Pricing with Pre-Scheduled Sales

Maxim Sinitsyn*

Abstract

In this paper, I introduce a framework of price promotions by firms which pre-schedule their sale dates. I set up a dynamic model of demand accumulation in which high- and low-valuation consumers enter the market every period. The high-valuation consumers buy immediately and leave the market, and the low-valuation consumers accumulate while waiting for the sale price. The firms schedule the dates of their promotions in advance. I find that often they employ mixed strategies, choosing the future promotion period according to a probability distribution function. If the firms can cancel their pre-scheduled sales, typically they are able to wait longer until holding sales by shifting the probability distribution towards later periods. Scheduling promotions in advance increases the firms' profits in comparison to the case when the promotion decision is made in the period when the promotion is offered.

1 Introduction

Temporary price promotions or sales remain an important marketing tool used by firms. In addition to continuing interest in this topic in the fields of marketing and industrial organization (see Narasimhan 2009 for a literature review), the behavior of promotional prices has recently attracted a lot of attention in the macroeconomic literature. Since the effects of fiscal and monetary policies crucially depend on the frequency of price adjustments,

*Department of Economics, University of California, San Diego, La Jolla, California, 92093, msinitsyn@ucsd.edu

papers in this literature explore how to properly account for the price variations caused by temporary sales (Nakamura and Steinsson 2008, Eichenbaum et al. 2011, Guimaraes and Sheedy 2011, Coibon et al. 2012, Kehoe and Madrigan 2012, Anderson et al. 2016).

There is an abundance of various theoretical frameworks for modeling sales: static models of price discrimination between consumers with different information about prices (Varian 1980), dynamic models of intertemporal price discrimination between consumers with different product valuations (Sobel 1984) or between consumers with different inventory holding costs (Blattberg et al. 1981), and multi-period models of clearance sales under demand uncertainty (Lazear 1986), just to name a few. However, almost all existing models of sales neglect to incorporate in them an important institutional feature, namely the fact that price promotions are typically set in advance.[1]

Firms often offer temporary price discounts in combination with other elements of the promotional mix such as in-store displays and feature advertising. In anticipation of high demand, it is necessary to stock up on inventory and perhaps train the sales force. All of these activities require preparation and are planned in advance. For example, BeautyMark Marketing, a firm specializing in designing marketing strategies for beauty businesses, advises to start planning promotions three months ahead of time (BeautyMark Marketing 2015). Blattberg and Neslin (1990) provide an example of a manufacturer's promotion calendar which shows the upcoming promotion events for the whole year. Consistent with this example, Anderson et al. (2016) report that the timing of trade promotions is planned up to a year in advance.

What are the firms' optimal promotional strategies if they have to set their sale dates in advance? How are these strategies affected by market characteristics? What is the impact of committing to a future promotion date on the firms' profits? How do firms adjust the strategies if cancellations of pre-scheduled sales are possible? The current paper develops the framework that allows to address these questions.

[1] Notable exceptions are Cui et al. (2008) and Villas-Boas and Villas-Boas (2008), which I discuss later in the introduction.

Following Sobel (1984), I set up a dynamic model of demand accumulation in which high- and low-valuation consumers enter the market every period. The high-valuation consumers buy the product immediately and leave the market, whereas the low-valuation consumers wait for a sale and come back the next period if they do not purchase in the current period. This leads to an accumulation of the low-valuation consumers.

There are two firms in the market. Each period, they could charge either a regular price that targets only the high-valuation consumers or a sale price at which both types of consumers purchase. The sale price, then, clears the market of the accumulated low-valuation consumers. The novel feature of my model is that the firms have to schedule their price promotions in advance. Thus, in the beginning of the first period, each firm chooses a future period, in which it will hold a sale.

In the marketing literature, a commitment to a future price path is explicitly modeled in Cui et al. (2008) who examine a monopolist manufacturer selling to retailers with different inventory costs. The optimal strategy of the manufacturer is to offer the lowest wholesale price in the first period of the pricing cycle and then to gradually increase it for an endogenously determined number of periods. The presence of the commitment to future prices, however, does not drive any of their results since, as the authors show, at no time point does the monopolist have an incentive to deviate from the preannounced prices. In contrast, the presence of the commitment to future prices is crucial in my model since in some periods, a firm would have preferred to keep its price high instead of holding a pre-scheduled sale or it would have preferred to hold a sale in a period, for which it did not schedule one.

Another paper that discusses the effect of a price commitment of a firm selling to different types of consumers is Villas-Boas and Villas-Boas (2008) who consider a monopolist determining the optimal time period between its sales in the presence of consumer learning and forgetting. In their model, each period a fraction of consumers becomes uninformed and forgets about whether or not they like the product. Being unsure about their utility, they do not purchase at a high price that the monopolist charges to capture the surplus from the informed consumers. When a sufficient number of uninformed consumers accumulates, the monopolist drops the price to let these consumers try the product and learn about the fit. In the future periods, the monopolist is able to capture a larger surplus from those who turn out to value the product highly. Villas-Boas and Villas-Boas (2008) argue that the ability to commit to a sequence of prices decreases the interval between sales.

The two models discussed above examined the optimal pricing strategies of a monopolist. In contrast to them, I study the interaction between two competing firms. I start my analysis by introducing a basic finite-horizon framework in which at the start of the first period, the firms commit to a future sale date. I show that the firms choose a sale period with certainty, i.e., a pure-strategy equilibrium exists, only in a two- or a three-period model with the number of low-valuation consumers entering each period being relatively large or the sale price being high. For the model with more than three periods, pure-strategy equilibria do not exist. The firms, then, can employ mixed strategies, choosing the period in which they will hold a sale according to a probability distribution function. The support of this function is bounded by the starting period—the earliest period in which the firms consider holding a sale—and the ending period by which a sale occurs with certainty. The starting period is chosen by the firms and is endogenous. In the finite-horizon model, the ending period is the last period. For all the periods in-between, the probability that a sale occurs in that period must be specified. These probabilities sum up to one. To satisfy the requirement of a mixed-strategy equilibrium, the resulting probability distribution must be chosen in such a way that each firm's ex ante expected profit is the same regardless of which period was selected for a price promotion.

Additional requirements are necessary to make the optimal strategies deviation-proof. First, the firms should not deviate to always charging the regular price and not scheduling a sale at all. Second, the firms should not deviate to scheduling a sale before the starting period. I find that for any number of periods in the model, the starting period and the corresponding probability distribution over sale periods are uniquely determined. This probability distribution depends only on the number of periods in the model, and not on the sale price or the sizes of the consumer segments. The expected per-period profit increases with the sale price and with the share of high-valuation consumers.

I consider several extensions of the basic framework. First, I examine the possibility of the firms being able to cancel their pre-scheduled sales. For the vast majority of model parameters, the ability to cancel sales leads the firms to shift the sale probabilities towards later periods. This results in the expected per-period profit being higher than the one in the original model without sale cancellations.

The second extension considers the possibility of the firms varying the sale price. Thus, in addition to selecting a sale period, the firms also choose a corresponding sale price for that period. In this variant of the model, the low-valuations consumers have heterogeneous tastes for the firms' products. I find that the firms charge lower sale prices in the earlier periods. When consumer heterogeneity is low, in addition to choosing a sale period randomly, the firms also randomize between sale prices.

Finally, in the concluding section, I consider an infinite-horizon model in which the firms make repeated sale scheduling decisions. In this model, the firms choose both the starting and the ending period for the sale probability distribution. I find that when the firms commit to a future sale date, they are able to increase their profits in comparison to the model in which the promotion decision is made in the period when the promotion is offered.

2 Basic Framework with Commitment

In this section, I set up a basic discrete-time dynamic framework with two firms competing in prices and specify how to incorporate pre-scheduling of sales into the firms' strategies. The demand side of the market is similar to Sobel (1984).[2] Each period, $2H + L$ consumers enter the market. There are $2H$ high-valuation loyal consumers who are willing to pay the

[2] Pesendorfer (2002) provides the empirical evidence for the assumptions and predictions of Sobel (1984) type models.

regular price for the product. They always buy from their favorite firm. Each firm has H loyal consumers. I normalize the regular price to 1.[3]

The remaining L consumers are switchers who have a low valuation for the product. They buy the product only if it is offered at a discounted price $p < 1$. The low-valuation consumers can purchase from either firm, but they incur a small search cost $\varepsilon > 0$ when they sample a price from an additional firm. As shown by Diamond (1971), the equilibrium search behavior of these consumers and the corresponding pricing strategies of the firms are as follows. The firms' optimal strategy when they want to target the low-valuation consumers is to charge exactly p. Correspondingly, when a low-valuation consumer encounters the sale price p, he knows that the rival firm will not offer a lower price, so he does not search. Knowing that the low-valuation consumers do not search after encountering price p, a firm that decides to target these consumers, has no incentives to lower its price below p. Hence, in equilibrium, the firms will either charge the regular price 1 or the sale price p. This allows me to focus on the timing of promotions rather than on the discount depth. In Section 6, I consider the low-valuation consumers who have zero search costs and show how to incorporate into the model the possibility of competition in sale prices.

