{\infty} \xi_j(s_j, s_{-j}) \exp(-s_j) ds_j = 0 \quad (19)$$

for all j and s_{-j} . Hence, we can rewrite the interim expected transfer as

$$\begin{aligned} t_i(a_i) &= - \int_{\mathbb{R}_+^{N-1}} \exp(a_i) \int_{s_i=a_i}^{\infty} \xi_i(s_i, s_{-i}) \exp(-s_i) ds_i \exp(-\Sigma s_{-i}) ds_{-i} \\ &= - \int_{\mathbb{R}_+^N} \xi_i(a_i + s_i, s_{-i}) \exp(-\Sigma s) ds \\ &= - \int_{\mathbb{R}_+^N} [\xi_i(a_i + s_i, s_{-i}) + \Sigma \xi_{-i}(a_i + s_i, s_{-i})] \exp(-\Sigma s) ds \\ &= - \int_{\mathbb{R}_+^N} \Xi_i(a_i + s_i, s_{-i}) \exp(-\Sigma s) ds, \end{aligned}$$

where we applied equation (19) to each $j \neq i$ in the third line, and used the assumption of $\Sigma \xi = \Xi$ in the fourth line.

Using the definition of Ξ , we get

$$\begin{aligned} t_i(a_i) &= - \int_{\{\bar{\eta} \cdot s + \bar{\eta}_i a_i \geq 1\}} (\bar{v} \nabla \cdot q(a_i + s_i, s_{-i}) - \Sigma \bar{\lambda}(1)) e^{-\Sigma s} ds - \int_{\{\bar{\eta} \cdot s + \bar{\eta}_i a_i < 1\}} (-\Sigma \bar{\lambda}(0)) e^{-\Sigma s} ds \\ &= - \int_{\{\bar{\eta} \cdot s + \bar{\eta}_i a_i \geq 1\}} (\bar{v} \nabla \cdot q(a_i + s_i, s_{-i}) - \Sigma \bar{\eta}) e^{-\Sigma s} ds + \Sigma \bar{\lambda}(0), \end{aligned}$$

where in the second line we used the fact that $\Sigma \bar{\lambda}(1) = \Sigma \bar{\eta} + \Sigma \bar{\lambda}(0)$, and $\{\bar{\eta} \cdot s + \bar{\eta}_i a_i \geq 1\}$ is a shorthand for $\{s \in \mathbb{R}_+^N \mid \bar{\eta} \cdot s + \bar{\eta}_i a_i \geq 1\}$.

Integrating by parts, we have

$$\begin{aligned}
& \int_{\{\bar{\eta} \cdot s + \bar{\eta}_i a_i \geq 1\}} \nabla \cdot q(a_i + s_i, s_{-i}) e^{-\Sigma s} ds \\
&= \sum_{j=1}^N \int_{\mathbb{R}_+^{N-1}} \int_{s_j = \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+}{\bar{\eta}_j}}^{\infty} \nabla_j q(a_i + s_i, s_{-i}) e^{-\Sigma s} ds_j ds_{-j} \\
&= \int_{\mathbb{R}_+^{N-1}} \left[-q_i \left(a_i + \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i}, s_{-i} \right) e^{-\frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i} - \Sigma s_{-i}} \right. \\
&\quad \left. + \int_{s_i = \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i}}^{\infty} q_i(a_i + s_i, s_{-i}) e^{-\Sigma s} ds_i \right] ds_{-i} \\
&\quad + \sum_{j \neq i} \int_{\mathbb{R}_+^{N-1}} \left[-q_j \left(a_i + s_i, \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+}{\bar{\eta}_j}, s_{-i-j} \right) e^{-\frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+}{\bar{\eta}_j} - \Sigma s_{-j}} \right. \\
&\quad \left. + \int_{s_j = \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+}{\bar{\eta}_j}}^{\infty} q_j(a_i + s_i, s_{-i}) e^{-\Sigma s} ds_j \right] ds_{-j} \\
&= - \int_{\mathbb{R}_+^{N-1}} q_i \left(a_i + \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i}, s_{-i} \right) e^{-\frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i} - \Sigma s_{-i}} ds_{-i} \\
&\quad - \sum_{j \neq i} \int_{\mathbb{R}_+^{N-1}} (1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+ e^{-\frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+}{\bar{\eta}_j} - \Sigma s_{-j}} ds_{-j} + \int_{\{\bar{\eta} \cdot s + \bar{\eta}_i a_i \geq 1\}} e^{-\Sigma s} ds,
\end{aligned}$$

where in the last line, we used the facts that $q_j \left(a_i + s_i, \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+}{\bar{\eta}_j}, s_{-i-j} \right) = (1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+$ and $\sum_{j=1}^N q_j(a_i + s_i, s_{-i}) = 1$ whenever $\bar{\eta} \cdot s + \bar{\eta}_i a_i \geq 1$.

