Econ 120A | **Ramu Ramanathan**
Spring 2003 | **Answers to Exam #2**

I.

Let X and Y be two random variables, with means μ_x and μ_y, Var(X) = σ_X^2 = E(X^2) – μ_X^2, Var(Y) = σ_Y^2 = E(Y^2) – μ_Y^2, and Cov (X, Y) = σ_{XY} . Now make the transformations $U = X + Y$, and $V = X - Y$.

(a) (3 points) Derive E(U) and E(V) in terms of μ_X and μ_Y.

$$E(U) = E(X+Y) = E(X) + E(Y) = \mu_X + \mu_Y$$

$$E(V) = E(X - Y) = E(X) - E(Y) = \mu_X - \mu_Y$$

(b) (5 points) Derive E(U V) in terms of μ_X, μ_Y, σ_X^2 and σ_Y^2.
[*Hint*: you want the expected value of U times V. Use the definitions at the top of the page.]

$$E(UV) = E[(X+Y)(X-Y)] = E(X^2 - Y^2) = E(X^2) - E(Y^2)$$

Since $\sigma_X^2 = E(X^2) - \mu_X^2$, *we have,* $E(X^2) = \sigma_X^2 + \mu_X^2$, *similarly for* $E(Y^2)$.

$$E(UV) = \sigma_X^2 + \mu_X^2 - (\sigma_Y^2 + \mu_Y^2)$$

(c) (6 points) Derive Cov (U, V) in terms of σ_X^2 and σ_Y^2 only.

$$Cov(U, V) = E(UV) - [E(U)\,E(V)]$$

$$= \sigma_X^2 + \mu_X^2 - (\sigma_Y^2 + \mu_Y^2) - (\mu_X + \mu_Y)(\mu_X - \mu_Y)$$

$$= \sigma_X^2 + \mu_X^2 - (\sigma_Y^2 + \mu_Y^2) - (\mu_X^2 - \mu_Y^2) = \sigma_X^2 - \sigma_Y^2$$

(d) (4 points) What is the condition under which U and V will be uncorrelated (that is, have zero correlation) and why?

Since uncorrelated means Cov(X,Y) = 0, required condition is $\sigma_X^2 = \sigma_Y^2$.

II.

Let $x_1, x_2, \ldots, x_n$ be a random sample drawn from a population with mean μ and variance σ^2. In other words, E(x_i) = μ, and Var (x_i) = σ^2 for i = 1, 2, …., n., and the x's are all independent of each other. Let $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ be the sample mean.

(a) (4 points) Show that E($\bar{x}$) = μ.

$$E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\, n\, \mu = \mu$$

It is possible to show that Var ($\bar{x}$) = $\frac{\sigma^2}{n}$, *but you need not prove it.*

(b) (4 points) Let Z = $\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}$. Show that E(Z) = 0.

$$E(Z) = E\left(\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}\right) = \frac{\sqrt{n}}{\sigma}E(\bar{x}-\mu) = 0 \text{ because } E(\bar{x}) = \mu.$$

(c) (4 points) Show that Var(Z) = 1.

$$Var(Z) = E\left[\left(\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}\right)^2\right] = \left(\frac{\sqrt{n}}{\sigma}\right)^2 E[(\bar{x}-\mu)^2]$$

$$= \frac{n}{\sigma^2}\, Var(\bar{x}) = \frac{n}{\sigma^2}\,\frac{\sigma^2}{n} = 1$$

III.

X and Y are random variables with means μ_x and μ_y, variances σ_X^2 and σ_Y^2, and covariance σ_{XY}, all fixed constants. Consider the new random variable $U = (Y - \mu_Y) - b(X - \mu_x)$, where b is a fixed constant that I choose. It follows that $U^2 = (Y - \mu_Y)^2 - 2b(X-\mu_x)(Y-\mu_Y) + b^2(X-\mu_x)^2$.

[*Hint*: Do not expand this expression. Keep the terms grouped as they are.]

(a) (4 points) Derive the expected value of U.

$$E(U) = E[(Y-\mu_Y) - b(X-\mu_x)] = E(Y) - \mu_Y - bE(X-\mu_x)$$
$$= E(Y) - \mu_Y - b[E(X) - \mu_x] = \mu_Y - \mu_Y - b[\mu_x - \mu_x] = 0$$

(b) (7 points) Derive σ_U^2, the variance of *U*, in terms of *b*, σ_X^2 , σ_Y^2 and the covariance σ_{XY} .

σ_U^2 = *E[U²] – [E(U)]² = E[U²] because the second term is zero.*

$$= E[(Y - \mu_Y)^2 - 2b(X-\mu_X)(Y - \mu_Y) + b^2(X-\mu_X)^2]$$

$$= E[(Y - \mu_Y)^2] - 2bE[(X-\mu_X)(Y - \mu_Y)] + b^2E[(X-\mu_X)^2]$$

$$= \sigma_Y^2 - 2b\sigma_{XY} + b^2 \sigma_X^2$$

(c) (6 points) Suppose I want to choose *b* in order to minimize σ_U^2 with respect to *b*, *treating everything else as fixed.* Derive the value of *b* that minimizes this variance. [*Hint*: Your answer will depend only on σ_X^2 and σ_{XY}].

$$\frac{\partial \sigma_U^2}{\partial b} = 0 = -2\sigma_{XY} + 2b\ \sigma_X^2 \quad \text{or} \quad b\ \sigma_X^2 = \sigma_{XY}$$

Solving for b, we get $b = \sigma_{XY}/\sigma_X^2$.

(d) (3 points) Be sure to check the second-order condition also.

$\dfrac{\partial^2 \sigma_U^2}{\partial b^2} = 2\ \sigma_X^2$ *which is positive and hence a minimum is attained.*