The strategy of a low-valuation consumer is then to check the price of a random firm. If it is p, the consumer purchases the product from that firm. If it is 1, the consumer checks the price of another firm. If it is p, the consumer makes a purchase. If it is also 1, i.e., neither firm holds a sale, the low-valuation consumers return next period and repeat their search. Therefore, the number of low-valuation consumers present in the market is given by nL, where n is the number of periods since the last sale held by either firm.

In contrast to Sobel's (1984) environment with contemporaneous pricing decisions, the current paper considers the framework with the firms setting their sales in advance. The issue of pre-scheduling sale dates and committing to them is central to the paper and warrants an elaboration. Anderson et al. (2016) report that in the consumer packaged goods industry, the schedule of sales and associated promotional activities is specified up

[3] At the same time, both firms' marginal costs are normalized to 0.

to a year in advance. To gain additional insight into the planning of price promotions, I interviewed several managers who were involved in making promotion decisions for the firms in three industries: consumer electronics, processed meats, and cookies. There are some differences between the industries, but the key features are common.

First, promotions are scheduled well in advance with the lead time ranging from one quarter to one year. This is necessary to allow sufficient time for planning and coordinating between multiple parties involved in a promotion. Frequently, promotions are accompanied by media advertising. It has to be purchased six to eight weeks in advance. A firm needs to increase its production in anticipation of high demand. Depending on the industry, the lead time on production increase ranges from four-six weeks to six months. In a vertical relationship, a promotion needs to be coordinated between a manufacturer and a retailer with agreements on additional marketing mix elements, such as in-store displays and feature advertising.

After scheduling their promotions, firms do retain some flexibility over them. The adjustments they make occur mostly due to competitor moves and changes in demand. However, these modifications usually involve changing only the discount depth, but not the weeks for which sales are scheduled. In some industries, even the decision about the sale price has to be done well in advance. For example, in the consumer packaged goods industry, the authors of Anderson et al. (2016) reported that when they were involved in manipulating the discount depth on a sample of products, their lead time was almost four months.

Having established that an effective price promotion requires a significant amount of advance planning, another question is whether it is possible to cancel a scheduled promotion if a firm realizes that it will not generate sufficient sales. It turns out that even though cancellations are possible, they are costly, and firms try to avoid them. One of the costs associated with a sale cancellation is the inventory cost. Although an undesirable sale is not going to generate the expected lift, it would still move more inventory than a regular price. Thus, cancelling a sale could impose significant inventory costs. These costs increase in importance if a product is perishable. Another cost of sale cancellations is the deterioration of a long-term relationship with retailers who had this promotion in their plans. In fact, an example of an actual sale cancellation given to me stemmed from logistics: the disruptions in the supply chain led to an insufficient inventory for holding a sale, so the firm had no choice but to cancel it. Given the sizeable costs associated with cancelling a pre-scheduled sale, in my basic framework, I assume that sales can not be cancelled. In section 5, I examine how the optimal strategies of the firms change if cancellations are possible.

In summary, the conversations with the managers confirmed that in order to run an effective price promotion, it is often necessary to plan it well in advance. To model this feature of real-life sales, I assume that each firm chooses a future sale date at the start of the first period. The firm's strategy, then, is given by $P_{t,T} = (0, \ldots, 0, \delta_t, \delta_{t+1}, \ldots, \delta_T)$, where δ_s is the probability of a sale in period s, t is the first period in which a sale is conducted with a positive probability, and T is the period by which a firm holds a sale with certainty. I will use the term "planning horizon" to refer to the set of periods $\{t, t+1, \ldots, T-1, T\}$. I assume that each firm commits to exactly one sale date before T; thus $\delta_t + \delta_{t+1} + \ldots + \delta_T = 1$.[4] In all periods before t, the firms charge the regular price.

As an illustration, in a 4-period game ($T = 4$) with the planning horizon starting in the second period ($t = 2$), each firm chooses between three potential price paths: $(1; p; 1; 1)$, $(1; 1; p; 1)$, or $(1; 1; 1; p)$. The following table depicts a normal form representation of this game.[5]

Table 1: Normal Form Representation of a Game with Pre-Scheduled Sales for *t=2 and T=4*

	$(1; p; 1; 1)$	$(1; 1; p; 1)$	$(1; 1; 1; p)$
$(1; p; 1; 1)$	$\cdot, \cdot$	$\cdot, \cdot$	$\cdot, \cdot$
$(1; 1; p; 1)$	$\cdot, \cdot$	$\cdot, \cdot$	$\cdot, \cdot$
$(1; 1; 1; p)$	$\cdot, \cdot$	$\cdot, \cdot$	$\cdot, \cdot$

[4] The possibility of pre-scheduling multiple sale dates is discussed in Section 7.

[5] The table does not contain the payoffs since its purpose is to illuminate the structure of the strategies.

Then, for example, the mixed strategy $(0; 0.2; 0.3; 0.5)$ represents the following action. At the start of the first period, the firm schedules its sale for the second period with probability 0.2, it schedules its sale for the third period with probability 0.3, and with the remaining probability 0.5, it schedules its sale for the last period. Note that the realization of this mixed strategy is a fixed sale date in exactly one future period.

How is the ending period of the planning horizon, T, chosen? It likely depends on the specific type of promotions we are trying to model. For periodic sales by supermarkets, it is reasonable to assume that T is endogenous, optimally chosen by the firms. I examined an infinite-horizon model, in which the firms choose T, in the online appendix. The findings from this model are presented in the concluding section.

Another possibility to consider is that T is exogenous, and the firms have to hold their sales before period T. This model represents, for example, clearance sales that take place before the end of a fashion season.[6] Thus, the basic set-up I examine in the paper is the finite-horizon model with T periods. In the beginning of the first period, each firm chooses a period in which it will hold a sale. The firms maximize their ex ante expected sum of profits from all T periods. I will search for symmetric Nash equilibria (NE) of this game.

Before solving the general version of this model, in the next section, I present an illustrative example that captures the main issues accompanying the firms' choice of a future promotion date.

3 Illustrative Example

In this section, I present a simple finite-horizon example, for which I explain how to compute the firms' optimal strategies and check that there are no deviations. I normalize H to 1 and set L equal to $2/3$. The regular price is 1, and the sale price is 0.5. Initially, I consider a three-period model, i.e., $T = 3$.

[6] Regardless of whether T is exogeneous or endogeneous, the choice of the starting period of the planning horizon, t, is endogenous.

In the beginning of the first period, each firm chooses a period in which it will hold a sale. It is easy to show that a symmetric pure-strategy NE does not exist for these parameters of the model. The firms can not both choose either period 1 or period 2 for their sales because the demand is not large enough to support two firms holding a sale. At the same time, both firms can not schedule a sale for period 3 since there is a profitable deviation for one of the firms to reschedule its sale for the second period. Note that a symmetric pure-strategy NE also does not exist for $T > 3$ since when both firms schedule a sale for the same period $s > 2$, each has an incentive to deviate to holding a sale in period $s - 1$.

Additionally, there are no asymmetric pure-strategy equilibria. First, consider the case when one firm never holds a sale and charges the reservation price every period. Then, the second firm's optimal strategy is to wait for the largest number of low-valuation consumers and hold a sale in period 3. Given this strategy, the first firm would like to deviate and schedule a sale for the second period. Thus, there are no pure-strategy equilibria with one of the firms never holding a sale. Now, consider the case of the firms choosing different periods for their sales. If there exists at least one extra period between the scheduled sales, a firm with the earlier sale date would like to move its sale date later, just one period before the sale date of the rival. If the sales are scheduled for consecutive periods, the firm with the later sale date gets the minimum amount of low-valuation consumers, which makes the sale unprofitable. Thus, this firm has a profitable deviation to charging the reservation price in every period. Therefore, asymmetric pure-strategy equilibria also do not exist. The same argument applies to the game with any number of periods.