Therefore, we have the following expression for the interim expected transfer:

$$\begin{aligned}
t_i(a_i) &= \underbrace{\bar{v} \int_{\mathbb{R}_+^{N-1}} q_i \left(a_i + \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i}, s_{-i} \right) e^{-\frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i} - \Sigma s_{-i}} ds_{-i}}_X \\
&\quad + \underbrace{\bar{v} \sum_{j \neq i} \int_{\mathbb{R}_+^{N-1}} (1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+ e^{-\frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j})^+}{\bar{\eta}_j} - \Sigma s_{-j}} ds_{-j}}_Y \\
&\quad - \underbrace{\bar{v} \int_{\{\bar{\eta} \cdot s + \bar{\eta}_i a_i \geq 1\}} (1 - \Sigma \bar{\eta}) e^{-\Sigma s} ds}_{Z} + \Sigma \bar{\lambda}(0).
\end{aligned} \tag{20}$$

Next, we show there is no incentive to locally deviate from truthtelling (equation (21)).

We calculate

$$\begin{aligned}\frac{\partial X}{\partial a_i} &= \int_{\{\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i \geq 1\}} \nabla_i q(a_i, s_{-i}) e^{-\Sigma s_{-i}} ds_{-i} \\ &\quad + \int_{\{\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i < 1\}} (1 - \bar{\eta}_{-i} \cdot s_{-i}) e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i}}{\bar{\eta}_i} - \Sigma s_{-i}} ds_{-i},\end{aligned}$$

where we used the fact that $q_i \left(a_i + \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i}, s_{-i} \right) = 1 - \bar{\eta}_{-i} \cdot s_{-i}$ if $\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i < 1$. Likewise,

$$\begin{aligned}\frac{\partial Y}{\partial a_i} &= \sum_{j \neq i} \left[\int_{\{\bar{\eta}_{-j} \cdot s_{-j} + \bar{\eta}_i a_i < 1\}} (-\bar{\eta}_i) e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j}}{\bar{\eta}_j} - \Sigma s_{-j}} ds_{-j} \right. \\ &\quad \left. + \int_{\{\bar{\eta}_{-j} \cdot s_{-j} + \bar{\eta}_i a_i < 1\}} (1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j}) e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j}}{\bar{\eta}_j} - \Sigma s_{-j}} \frac{\bar{\eta}_i}{\bar{\eta}_j} ds_{-j} \right]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial Z}{\partial a_i} &= \frac{\partial}{\partial a_i} \left[\int_{\mathbb{R}_+^{N-1}} \left(\int_{s_i = \frac{(1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i})^+}{\bar{\eta}_i}}^{\infty} (1 - \Sigma \bar{\eta}) e^{-s_i} ds_i \right) e^{-\Sigma s_{-i}} ds_{-i} \right] \\ &= \int_{\{\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i < 1\}} (1 - \Sigma \bar{\eta}) e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i}}{\bar{\eta}_i} - \Sigma s_{-i}} ds_{-i}.\end{aligned}$$

In the expression for $\frac{\partial Y}{\partial a_i}$, we change the variables from $s_{-j} = (s_{-i-j}, s_i)$ to $s_{-i} = (s_{-i-j}, s_j)$ by leaving s_{-i-j} unchanged and defining $s_j = \frac{1 - \bar{\eta}_i a_i - \bar{\eta}_i s_i - \bar{\eta}_{-i-j} \cdot s_{-i-j}}{\bar{\eta}_j}$. This change of variable implies:

$$\begin{aligned}&\int_{\{\bar{\eta}_{-j} \cdot s_{-j} + \bar{\eta}_i a_i < 1\}} (-\bar{\eta}_i) e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j}}{\bar{\eta}_j} - \Sigma s_{-j}} ds_{-j} \\ &= \int_{\{\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i < 1\}} (-\bar{\eta}_j) e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i}}{\bar{\eta}_i} - \Sigma s_{-i}} ds_{-i}\end{aligned}$$

and

$$\begin{aligned}&\int_{\{\bar{\eta}_{-j} \cdot s_{-j} + \bar{\eta}_i a_i < 1\}} (1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j}) e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-j} \cdot s_{-j}}{\bar{\eta}_j} - \Sigma s_{-j}} \frac{\bar{\eta}_i}{\bar{\eta}_j} ds_{-j} \\ &= \int_{\{\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i < 1\}} \bar{\eta}_j s_j e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i}}{\bar{\eta}_i} - \Sigma s_{-i}} ds_{-i}.\end{aligned}$$

