II. The Arrow-Debreu Model of Competitive Equilibrium - Definition and Existence
A. Existence of General Equilibrium in a simple model

Overview: The issue of 'existence' of general equilibrium is to demonstrate sufficient conditions on a competitive economy so that there is an array of market-clearing prices. The treatment of this section, and of the textbook chapter 5, outlines the building blocks of a mathematical proof of existence. Sufficient conditions are essentially that (i) prices for N goods can be represented as points on the unit simplex in R^N, essentially that only *relative* prices matter in determining supply and demand, (ii) supply decisions of firms and demand decisions of households can be represented as continuous functions from price space into R^N, (iii) that Walras's Law holds (the market value of unsatisfied excess demands and supplies nets out to zero); this property follows from firm profits being rebated to shareholders and all households fully spending their income. Then a well-constructed price adjustment process, raising the price of goods in excess demand and reducing those of goods in excess supply, can be represented as a continuous function from the price simplex into itself. Applying the Brouwer Fixed Point Theorem results in a fixed point of the price adjustment process. It is then a consequence of Walras's Law that the fixed point is a market-clearing price vector.

Speaking at a memorial conference in honor of Gerard Debreu in 2005, Prof. Hugo Sonnenschein commented

The Arrow-Debreu model, as communicated in *Theory of Value*

> changed basic thinking, and it quickly became the standard model of price theory. It is the benchmark model in Finance, International Trade, Public Finance, Transportation, and even macroeconomics. ... In rather short order it was no longer as it is in Marshall, Hicks, and Samuelson; rather it became as it is in *Theory of Value*.

See Chapter 5 of *General Equilibrium Theory: An Introduction, 2nd ed.*

N goods in the economy.

A typical array of prices is an N-dimensional vector

$$p = (p_1, p_2, p_3, \ldots, p_{N-1}, p_N) = (3, 1, 5, \ldots, 0.5, 10).$$

Assume only <u>relative prices</u> (price ratios) matter here, not the numerical values of prices. This is essentially assuming that there is no money, no monetary instrument held as wealth in which prices are denominated.

The price space: The unit simplex in $\mathbf{R}^N$, is

$$P = \left\{ p \mid p \in R^N, p_i \geq 0, i = 1, \ldots, N, \sum_{i=1}^{N} p_i = 1 \right\}. \tag{5.1}$$

The unit simplex is a (generalized) triangle in N-space. It's called "unit" because the co-ordinates add to 1. It's a "simplex" because it has that generalized triangle specification.

For each household $i \in H$, we define a demand function, $D^i : P \rightarrow \mathbf{R}^N$.

For each firm $j \in F$, a supply function, $S^j : P \rightarrow \mathbf{R}^N$.

Positive co-ordinates in $S^j(p)$ are outputs, negative co-ordinates are inputs.

$p \cdot S^j(p) \equiv \sum_{n=1}^{N} p_n S_n^j(p) \equiv$ profits of firm j.

The economy has an initial endowment of resources $r \in \mathbf{R}_+^N$ that is also supplied to the economy.

The market excess demand function is defined as

$$Z(p) = \sum_{i \in H} D^i(p) - \sum_{j \in F} S^j(p) - r, \tag{5.2}$$

$$Z : P \rightarrow \mathbf{R}^N \tag{5.3}$$

$Z(p) \equiv (Z_1(p), Z_2(p), Z_3(p), \ldots, Z_N(p))$, where $Z_k(p)$ is the excess demand for good k. When $Z_k(p)$, the excess demand for good k, is negative, we will say that good k is in excess supply.

There are two principal assumptions: Walras's Law and Continuity of $Z(p)$:

Walras's Law: For all $p \in P$,

$$p \cdot Z(p) = \sum_{n=1}^{N} p_n \cdot Z_n(p) = \sum_{i \in H} p \cdot D^i(p) - \sum_{j \in F} p \cdot S^j(p) - p \cdot r = 0.$$

The economic basis for Walras's Law involves the assumption of scarcity and the structure of household budget constraints. $\sum_{i \in H} p \cdot D^i(p)$ is the value of aggregate household expenditure. The term $\sum_{j \in F} p \cdot S^j(p) + p \cdot r$ is the value of aggregate household income (value of firm profits plus the value of endowment). Walras's Law says that expenditure equals income.

Continuity:

$$Z : P \rightarrow \mathbf{R}^N, Z(p) \text{ is a continuous function for all } p \in P.$$

That is, small changes in p result in small changes in $Z(p)$ everywhere in P.

We assume in this discussiion that $Z(p)$ is well defined and fulfills Walras's Law and Continuity. As mathematical theorists, part of our job is to derive these properties from more elementary properties during the next few weeks (so that we can be sure of their generality).

Definition: $p^o \in P$ is said to be an equilibrium price vector if $Z(p^o) \leq 0$ (0 is the zero vector; the inequality applies coordinatewise) with $p^o_k = 0$ for k such that $Z_k(p^o) < 0$. That is, p^o is an equilibrium price vector if supply equals demand in all markets (with possible excess supply of free goods).

Theorem 5.1 (&9.3) Brouwer Fixed-Point Theorem: Let $f(\cdot)$ be a continuous function, $f : P \rightarrow P$. Then there is $x^* \in P$ so that $f(x^*) = x^*$.