In the absence of pure-strategy equilibria, we need to search for a mixed-strategy equilibrium. I start by noting that no firm should place a positive probability on scheduling a promotion for the first period since even if it were the only firm offering a promotion in that period, it would earn less than 1. Therefore, I will solve for the strategies of the type $P_{2,3} = (0, \delta_2, 1 - \delta_2)$. In this equilibrium, δ_2 is determined in such a way that the sum of the profits from all three periods is the same regardless of whether the firm sets a sale date for period 2 or for period 3. In all periods except for the sale period, the firm charges the regular price and earns profit 1; therefore, it is only necessary to equate the expected profits from the periods in which the firm holds a sale. If the focal firm holds a sale in the second period, its rival offers a sale in the same period with probability δ_2, and the low-valuation consumers are split between the firms. With probability $1-\delta_2$, the rival charges the regular price, and all 4/3 low-valuation consumers are captured by the focal firm. Thus, the focal firm's expected profit from a sale in the second period is $0.5(\delta_2(1+2/3)+(1-\delta_2)(1+4/3)) = 0.5(7/3-2\delta_2/3)$. If the focal firm holds a sale in the third period, its rival held a sale in the previous period with probability δ_2, so the focal firm captures only 2/3 new low-valuation consumers. With probability $1-\delta_2$, the rival holds a sale in the same period and both firms get 3/3 low-valuation consumers. Then, the focal firm's expected profit from a sale in the third period is $0.5(\delta_2(1+2/3)+(1-\delta_2)(1+1)) = 0.5(2-\delta_2/3)$. The profit from scheduling a sale for the second period is equal to the profit from scheduling a sale for the third period when $\delta_2 = 1$, i.e., when both firms schedule a sale for the second period with certainty. However, I have already established that the firms will not follow such strategy since each would rather charge the regular price and sell only to its high-valuation consumers. This means that in the three-period model, there are no mixed strategy equilibria with the firms committing to holding a sale.[7]

Now, consider the four-period model, i.e., $T = 4$. I will solve for the strategies of the type $P_{2,4} = (0, \delta_2, \delta_3, 1-\delta_2-\delta_3)$. As argued above, since the firms earn 1 in all periods in which they charge the regular price, in order to solve for the mixed-strategy equilibrium, it is only necessary to equate the expected profits from the sale periods. The focal firm's expected profit from holding a sale in the second period is $\pi_2 = 0.5(\delta_2(1+2/3)+(1-\delta_2)(1+4/3)) = 0.5(7/3-2\delta_2/3)$. If the focal firm schedules a sale for the third period, with probability δ_2,

[7] I show in the online appendix that the only equilibrium that exists for this case is the one where the firms schedule a sale for the second period with probability 1/2, a sale for the third period with probability 1/3, and with the remaining probability 1/6 do not hold a sale at all. Since this mixed strategy includes the possibility of never holding a sale, its expected per-period profit is one. In the paper, I will not study such equilibria, focusing instead on the ones in which holding a sale dominates the strategy of always charging the regular prices.

the rival held its sale in the previous period, so the focal firm sells to 2/3 new low-valuation consumers. With probability δ_3, the rival also holds a sale in the third period, so the low-valuation consumers are split, and the focal firm sells to 3/3 of them. With the remaining probability $1 - \delta_2 - \delta_3$, the rival holds a sale in the next period, so the focal firm gets all 6/3 low-valuation consumers. The expected profit from holding a sale in the third period is, then, $\pi_3 = 0.5(\delta_2(1 + 2/3) + \delta_3(1 + 3/3) + (1 - \delta_2 - \delta_3)(1 + 6/3)) = 0.5(3 - 4\delta_2/3 - \delta_3)$.

When the focal firm holds a sale in the fourth period, if the rival held a sale in period 2, the focal firm sells to 4/3 low-valuation consumers, and if the rival held a sale in period 3, the focal firm sells to 2/3 low-valuation consumers. If the rival also holds a sale in the fourth period, the low-valuation consumers are split, and the focal firm sells to 4/3 of them. The expected profit from holding a sale in the fourth period is, then, $\pi_4 = 0.5(\delta_2(1 + 4/3) + \delta_3(1 + 2/3) + (1 - \delta_2 - \delta_3)(1 + 4/3)) = 0.5(7/3 - 2\delta_3/3)$.

Equating the expected profits π_2 and π_4, we get $\delta_2 = \delta_3$. Using this equality and equating π_2 and π_3, we obtain $\delta_2 = 2/5$. Thus, the solution is $(0, 2/5, 2/5, 1/5)$. Since $\pi_2 = \pi_3 = \pi_4$, we can use π_2 to find the expected profit in a sale period: $\pi_2 = 0.5(7/3 - 2\delta_2/3) = 0.5(7/3 - 4/15) = 31/30 \approx 1.03$. This profit is greater than one; therefore, the firms commit to a future sale date and do not deviate to charging the regular price in every period.

It is also necessary to consider whether it is possible to have a planning horizon that starts in the third period, i.e., $P_{3,4} = (0, 0, \delta_3, 1 - \delta_3)$. The focal firm's expected profit from holding a sale in period 3 is $\pi_3 = 0.5(\delta_3(1 + 3/3) + (1 - \delta_3)(1 + 6/3)) = 0.5(3 - \delta_3)$. The focal firm's expected profit from holding a sale in period 4 is $\pi_4 = 0.5(\delta_3(1 + 2/3) + (1 - \delta_3)(1 + 4/3)) = 0.5(7/3 - 2\delta_3/3)$. Equating these expected profits, we obtain $3 - \delta_3 = 7/3 - 2\delta_3/3$, from where $\delta_3 = 2$. This means that it is not possible to have a planning horizon starting in period 3, and the only equilibrium strategy for the four-period model is $P_{2,4} = (0, 2/5, 2/5, 1/5)$.

Now, consider the five-period model, i.e., $T = 5$. First, examine the strategies with the planning horizon starting in the second period $P_{2,5} = (0, \delta_2, \delta_3, \delta_4, 1 - \delta_2 - \delta_3 - \delta_4)$. The solution for this case is $(0, -2/13, 10/13, -1/13, 6/13)$, and it includes negative probabilities.[8] Thus, a strategy of the type $(0, \delta_2, \delta_3, \delta_4, 1-\delta_2-\delta_3-\delta_4)$ can not be used by the firms. Then, instead of starting their planning horizon in the second period, the firms can start it in the third period. Consider $P_{3,5} = (0, 0, \delta_3, \delta_4, 1 - \delta_3 - \delta_4)$. A NE that uses such strategies exists, and the equilibrium strategies are $(0, 0, 10/17, 1/17, 6/17)$. The corresponding expected profit in the sale period is $41/34 \approx 1.21$. Since with these strategies, the firms do not hold a sale in the second period, it is necessary to check that they do not have a profitable deviation to charging the sale price in the second period. Indeed, if one firm deviates to holding a sale in the second period, it will sell to all 4/3 low-valuation consumers and will earn $0.5(1 + 4/3) = 7/6$. This is smaller than 41/34; therefore, the firms do not have an incentive to deviate to holding a sale before the start of the planning horizon. It is easy to verify that the strategy of the type $P_{4,5} = (0, 0, 0, \delta_4, 1 - \delta_4)$ does not exist for the five-period model; thus, the only equilibrium strategy is $P_{3,5} = (0, 0, 10/17, 1/17, 6/17)$.

Similar analysis of the six-period model reveals that the only equilibrium strategy is $P_{3,6} = (0, 0, 10/21, 3/12, 6/21, 2/21)$. It gives the firms the expected profit of $53/42 \approx 1.262$ in a sale period. In the next section, I derive the closed-form expressions for the equilibrium strategies in the basic model for any exogenous T.

4 Equilibrium Strategies in the Basic Model

In this section, I examine the outcome of a finite-horizon model with $T \geq 2$ periods. In the beginning of the first period, each firm chooses a period in which it will hold a sale. This is the only decision the firms make in this model. First, I consider the case of both firms choosing their sale periods with certainty. The set of model parameters for which a pure-strategy symmetric NE exists is established in the following lemma.[9]

[8] This and the other equilibrium strategies presented in the rest of this section were calculated using the formulas derived in the next section.

