Econ 120C **Ramu Ramanathan**
Fall 2003 **Final Exam Answers**

The campus grading regulations prohibit regarding of final exams. Therefore, read all parts of questions carefully and answer fully. Maximum number of points is 70.

I. Consider the following two-equation model in which the *y*s are endogenous and the *x*s are exogenous:

$$\text{(1)} \quad y_1 = a_1 y_2 + a_2 x_1 + u$$

$$\text{(2)} \quad y_2 = b_1 y_1 + b_2 x_2 + b_3 x_3 + v$$

Ia (5 points) Check the order condition for each equation and state whether the condition is satisfied for exact or over identification. Explain your answers.

In the first equation, two variables are missing. This is one more than is needed and hence the equation is overidentified. The second equation has one variable missing and is exactly identified.

Ib(10 points) Explicitly derive the reduced forms for the two endogenous variables. You <u>need not</u> try to solve backwards for the original parameters.

Substituting for y_1 *from the first equation into the second we get,*

$$y_2 = b_1(a_1 y_2 + a_2 x_1 + u) + b_2 x_2 + b_3 x_3 + v$$

This gives $y_2(1-a_1 b_1) = b_1 a_2 x_1 + b_2 x_2 + b_3 x_3 + error$. *Dividing by* $1-a_1 b_1$, *we get*

$$y_2 = \frac{b_1 a_2}{1-a_1 b_1} x_1 + \frac{b_2}{1-a_1 b_1} x_2 + \frac{b_3}{1-a_1 b_1} x_3 + error$$

$$= p_1 x_1 + p_2 x_2 + p_3 x_3 + error$$

Substituting this into the first equation, we have

$$y_1 = a_1(p_1 x_1 + p_2 x_2 + p_3 x_3) + a_2 x_1 + error$$
$$= (a_1 p_1 + a\ ) x1 + a_1 p_2 x_2 + a_1 p_3 x_3 + error$$
$$= m_1 x_1 + m_2 x_2 + m_3 x_3 + error$$

Ic (5+5 points) Describe carefully how you would apply the two-stage least squares to each of the equations.

Equation (1)

In the first stage, regress y_2 against x_1, x_2, and x_3, and save $\hat{y}_2$. Next regress y_1 against $\hat{y}_2$ and x_1, with no constant term.

Equation (2)

In the first stage, regress y_1 against x_1, x_2, and x_3, and save $\hat{y}_1$. Next regress y_2 against $\hat{y}_1$, x_2, and x_3, with no constant term.

II.

The Justice Department hires you to relate the probability of conviction (p_t) of an individual accused of a violent crime as a function of the number of previous convictions (CONVICT) for any crime, race (RACE), income (INCOME), and age (AGE). You have data on these variables and on Y_t, which takes the value 1 with probability p_t if convicted, and 0 with probability $1 - p_t$ if not convicted. You decide to use the Linear Probability Model

$$Y_t = b_1 + b_2 \, CONVICT_t + b_3 \, RACE_t + b_4 INCOME_t + b_5 AGE_t + u_t$$

IIa. (15 points) Treating this as a binomial process rather than as a normal process, carefully show first that for $E(u_t)$ to be zero, $p_t = b_1 + b_2 \, CONVICT_t + b_3 \, RACE_t + b_4 INCOME_t + b_5 AGE_t$. Next show that the variance of u_t is given by $s_t^2 = p_t(1 - p_t)$.

When $Y_t = 1$, we have $u_t = 1 - b_1 - b_2 \, CONVICT_t - b_3 \, RACE_t - b_4 INCOME_t - b_5 AGE_t$, with probability p_t. When $Y_t = 0$, $u_t = - b_1 - b_2 \, CONVICT_t - b_3 \, RACE_t - b_4 INCOME_t - b_5 AGE_t$, with probability $1 - p_t$.