Theorem 5.2: Let Walras's Law and Continuity be fulfilled. Then there is $p^* \in P$ so that p^* is an equilibrium.

Proof: Let $T : P \to P$, where $T(p) = (T_1(p), T_2(p), \ldots, T_k(p), \ldots, T_N(p))$. $T_k(p)$ is the adjusted price of good k, adjusted by the auctioneer trying to bring supply and demand into balance. Let $\gamma^k > 0$. The adjustment process of the kth price can be represented as $T_k(p)$, defined as follows:

$$T_k(p) \equiv \frac{\max[0, p_k + \gamma^k Z_k(p)]}{\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)]}. \tag{5.4}$$

The function T is a price adjustment function. It raises the relative price of goods in excess demand and reduces the price of goods in excess supply while keeping the price vector on the simplex. In order for T to be well defined, the denominator must be nonzero, that is,

$$\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)] \neq 0. \tag{5.5}$$

(5.5)follows from Walras's Law. For the sum in the denominator to be zero or negative, all goods would have to be in excess supply simultaneously, which is contrary to our notions of scarcity and– it turns out– to Walras's Law as well.

Suppose, contrary to (5.5), $\sum_{n=1}^{N} \max[0, p_n + \gamma^n Z_n(p)] = 0$. Then $p_n + \gamma^n Z_n(p) \leq 0$ all $n = 1, \ldots, N$, and for each $p_n > 0$ we have $Z_n(p) < 0$. Then $\sum_{n=1}^{N} p_n Z_n(p) < 0$. But Walras's Law says $\sum_{n=1}^{N} p_n Z_n(p) = 0$. The contradiction proves (5.5).

Recall that $Z(\cdot)$ is a continuous function. The operations of max[ ], sum, and division by a nonzero continuous function maintain continuity. Hence, $T(p)$ is a continuous function from the simplex into itself.

By the Brouwer Fixed-Point Theorem there is $p^* \in P$ so that $T(p^*) = p^*$.

We must show that p^* is not just the stopping point of the price adjustment process, but that it actually does represent general equilibrium prices for the economy.

Since $T(p^*) = p^*$, for each good k, $T_k(p^*) = p_k^*$. That is, for all $k = 1, \ldots, N$,

$$p_k^* = \frac{\max[0, p_k^* + \gamma^k Z_k(p^*)]}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]}. \tag{5.6}$$

For each k, either

$$p_k^* = 0 \qquad \text{(Case 1)} \tag{5.7}$$

or

$$p_k^* = \frac{p_k^* + \gamma^k Z_k(p^*)}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]} > 0 \qquad \text{(Case 2)}. \tag{5.8}$$

$p_k^* = 0 = \max[0, p_k^* + \gamma^k Z_k(p^*)]$. Hence, $0 \geq p_k^* + \gamma^k Z_k(p^*) = \gamma^k Z_k(p^*)$ and $Z_k(p^*) \leq 0$. This is the case of free goods with market clearing or with excess supply in equilibrium.

To avoid repeated messy notation, let

$$\lambda = \frac{1}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n Z_n(p^*)]} \tag{5.9}$$

so that $T_k(p^*) = \lambda(p_k^* + \gamma^k Z_k(p^*))$. Note that $\lambda > 0$, by the argument demonstrating (5.5). Since p^* is the fixed point of T we have $p_k^* = \lambda(p_k^* + \gamma^k Z_k(p^*)) > 0$. This expression is true for all k with $p_k^* > 0$, and λ is the same for all k. Let's perform some algebra on this expression. We first combine terms in p_k^*:

$$(1 - \lambda)p_k^* = \lambda\gamma^k Z_k(p^*), \tag{5.10}$$

then multiply through by $Z_k(p^*)$ to get

$$(1 - \lambda)p_k^* Z_k(p^*) = \lambda\gamma^k (Z_k(p^*))^2, \tag{5.11}$$

and now sum over all k in Case 2, obtaining

$$(1-\lambda)\sum_{k\in \text{Case2}} p_k^* Z_k(p^*) = \lambda \sum_{k\in \text{Case2}} \gamma^k (Z_k(p^*))^2. \tag{5.12}$$

Walras's Law says

$$0 = \sum_{k=1}^{N} p_k^* Z_k(p^*) = \sum_{k\in \text{Case1}} p_k^* Z_k(p^*) + \sum_{k\in \text{Case2}} p_k^* Z_k(p^*). \tag{5.13}$$

But for $k \in$ Case 1, $p_k^* Z_k(p^*) = 0$, and so

$$0 = \sum_{k\in \text{Case1}} p_k^* Z_k(p^*). \tag{5.14}$$

Therefore,

$$\sum_{k\in \text{Case2}} p_k^* Z_k(p^*) = 0. \tag{5.15}$$

Hence, from (5.11) we have

$$0 = (1-\lambda)\cdot \sum_{k\in \text{Case2}} p_k^* Z_k(p^*) = \lambda \cdot \sum_{k\in \text{Case2}} \gamma^k (Z_k(p^*))^2. \tag{5.16}$$

Using Walras's Law, we established that the left-hand side equals 0, but the right-hand side can be zero only if $Z_k(p^*) = 0$ for all k such that $p_k^* > 0$ (k in Case 2). Thus, p^* is an equilibrium. This concludes the proof.

QED