[9] All proofs are in the Appendix.

Lemma 1 *There exists a pure-strategy symmetric NE for the finite-horizon model with pre-scheduled sales if and only if* $p \geq \frac{H}{H+L}$ *and* $T \in \{2, 3\}$. *The sale is scheduled for the second period.*

This lemma shows that a pure-strategy NE exists only for the two-period and the three-period models if the number of low-valuation consumers L is relatively high and/or the sale price p is large. If a pure-strategy NE does not exist, it is necessary to search for the mixed-strategy NE.

Assume that there exists a symmetric mixed-strategy NE in which each firm uses the strategy $P_{t,T} = (0, \ldots, 0, \delta_t, \ldots, \delta_T)$ with a positive probability of a sale in any period within the planning horizon $\{t, \ldots, T\}$. One of the requirements of this mixed-strategy NE is that a firm should be indifferent between scheduling its sale for any period between t and T. This means that the expected profits from holding a sale in any period between t and T should be equal. The following lemma shows that in this case, the sale probabilities are connected by a simple recursive formula.

Lemma 2 *In any symmetric mixed-strategy NE, if a firm's expected profits in periods* s *and* $s+1$ *are equal, then the sale probabilities in those periods are linked by the following recursive formula:*

$$\delta_{s+1} = \frac{2}{s+1} - \delta_s \frac{s}{s+1}. \tag{1}$$

Using (1), it is possible to express all sale probabilities in terms of the sale probability in the starting period of the planning horizon, δ_t, as shown in the following lemma.

Lemma 3 *In any symmetric mixed-strategy NE, if a firm's expected profits are equal in all periods between* t *and* T, *then the sale probability in any period* $s \geq t$ *can be defined through* δ_t *in the following way:*

$$\begin{aligned} \textit{if } s-t \textit{ is odd, then } \delta_s &= \frac{2}{s} - \delta_t \frac{t}{s}; \\ \textit{if } s-t \textit{ is even, then } \delta_s &= \delta_t \frac{t}{s}. \end{aligned} \tag{2}$$

It is necessary for all sale probabilities to be non-negative. The next lemma outlines the restrictions on δ_t that guarantee that. Conveniently, these restrictions also ensure that the firms do not have incentives to deviate from their equilibrium strategies by holding a sale in any period before t.

Lemma 4 *The sale probabilities defined in (2) are non-negative if and only if* $\delta_t \geq 0$ *and* $\delta_t \leq 2/t$. *The latter restriction also ensures that the firms do not schedule a sale for any period before* t.

Equations (2) define all sale probabilities in terms of δ_t. Since these probabilities have to sum up to one, it is possible to obtain the closed-form expression for δ_t as shown in the next lemma.

Lemma 5 *The sale probabilities defined in (2) sum up to* 1 *if and only if the sale probability in the starting period of the planning horizon,* δ_t, *is given by the following formula:*

$$\begin{aligned} \textit{if } T-t \textit{ is odd, then } \delta_t &= \frac{1-2\left(\frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T}\right)}{t\left(\frac{1}{t(t+1)}+\frac{1}{(t+2)(t+3)}+\ldots+\frac{1}{(T-1)T}\right)}; \\ \textit{if } T-t \textit{ is even, then } \delta_t &= \frac{1-2\left(\frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T-1}\right)}{t\left(\frac{1}{t(t+1)}+\frac{1}{(t+2)(t+3)}+\ldots+\frac{1}{(T-2)(T-1)}+\frac{1}{T}\right)} \end{aligned} \tag{3}$$

Since (3) gives the closed-form expression for δ_t, it is possible to write out the conditions from Lemma 4 ($\delta_t \geq 0$ and $\delta_t \leq 2/t$) in terms of t and T. These conditions are given in the next lemma and ensure that all probabilities are non-negative.

Lemma 6 *The sale probabilities defined in (2) are non-negative if and only if the following conditions are satisfied:*

$$\begin{aligned} \textit{if } T-t \textit{ is odd, } \quad \frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T} &\leq \frac{1}{2} \leq \frac{1}{t}+\frac{1}{t+2}+\ldots+\frac{1}{T-1}; \\ \textit{if } T-t \textit{ is even, } \quad \frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T-1} &\leq \frac{1}{2} \leq \frac{1}{t}+\frac{1}{t+2}+\ldots+\frac{1}{T}. \end{aligned} \tag{4}$$

Inequalities (4) can be used to identify the starting period of the planning horizon. The following proposition shows that this starting period always exists and is unique.

Proposition 1 *For any given T, there exists a unique value of t that satisfies the conditions of Lemma 6.*

Taken together, we can use the following procedure to solve for the strategies that guarantee that a firms' expected payoffs from holding a sale in any period within the planning horizon are equal. For any T, it is possible to use (4) to find the starting period of the planning horizon t. As the previous proposition indicates, there will be only one such value of t. Then, using (3), we can find the sale probability in period t. Finally, using (2), we can find the sale probabilities in all periods. It is interesting to note that these probabilities depend only on the starting and the ending periods of the planning horizon, t and T, and do not depend on the other parameters of the model: p, H, and L. Since t is uniquely determined by T, the equilibrium strategies depend only on the number of periods in the model, T. The following table shows these strategies for $T \leq 13$.

Table 2: Equilibrium Strategies for $T \leq 13$

t (Starting period)	T (Ending period)	Strategy
2	4	$(0; 0.4; 0.4; 0.2)$
3	5	$(0; 0; 0.59; 0.06; 0.35)$
3	6	$(0; 0; 0.48; 0.14; 0.29; 0.1)$
3	7	$(0; 0; 0.21; 0.34; 0.13; 0.23; 0.09)$
4	8	$(0; 0; 0; 0.4; 0.08; 0.26; 0.06; 0.2)$
4	9	$(0; 0; 0; 0.26; 0.19; 0.18; 0.14; 0.13; 0.11)$
4	10	$(0; 0; 0; 0.12; 0.3; 0.08; 0.22; 0.06; 0.17; 0.05)$
5	11	$(0; 0; 0; 0; 0.28; 0.1; 0.2; 0.07; 0.16; 0.06; 0.13)$
5	12	$(0; 0; 0; 0; 0.14; 0.21; 0.1; 0.16; 0.08; 0.13; 0.07; 0.11)$
5	13	$(0; 0; 0; 0; 0.07; 0.28; 0.05; 0.21; 0.04; 0.17; 0.03; 0.14; 0.03)$

The length of the planning horizon, $T-t$, weakly increases as T increases. This happens because with an increase in T, all terms in the sum in the right-hand side of inequalities (4) get smaller, so more of these terms are required in order for the sum to exceed 0.5. In fact, it is possible to evaluate the length of the planning horizon using the following approximation. In order to satisfy (4), as T becomes large, $\frac{1}{t} + \frac{1}{t+2} + \ldots + \frac{1}{T}$ gets infinitely close to 1/2.

So, $\frac{1}{2} \approx \frac{1}{t} + \frac{1}{t+2} + \ldots + \frac{1}{T} \approx \frac{1}{2}\int_t^T \frac{1}{x}dx = \frac{1}{2}(\ln(T) - \ln(t))$, from where $\ln(T) - \ln(t) \approx 1$, which means that $T/t \approx e \approx 2.7183$. Therefore, t approaches $T/2.7183$ as T increases. For example, $t = 100$ for $T = 270$ and $t = 1000$ for $T = 2718$.

So far, we have constructed a set of strategies that satisfy two conditions: 1) in the beginning of period 1, when pre-scheduling its sale, each firm is indifferent between holding a sale in any period within the planning horizon; and 2) the firms do not deviate to holding a sale before the start of the planning horizon, i.e., before period t.