Combining the above expressions of $\frac{\partial X}{\partial a_i}$, $\frac{\partial Y}{\partial a_i}$ and $\frac{\partial Z}{\partial a_i}$ with equation (20), we get

$$\begin{aligned}
t'_i(a_i) &= \bar{v} \int_{\{\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i \geq 1\}} \nabla_i q(a_i, s_{-i}) e^{-\Sigma s_{-i}} ds_{-i} \\
&\quad + \bar{v} \int_{\{\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i < 1\}} \bar{\eta}_i e^{-\frac{1 - \bar{\eta}_i a_i - \bar{\eta}_{-i} \cdot s_{-i}}{\bar{\eta}_i} - \Sigma s_{-i}} ds_{-i} \\
&= \int_{\mathbb{R}_+^{N-1}} \nabla_i q(a_i, s_{-i}) \bar{w}_i(a_i, s_{-i}) e^{-\Sigma s_{-i}} ds_{-i},
\end{aligned} \tag{21}$$

where in the second equality we used the fact that $\nabla_i q(a_i, s_{-i}) = \bar{\eta}_i$ and $\bar{w}_i(a_i, s_{-i}) = \bar{v} e^{\frac{\bar{\eta}_i a_i + \bar{\eta}_{-i} \cdot s_{-i} - 1}{\bar{\eta}_i}}$ if $\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i < 1$, and $\bar{w}_i(a_i, s_{-i}) = \bar{v}$ if $\bar{\eta}_{-i} \cdot s_{-i} + \bar{\eta}_i a_i \geq 1$.

Finally, suppose a buyer receives a signal s_i in $\bar{I}$; by bidding s'_i instead of s_i in M , their interim expected transfer is changed by

$$\begin{aligned}
t_i(s'_i) - t_i(s_i) &= \int_{a_i=s_i}^{s'_i} t'_i(a_i) da_i \\
&= \int_{\mathbb{R}_+^{N-1}} \int_{a_i=s_i}^{s'_i} \nabla_i q(a_i, s_{-i}) \bar{w}_i(a_i, s_{-i}) da_i e^{-\Sigma s_{-i}} ds_{-i} \\
&\geq \int_{\mathbb{R}_+^{N-1}} \int_{a_i=s_i}^{s'_i} \nabla_i q(a_i, s_{-i}) \bar{w}_i(s_i, s_{-i}) da_i e^{-\Sigma s_{-i}} ds_{-i} \\
&= \int_{\mathbb{R}_+^{N-1}} (q_i(s'_i, s_{-i}) - q_i(s_i, s_{-i})) \bar{w}_i(s_i, s_{-i}) e^{-\Sigma s_{-i}} ds_{-i}
\end{aligned}$$

where we applied (21) and exchanged the order of integration in the second line, and the inequality in the third line follows because $\nabla_i q(a_i, s_{-i}) \geq 0$ and $\bar{w}_i(a_i, s_{-i})$ increases with a_i . This shows that the truthtelling $\bar{b}$ is an equilibrium of $(M, \bar{I})$. $\square$

Proof of Theorem 3. Fix a mechanism M that satisfies the hypotheses of Theorem 3. Proposition 1 implies Condition 1 for $(M, \bar{I}, \bar{b})$ to be a strong maxmin solution, Proposition 3 implies Condition 2, and Proposition 4 implies Condition 3. This proves the first part of Theorem 3.

Now fix an allocation rule q that satisfies Condition 1 of Theorem 3, and let Ξ be its associated aggregate excess growth function. Let Z denote the set of permutations of $\{1, \dots, N\}$ with a typical element ζ . We denote by

$$[\zeta \leq k] = \{j \mid \zeta(j) \leq k\},$$

and analogously define $[\zeta > k]$. Next, let

$$\tau_{\zeta,k}(a) = \int_{\mathbb{R}_+^{N-k}} \Xi(a_{[\zeta \leq k]}, x_{[\zeta > k]}) \exp(-\Sigma x_{[\zeta > k]}) dx_{[\zeta > k]}, \quad (22)$$

and

$$\xi_i(a) = \frac{1}{N!} \sum_{\zeta \in Z} [\tau_{\zeta, \zeta(i)}(a) - \tau_{\zeta, \zeta(i)-1}(a)]. \quad (23)$$

Finally, define the transfer rule:

$$t_i(a) = \exp(a_i) \int_{x_i=0}^{a_i} \xi_i(x_i, a_{-i}) \exp(-x_i) dx_i. \quad (24)$$

We show that Condition 2 of Theorem 3 is satisfied. Equation (24) implies that

$$\xi_i(a) = \nabla_i t(a) - t_i(a)$$

for all $a \in \mathbb{R}_+^N$. Given the definition of $\xi_i(a)$ in (23), $\nabla \cdot t(a) - \Sigma t_i(a) = \Xi(a)$ follows by telescoping the summation over i for each fixed permutation ζ and noticing that $\tau_{\zeta, N}(a) = \Xi(a)$ and $\tau_{\zeta, 0}(a) = 0$ (by Lemma 7). Finally, to show that t is bounded, by equation (24) and the fact that Ξ is bounded it suffices to show that

$$\int_{x_i=0}^{\infty} \xi_i(x_i, a_{-i}) \exp(-x_i) dx_i = 0,$$

for every $a_{-i} \in \mathbb{R}_+^{N-1}$. The above equation follows from the definition of ξ_i in (23) since it is easy to see that

$$\int_{x_i=0}^{\infty} \tau_{\zeta, \zeta(i)}(x_i, a_{-i}) \exp(-x_i) dx_i = \tau_{\zeta, \zeta(i)-1}(a_i, a_{-i}),$$

where the right-hand side does not depend on a_i . □