We have $0 = E(u_t) = p_t(1 - b_1 - b_2 \, CONVICT_t - b_3 \, RACE_t - b_4 INCOME_t - b_5 AGE_t) - (1 - p_t)(b_1 + b_2 \, CONVICT_t + b_3 \, RACE_t + b_4 INCOME_t + b_5 AGE_t)$

$= p_t - b_1 - b_2 \, CONVICT_t - b_3 \, RACE_t - b_4 INCOME_t - b_5 AGE_t = 0.$

Solving for p_t we get $p_t = b_1 + b_2 \, CONVICT_t + b_3 \, RACE_t + b_4 INCOME_t + b_5 AGE_t$.

We see that u_t takes the value $1 - p_t$ with probability p_t and the value $- p_t$ with probability $1 - p_t$. Therefore, $s_t^2 = E(u_t^2) = (1 - p_t)^2 p_t + p_t^2 (1 - p_t) = p_t(1 - p_t)(1 - p_t + p_t) = p_t(1 - p_t)$.

IIb (10 points) Describe in full detail how the weighted least squares method can be used to estimate the b s. Ignore any problems that might arise in this procedure.

First regress Y_t against a constant, $CONVICT_t$, $RACE_t$, $INCOME_t$, and AGE_t, and save $\hat{Y}_t$ which is also $\hat{p}_t$. Next compute the variance as $\hat{s}_t^2 = \hat{Y}_t(1-\hat{Y}_t)$. Then compute the weight $w_t = \frac{1}{\sqrt{\hat{s}_t^2}}$. Finally regress $w_t Y_t$ against w_t, $w_t CONVICT_t$, $w_t RACE_t$, $w_t INCOME_t$, and $w_t AGE_t$.

IIc (5 points) Explain what the drawbacks of this method are that make the approach unpopular.

First of all, when you regress Y_t against a constant, $CONVICT_t$, $RACE_t$, $INCOME_t$, and AGE_t, and save $\hat{Y}_t$, there is no guarantee that $0 < \hat{Y}_t < 1$. Second after getting the WLS estimators, there is no guarantee that $\hat{Y}_t$ obtained using the new estimates will give $\hat{Y}_t$ between 0 and 1.

III. (15 points) Describe in full how the Probit Model can be used for the model in II, instead of the linear probability function.

For the probit model, we assume that there is a response function $Y_t^* = b_1 + b_2\ CONVICT_t + b_3\ RACE_t + b_4 INCOME_t + b_5 AGE_t + u_t$. We assume that u_t is not binomial but normal with mean 0 and variance s_t^2. Thus, u_t/s_t is the standard N(0, 1). $Y_t=1$, if $Y_t^* > 0$, and $Y_t= 0$, if $Y_t^* \le 0$. If $Z = u_t/s_t$ is the standard normal and F(z) is $P(Z \le z)$, the cumulative distribution function of the standard normal, then we have

$P(Y_t = 0) = P(Y_t^* \le 0) = P(u_t \le = - b_1 - b_2\ CONVICT_t - b_3\ RACE_t - b_4 INCOME_t - b_5 AGE_t) = P[u_t/s \le (1/s)(- b_1 - b_2\ CONVICT_t - b_3\ RACE_t - b_4 INCOME_t - b_5 AGE_t)]$

$$= F[(1/s_t)(- b_1 - b_2\ CONVICT_t - b_3\ RACE_t - b_4 INCOME_t - b_5 AGE_t)]$$

Call the express inside the function F[], Q. Then $P(Y_t = 0) = F(Q)$.

Using the same procedure, we have $P(Y_t = 1) = P(Y_t^* > 0) = 1 - P(Y_t^* \le 0) = 1 - F(Q)$. The above is a for a typical person. To get the full likelihood function, we multiply these for the two groups of accused, those for whom $Y_t = 1$ and those for whom $Y_t = 0$. The full likelihood function is therefore given by

$$L = \prod_{Y_t=0} F(Q) \prod_{Y_t=1} [1 - F(Q)]$$

where Π denotes the product of terms. L is a function of the unknown b s and s. The parameters are estimated by maximizing this highly nonlinear function with respect to the b s and s. Many standard regression programs, including GRETL, carry out this procedure.