There is another condition that needs to be satisfied in order to ensure that there are no profitable deviations. By committing to a future sale date, the firms must earn an expected profit of at least H per period. If this were not the case, the firms could earn a higher profit by charging the regular price each period and not offering promotions. Since the firms earn exactly H in each period they charge the regular price, it is necessary to check that the expected profit in the sale period is at least H. The expected profit from holding a sale is the same regardless of which period between t and T is chosen for a promotion. Thus, this no-deviation condition is satisfied by checking, for example, that the expected profit in the starting period of the planning horizon, t, is at least H:

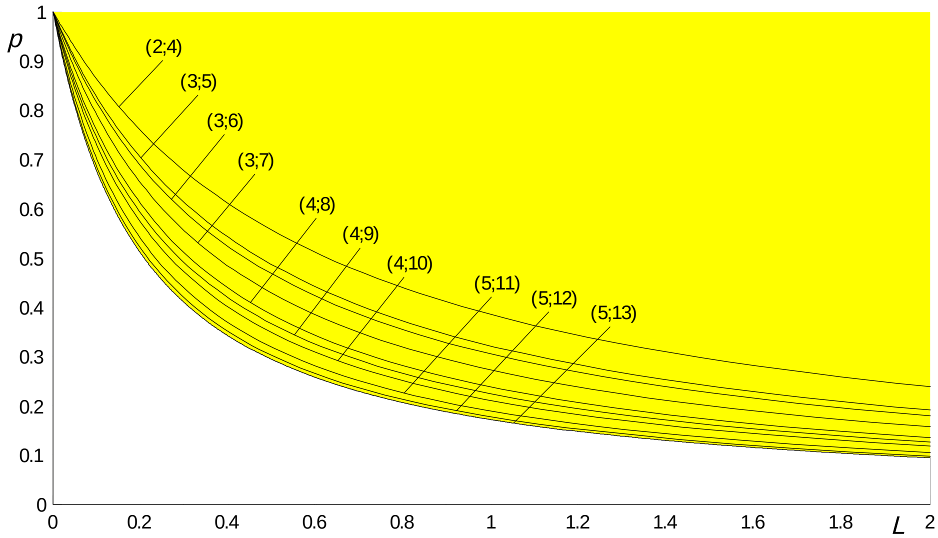
$$\pi_t = p\left(H + \delta_t \frac{tL}{2} + (1 - \delta_t)\, tL\right) \geq H. \tag{5}$$

In the analysis that follows, it is convenient to normalize H to 1. Thus, we treat L as the ratio of low-valuation to high-valuation consumers. Given that the regular price is also 1, this normalizes the per-period profit from charging the regular price to 1. Then, (5) can be rewritten as $p\left(1 + tL - \delta_t \frac{tL}{2}\right) \geq 1$, which can be further simplified to

$$p \geq \frac{2}{2 + tL(2 - \delta_t)}. \tag{6}$$

Each strategy $P_{t,T}$ has a different value for δ_t given by (3). These values need to be substituted into (6). The restrictions imposed by inequalities (6) can be seen on a graph, where L is on the horizontal axis and p is on the vertical axis. Figure 1 graphs the equations $p = \frac{2}{2+tL(2-\delta_t)}$ using the values of t and δ_t from the strategies $P_{t,T}$ for $T \leq 13$.

Figure 1: Regions, in which the firms earn at least H in the sale period



Note. The starting and the ending periods of the planning horizon are in parentheses.

For each $P_{t,T}$, the firms will earn at least 1 during a sale period when the model parameters fall into the region above the curve $p = \frac{2}{2+tL(2-\delta_t)}$. We can observe from Figure 1 that $\frac{2}{2+tL(2-\delta_t)}$ decreases when T increases. This reflects the fact that it is easier for the firms to earn a profit of at least 1 when they can schedule the promotions in the later periods that reach a larger number of low-valuations consumers. Additionally, an increase in T leads to an increase in t. It is easier for the firms to earn a profit of at least 1 when the planning horizon starts later since there are more accumulated low-valuation consumers. In fact, since δ_t is bounded by 1 and t increases unboundedly with T, $\frac{2}{2+tL(2-\delta_t)}$ converges to zero as T increases. Therefore, a deviation-proof solution exists for any parameters of the model as long as T is large enough.

Lastly, we calculate the expected per-period profit the firms earn when they pre-schedule sales. The expected profit from a period with a sale is identical for all periods within the planning horizon, and for period t, it is given in (5) as $\pi_t = p\left(H + \delta_t \frac{tL}{2} + (1-\delta_t)\, tL\right) = p\,(H + tL(1-\delta_t/2))$. The value of δ_t is obtained from (3), and it does not depend on the values of p and L. The profit in all other $T-1$ periods is H, so the firm's expected per-period profit is $\pi = \frac{(T-1)H+\pi_t}{T} = \frac{(T-1)H+p(H+tL(1-\delta_t/2))}{T}$.

Not surprisingly, π increases with p, i.e., the higher is the sale price that captures the low-valuation consumers, the higher are the profits. To examine the effect of the composition of consumers on π, I set the market size (number of consumers entering each period), $2H + L$, to M. Then, $L = M - 2H$. Substitute it into π to obtain $\pi = \frac{(T-1)H+p(H+t(M-2H)(1-\delta_t/2))}{T}$. The derivative of this profit with respect to H is equal to $\frac{1}{T}(T-1+p(1-2t(1-\delta_t/2)))$, which is greater than zero for all strategies from Table 1 regardless of the value of p. Predictably, the firms benefit from a larger share of high-valuation consumers.

For this basic setup, I assumed that each firm makes a decision about the timing of its sale once in the beginning of period 1 and does not have any decisions to make thereafter. In the next section, I examine how the equilibrium strategies change if the firms have the flexibility to cancel pre-scheduled sales.

5 Possibility of Cancelling Pre-Scheduled Sales

For the strategies found in the previous section, the ex ante expected profits from holding a sale in any period within a planning horizon are identical and are larger than H. However, the realization of this strategies might lead to a firm holding an "unprofitable" sale that results in a profit smaller than H. For example, if a firm scheduled its sale for period s, but in period $s-1$, the rival held a sale, the scheduled sale of the focal firm generates a profit of only $p(H+L)$, which, for a large set of model parameters, is smaller than H. In that case, the firm would prefer to cancel its sale and charge the regular price in period s. In Section 2, I discussed the potential costs associated with cancelling a sale. In this section, I assume that these costs are sufficiently small to allow sale cancellations.[10]

The timing of the model with the possibility of sale cancellations is as follows. At the

[10] Whereas a firm can cancel a sale, it can not offer another sale date in the future. Allowing firms to both cancel and reschedule sale dates would make the model equivalent to Sobel's (1984) contemporaneous pricing setup.

start of the first period, each firm chooses a period in which it will hold a sale. Afterwards, in any period s, each firm, after observing the price charged by the rival in this period, has an option of cancelling its scheduled sale.

Take a firm that has its sale pre-scheduled for some period s. Consider the possibility that in period $k < s$, it observes that the rival firm held its sale. The focal firm knows now the exact profit from its sale in period s. There will be $s - k$ accumulated low-valuation consumers, and the profit in that period will be $p(H + (s - k)L)$. If this profit is less than H, the focal firm would be better off cancelling its sale. The inequality $p(H+(s-k)L) < H$ is equivalent to

$$s - k < \frac{H}{pL}(1 - p). \tag{7}$$

This means that if the difference between the period in which the rival held its sale and the pre-scheduled sale period of the focal firm is less than $\frac{H}{pL}(1 - p)$, the focal firm cancels its sale and charges the regular price.

Take c to be the largest integer that is smaller than $\frac{H}{pL}(1-p)$. Then, a firm would cancel its sale scheduled for period s if and only if the rival held its sale within c periods of period s. The following lemma specifies the recursive formula that connects the sale probabilities.

Lemma 7 *In the finite-horizon model with sale cancellations, in any symmetric mixed-strategy NE, if a firm's expected profits in periods s and $s + 1$ are equal, then the sale probabilities in these periods are linked by the following recursive formula:*

$$\delta_{s+1} = \frac{2}{s+1}\left(1 - \delta_{s-c}\left(A - c\right) - \delta_{s-c+1} - \delta_{s-c+2} - \ldots - \delta_{s-1} + \delta_s\left(A - 1 - \frac{s}{2}\right)\right), \tag{8}$$

where $A = \frac{H}{pL}(1 - p)$ and c is the largest integer satisfying $c < A$.[11]

Given the probability of a sale in the starting period of the planning horizon, δ_t, it is possible to express all subsequent sale probabilities in terms of it. Setting the sum of these probabilities to one allows us to solve for δ_t. There is no closed-form solution for δ_t;

[11] If $c = 1$, the formula is ambiguous as there are two different terms containing δ_{s-1}. As could be seen from the proof, the proper relationship for this case is $\delta_{s+1} = \frac{2}{s+1}\left(1 - \delta_{s-1}\left(A - 1\right) + \delta_s\left(A - 1 - \frac{s}{2}\right)\right)$.

therefore, I solve for the equilibrium strategies numerically. After confirming that all sale probabilities are non-negative, I verify that the firms do not have a profitable deviation to holding a sale in period $t-1$.

Unlike the equilibrium strategies in the previous section, the strategies in the model with a possibility of sale cancellations do depend on parameters p, L, and H. A comparison of these strategies to the strategies from the model without sale cancellations is easier if instead of analyzing the differences in the probability distributions, we examine only the change in the probability of a sale in the starting period of the planning horizon, δ_t. There is a negative relationship between δ_t and the expected profit in a sale period. A larger δ_t implies a larger probability of having a joint sale and splitting the low-valuation consumers with the rival in period t. This decreases the expected profit from holding a sale in period t.[12] Thus, in order to determine the effect of sale cancellations on the expected profit, it is enough to examine the change in δ_t.

To follow the example in Section 3, I will fix H at 1, L at 2/3, and will allow p to vary. Re-write $c < \frac{H}{pL}(1-p)$ as $\frac{H}{pL} > \frac{H+cL}{L}$, from where

$$p < \frac{1}{1+cL/H}. \tag{9}$$

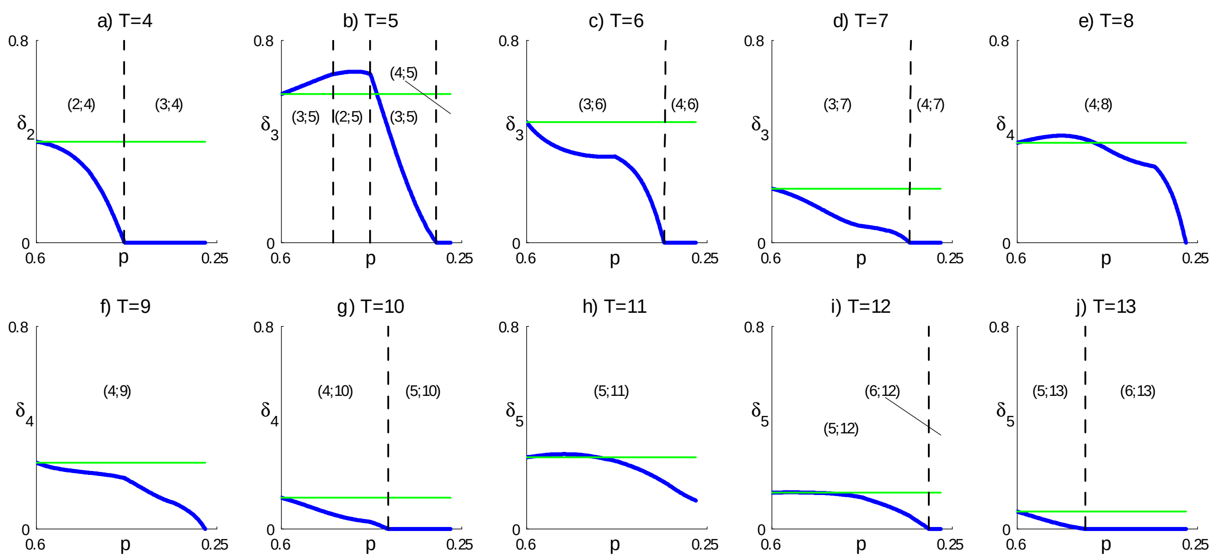
Consider $c = 1$ and take the value of p just high enough so that the firm is indifferent between charging the regular price and holding a sale given that the rival held a sale in the previous period. From (9), this price is $1/(1+L/H)$. For this value of p, constant A from Lemma 7 is $A = \frac{H(1+L/H)}{L} - \frac{H}{L} = 1$. The recursive formula from (8) becomes $\delta_{s+1} = \frac{2}{s+1}\left(1 - \delta_{s-1} \times 0 + \delta_s\left(-\frac{s}{2}\right)\right) = \frac{2}{s+1}\left(1 - \delta_s \frac{s}{2}\right) = \frac{2}{s+1} - \delta_s \frac{s}{s+1}$. This is exactly the same recursive relationship as in (1) for the model without a possibility of sale cancellations. Therefore, the solutions for δ_t would also be identical for these two models.

In summary, when $p = \frac{1}{1+L/H} = \frac{1}{1+2/3} = 0.6$, the solutions for the models with and without sale cancellations are identical. For the values of p larger than 0.6, these models also give identical results since the firms do not have incentives to cancel their sales even

[12] Recall that the expected profit from holding a sale in period t is equal to the expected profit from holding a sale in any period within the planning horizon.

if a rival held its sale in the previous period. Hence, the interesting case is that of $p < 0.6$. I will start with $p = 0.6$ and examine what happens to δ_t as p decreases. The equilibrium strategy in the model without sale cancellations remains the same, but the strategy in the model with sale cancellations changes. Figure 2 illustrates this change in δ_t for the values of T between 4 and 13.[13] The planning horizons used by the firms are shown in parentheses and the value of δ_t corresponding to $p = 0.6$ is shown by the horizontal line.

Figure 2: Change in δ_t for Various (t, T) in the Model with Sale Cancellations



The most interesting case happens for $T = 5$ (Figure 2b). For the model without sale cancellations, the planning horizon is $(3; 5)$ and δ_3 is 0.59. As p decreases, the probability of δ_3 in the model with sale cancellations increases, eventually reaching the level, at which it becomes more profitable for the firms to deviate to holding a sale in period 2. At this point, the firms switch to planning horizon $(2; 5)$. Figure 2b) shows the value of $\delta_2 + \delta_3$ in this region. Eventually, probability δ_2 falls to zero, and the firms switch back to using planning horizon $(3; 5)$. Probability δ_3 declines, dropping below the initial level of δ_3, and

[13] The lower bound on price p in this figure is $3/11 = \frac{1}{1+4\times(2/3)}$, which corresponds to the case of the firms cancelling their sales if the rival held a sale within 4 periods.

ultimately falling to zero, at which point the firms switch to using planning horizon $(4; 5)$.

A similar behavior of δ_t, where it first increases and then decreases, falling below its initial level, is present in three other cases: $T = 8$, 11, and 12 (Figures 2e), h), and i). For $T = 6$, δ_t first decreases, then increases within a small region, and then decreases again, always staying below its initial level (Figure 2c). Finally, for the remaining five cases ($T = 4$, 7, 9, 10, and 13), δ_t always decreases, eventually reaching zero, at which point the firms switch to the planning horizon $(t + 1, T)$.

In conclusion, whereas it is possible that for some model parameters, especially when p is large, allowing for sale cancellations increases δ_t, for the vast majority of model parameters, δ_t is smaller in the model with sale cancellations. This corresponds to an increased expected profit from a sale. The intuition for this result is as follows.

If a firm schedules its sale for some period toward the end of the planning horizon, it benefits by selling to more low-valuation consumers if it is the first firm to hold a sale. However, scheduling a sale for these later periods also increases the risk of holding a sale after the one of the rival and facing fewer low-valuation consumers. If a firm can cancel its unprofitable sale, it faces a higher floor on its profit in the sale period. This decreases the potential loss from holding a sale after the rival. The firms respond by shifting probability towards the later periods of the planning horizon, which results in an increase in the expected profit.

So far, the focus of the paper has been on the timing of promotions, with the sale price optimally set at p by both firms. In the next section, I will show how to incorporate into the framework the possibility of the firms choosing different discount levels.

6 Endogenous Sale Price

In this section, I will examine how the optimal strategies change when the firms can choose different levels of promotion depth. In this version of the model, in the beginning of the first period, the firms choose not only the period in which they will hold a sale, but also the sale price.

As was discussed in Section 2, for the current demand structure, the firms do not have any incentive to lower the sale price below p. Thus, in order to generate competition in prices, it is necessary to modify the demand. I assume that all low-valuation consumers have a search cost of zero, which allows them to compare the prices of both firms even if the first firm they sample charges a sale price. Consistent with the prevalent empirical practice, I assume a logit specification for the demand of the low-valuation consumers. That is, if both firms offer sale prices $p_i \leq p$ and $p_j \leq p$ in the same period, then the share of the low-valuation consumers captured by firm i is

$$\mathrm{Pr}_i = \frac{e^{-p_i/\mu}}{e^{-p_i/\mu} + e^{-p_j/\mu}}, \tag{10}$$

where μ is a measure of consumer heterogeneity.

The firms' strategies, then, involve not only the probabilities of holding a sale in any period between t and T, $P_{t,T} = (0, \ldots, 0, \delta_t, \delta_{t+1}, \ldots, \delta_T)$, but also the corresponding sale prices for each period within the planning horizon, $(p_t, p_{t+1}, \ldots, p_T)$. Note that only one of these prices will be used for the sale pre-scheduled for period $s \in \{t, ..., T\}$, and the regular price will be charged in all other periods.

With these strategies, the expected profit from scheduling a sale for period s can be obtained from equation (11) in the proof of Lemma 2 by adding two modifications. First, the sale price is now p_s instead of p. Second, (11) specifies that if both firms held a sale in this period, the low-valuation consumers would be split equally, with each firm capturing $\frac{sL}{2}$ of them. Now, firm i's share depends on its price and the price of the rival and is given by (10), so firm i obtains $\mathrm{Pr}_i \, sL$ low-valuation consumers if both firms schedule a sale for period s.

In order to solve for the optimal prices and probabilities, we need to set up a standard system of equations. First, the expected profits from holding a sale in any period within the planning horizon have to be equal to each other. In addition, we now also have the first-order conditions that ensure that each sale price maximizes the expected profit from holding a sale in that period. That is, $\frac{\partial \pi_s}{\partial p_s} = 0$, unless $p_s = p$, in which case we need to check that $\frac{\partial \pi_s}{\partial p_s} > 0$, so that neither firm has an incentive to undercut the rival. This system of equations has to be solved numerically. Finally, since the objective function is not quasiconcave, it is necessary to check that the obtained solution is a global maximum. This analysis is conducted by computing the values of the profit function for all possible sale prices between 0 and p with a grid size of 0.001 and confirming that these values do not exceed the expected profit at the solution point.

A firm gains from charging a lower sale price by increasing the number of low-valuation consumers captured by the sale if the rival also scheduled a sale for this period. On the other hand, if the rival scheduled its sale for any other period, the number of low-valuations consumers captured by the sale does not depend on the depth of the discount, so a lower sale price decreases the profit. The firms will set their sale prices by balancing these two effects.

I will use the example from Section 3 to illustrate how the possibility of charging different sale prices affects the optimal strategies. Recall that in this example, the number of low-valuation consumers entering each period is 2/3 (H is normalized to 1), and they consider buying the product only if its price is less than or equal to $p = 0.5$. The last equilibrium strategy computed in Section 3 was $P_{3,6} = (0; 0; 0.48; 0.14; 0.29; 0.1)$. The expected profit in a sale period from using this strategy was found to be 1.262. Table 3 below illustrates the optimal strategies for the planning horizon starting in period 3 and ending in period 6 for the relatively high levels of consumer heterogeneity μ. The last column in Table 3 shows the expected profit in a sale period.

When consumer heterogeneity μ is large, the preferences of low-valuation consumers are quite dispersed and their demand elasticity is small. Thus, when the rival schedules its sale for the same period, the focal firm does not capture many additional low-valuation consumers by decreasing its sale price below 0.5. This modest gain is outweighed by a loss from smaller revenues obtained when the focal firm is the only one to hold a sale that

Table 3: Equilibrium Strategies with Endogenous Sale Prices for High Levels of Consumer Heterogeneity

μ	Probabilities	Prices	Profit
0.05	$(0; 0; 0.48; 0.14; 0.29; 0.1)$	$(0.5; 0.5; 0.5; 0.5)$	1.262
0.045	$(0; 0; 0.46; 0.16; 0.27; 0.11)$	$(0.498; 0.5; 0.5; 0.5)$	1.266
0.04	$(0; 0; 0.42; 0.21; 0.23; 0.14)$	$(0.494; 0.5; 0.5; 0.5)$	1.275
0.035	$(0; 0; 0.38; 0.26; 0.19; 0.17)$	$(0.49; 0.5; 0.5; 0.5)$	1.285
0.034	$(0; 0; 0.37; 0.26; 0.2; 0.17)$	$(0.486; 0.496; 0.5; 0.5)$	1.278
0.0335	$(0; 0; 0.37; 0.26; 0.21; 0.16)$	$(0.484; 0.493; 0.499; 0.5)$	1.275
0.033	$(0; 0; 0.36; 0.26; 0.21; 0.17)$	$(0.48; 0.489; 0.494; 0.497)$	1.267

period. Therefore, the firms charge the largest possible sale price that captures the low-valuation consumers, i.e., price 0.5. Thus, for large levels of consumer heterogeneity, the equilibrium strategies in the model with the firms choosing the sale price are exactly the same as the strategies in the model with the sale price exogenously set at 0.5.

As μ gets smaller, the portion of low-valuation consumers captured by a decrease in a sale price gets larger. When the benefit from a larger demand outweighs the loss from a smaller price, the firms start charging the sale prices that are smaller than 0.5. This benefit is larger in the period with the highest sale probability since the probability of having both firms scheduling sales for that period is the largest. Therefore, such period will be the first one in which the sale price decreases. As we can see from Table 3, when $\mu = 0.045$, the period with the largest sale probability is period 3, and the firms decrease this period's sale price p_3 to 0.498.

At the same time, since the profitability of charging p_3 decreases as this price gets smaller, the firms shift the sale probability away from period 3 into the later periods, which contain higher sale prices. Interestingly, this leads to an increase in the expected profit in the sale period. For example, when $\mu = 0.035$, the sale price in period 3 drops to 0.49, but because of the change in the probability distribution $P_{3,6}$, the expected profit increases to 1.285.

For the smaller values of μ, there is a larger downward pressure on prices, and the firms start decreasing their sale prices in other periods as well. When μ reaches 0.033, the sale prices in all periods drop below 0.5. As μ decreases further, the probability distribution $P_{3,6}$ does not change much, but all sale prices decrease, lowering the expected profit. Within an equilibrium strategy, there is a negative relationship between the probability of a sale and a sale price. A larger probability of a sale in a period corresponds to a smaller sale price.

Eventually, the sale prices drop low enough so that the firms find it profitable to deviate to charging sale price 0.5. The reason for this deviation is illustrated by comparing the costs and benefits of charging a sale price $p_s < 0.5$ instead of 0.5. The benefit of charging p_s is that it captures one half of the low-valuation consumers if the rival holds a sale in the same period whereas by charging 0.5 the focal firm loses a significant portion of them to the rival. The cost of charging a low sale price is that the profit from the high-valuation consumers and from the low-valuation consumers when the rival holds a sale in a different period is smaller than the profit obtained by charging 0.5. As the sale price decreases, the cost of charging p_s rises, eventually leading the firms to deviate to charging 0.5.

In the example studied above, this happens when μ reaches 0.0324. The lowest sale price is charged in period 3, and this is the period in which the firms can no longer use a single price $p_3 < 0.5$ in the equilibrium. The firms can not charge 0.5 with certainty in period 3 either as this price would be undercut. Therefore, in this period, we need to search for a mixed-strategy equilibrium in prices. Note that there are two separate mixing processes that are combined in one strategy: the firms mix between the periods for which they pre-schedule a sale and, if a sale is pre-scheduled for period 3, they also mix between the sale prices in that period.

Since the demand functions of the low-valuation consumers are continuous, I rely on the result from Sinitsyn (2008a) that states that for such demands, the support of the mixed-strategy equilibria is finite. The solution techniques for how to find these equilibria are described in Sinitsyn (2008b and 2009), and I use them to find the equilibrium strategies here.

When μ drops to 0.0324, in the third period, the firms choose between two possible sale prices: they charge 0.5 with a small probability and charge $p_3 < 0.5$ with the remaining probability. As μ keeps decreasing, the probability of charging 0.5 increases. For $\mu = 0.032$, the firms charge 0.5 with probability 0.06 and charge 0.468 with probability 0.94. Meanwhile, the sale prices in other periods also decrease, eventually leading the firms to switch to mixed strategies in those periods as well. In Table 4, I illustrate the firms' equilibrium strategies that involve mixing between sale prices. For each μ, $P_{3,6}$ shows the sale probabilities for all periods within the planning horizon, and p_s shows the sale prices and corresponding probabilities of charging them in each period $s \in \{3, 4, 5, 6\}$.

Table 4: Equilibrium Strategies with Endogenous Sale Prices for Low Levels of Consumer Heterogeneity

μ		Period 3	Period 4	Period 5	Period 6	Profit
	$P_{3,6}$	0.37	0.26	0.2	0.17	
0.032	p_s	$\begin{pmatrix} 0.468; & \mathit{0.94} \\ 0.5; & \mathit{0.06} \end{pmatrix}$	$(0.478; \mathit{1})$	$(0.482; \mathit{1})$	$(0.484; \mathit{1})$	1.236
	$P_{3,6}$	0.38	0.25	0.2	0.17	
0.03	p_s	$\begin{pmatrix} 0.455; & \mathit{0.83} \\ 0.5; & \mathit{0.17} \end{pmatrix}$	$\begin{pmatrix} 0.468; & \mathit{0.91} \\ 0.5; & \mathit{0.09} \end{pmatrix}$	$\begin{pmatrix} 0.47; & \mathit{0.92} \\ 0.5; & \mathit{0.08} \end{pmatrix}$	$\begin{pmatrix} 0.47; & \mathit{0.93} \\ 0.5; & \mathit{0.07} \end{pmatrix}$	1.211
	$P_{3,6}$	0.39	0.25	0.2	0.16	
0.025	p_s	$\begin{pmatrix} 0.441; & \mathit{0.7} \\ 0.5; & \mathit{0.3} \end{pmatrix}$	$\begin{pmatrix} 0.456; & \mathit{0.76} \\ 0.5; & \mathit{0.24} \end{pmatrix}$	$\begin{pmatrix} 0.454; & \mathit{0.75} \\ 0.5; & \mathit{0.25} \end{pmatrix}$	$\begin{pmatrix} 0.454; & \mathit{0.75} \\ 0.5; & \mathit{0.25} \end{pmatrix}$	1.194
	$P_{3,6}$	0.39	0.24	0.2	0.17	
0.02	p_s	$\begin{pmatrix} 0.433; & \mathit{0.59} \\ 0.489; & \mathit{0.06} \\ 0.5; & \mathit{0.35} \end{pmatrix}$	$\begin{pmatrix} 0.449; & \mathit{0.65} \\ 0.5; & \mathit{0.35} \end{pmatrix}$	$\begin{pmatrix} 0.445; & \mathit{0.63} \\ 0.5; & \mathit{0.37} \end{pmatrix}$	$\begin{pmatrix} 0.446; & \mathit{0.64} \\ 0.5; & \mathit{0.36} \end{pmatrix}$	1.192
	$P_{3,6}$	0.39	0.25	0.2	0.16	
0.015	p_s	$\begin{pmatrix} 0.418; & \mathit{0.47} \\ 0.462; & \mathit{0.26} \\ 0.5; & \mathit{0.27} \end{pmatrix}$	$\begin{pmatrix} 0.432; & \mathit{0.51} \\ 0.475; & \mathit{0.23} \\ 0.5; & \mathit{0.26} \end{pmatrix}$	$\begin{pmatrix} 0.431; & \mathit{0.51} \\ 0.473; & \mathit{0.24} \\ 0.5; & \mathit{0.25} \end{pmatrix}$	$\begin{pmatrix} 0.431; & \mathit{0.51} \\ 0.473; & \mathit{0.24} \\ 0.5; & \mathit{0.25} \end{pmatrix}$	1.173

Note. Probabilities of charging the corresponding prices are in italics.

Table 4 illustrates that as μ gets smaller, the sale probabilities within $P_{3,6}$ do not change much. The pricing strategies change more significantly. A decrease in μ leads to a decrease in the lowest sale price used by the firms in each period. Eventually, the gap between this price and 0.5 becomes sufficiently large so that the firms find it profitable to deviate to charging another sale price within this gap. At this point, this new price is added to the support of the mixed-strategy equilibrium. For example, when $\mu = 0.02$, in the third period, the firms hold a sale with probability 0.39. Conditional on holding a sale, they charge 0.433 with probability 0.59, 0.489 with probability 0.06, and 0.5 with the remaining probability 0.35.

As consumer heterogeneity keeps decreasing, the firms add the third price to the support of the mixed-strategy equilibria in other periods. The last equilibrium presented in Table 4, for $\mu = 0.015$, has the firms using 3 sale prices in all periods. Further decrease in μ leads to more prices added to the equilibria. This negative relationship between consumer heterogeneity and the number of prices the firms use is a standard feature of the models of sales with heterogeneous consumers (Sinitsyn 2008b, 2009). A decrease in consumer heterogeneity also leads to a decrease in the expected profit.

In summary, introducing competition in sale prices into the framework does not affect the equilibrium strategies if the level of consumer heterogeneity is large. For the lower levels of heterogeneity, the firms decrease the sale prices. Interestingly, a decrease in a sale price in only one of the sale periods might be accompanied by an increase in the expected profit. This happens because the firms shift the sale probability away from the earlier period with a lower sale price to the later periods with higher sale prices and more accumulated low-valuation consumers. Eventually, however, with a continued decrease in sale prices, the expected profit starts to decline. The firms switch to using mixed strategies for their sale prices within the sale period. As consumer heterogeneity keeps decreasing, the sale prices the firms use and the expected profit decline.

7 Managerial Guidelines and Concluding Remarks

The methodological contribution of this paper is the development of a dynamic pricing framework that incorporates firms' commitment to future sale dates. The flexibility of this framework allows for straightforward extensions, two of which—a possibility of sale cancellations and endogenous sale prices—are examined in the paper. The qualitative findings of this paper offer a guidance to pricing managers on how to schedule their sales. First, the sale dates should be unpredictable to rivals. If a sale date is known to a rival, it can offer its promotion in a previous time period, removing all accumulated low-valuation consumers from the market. If a firm operates in an environment where sale cancellations are not too costly, it will benefit by shifting its sales towards the latter periods within the planning horizon. Finally, if a firm schedules a sale for the period, in which it thinks there is a large chance of a sale by a rival, it should charge a smaller sale price in that period.

There are multiple extensions that would enrich this framework and capture additional features of price promotions. One variation, which I examine in the online appendix, is an infinite-horizon model with pre-scheduled sales. In this model, the firms choose both the starting and the ending period of the planning horizon. This extension is suitable for studying periodic sales by supermarkets in the environment where there is no natural date, by which a sale must be held.

Similar to the finite-horizon model, the firms' strategies are probability distributions over the periods, in which they hold a sale. After holding their pre-scheduled sales, the firms schedule the next one according to their equilibrium strategy. Since in this version of the model, the firms set their sale dates repeatedly, it is necessary to verify that they do not deviate to scheduling an extra sale date within the support of the probability distribution functions.

The no-deviation requirements impose intuitive restrictions on the support of the probability distribution in the infinite-horizon model. The firms would like to push the starting period later since this accumulates more low-valuation consumers and increases their profits from holding a sale; thus, preventing a deviation to always charging the regular price. However, as the starting period is pushed later, the incentives to deviate by scheduling a sale right before this period rise. To counter the possibility of such deviation, the firms push the ending period later as well, expanding the difference between the starting and the ending periods. This increases the expected profit from waiting until the pre-scheduled sale period for two reasons. First, the firms benefit from being able to sell in the later periods with more accumulated low-valuation consumers. Second, as the probability distribution gets spread over a larger number of periods, the sale probabilities in each particular period decrease. This makes it less likely that a firm schedules its sale one period after the rival's sale and has to sell to the smallest possible number of low-valuation consumers. However, the difference between the starting and the ending period can not become too large since the firms would have an incentive to set an extra sale date between these periods.

I solve for the optimal strategies of the firms and find that for the vast majority of the parameters of the model, the support of the probability distribution starts between the second and the fifth period after the previous sale and ends by the thirteenth period. When the firms commit to a future sale date, they are able to increase their profits in comparison to the model in which the promotion decision is made in the period when the promotion is offered. For example, when the number of low-valuation consumers entering each period is equal to the number of high-valuation consumers, by pre-scheduling sales the firms can increase their per-period profits by up to 15%. Thus, in addition to being logistically necessary, scheduling promotions in advance also increases profits of the competing firms.

Another useful extension of my framework would allow the firms to schedule several sale dates in the future. In the original model and in the infinite-horizon extension, the firms are able to schedule only one sale at a time, whereas often firms plan their promotions up to a year in advance, with the planning horizon including multiple sale occasions (Blattberg and Neslin 1990, Anderson et al. 2016).

The framework presented in this paper does not consider the interactions between retailers and manufacturers. For example, trade promotions are set up jointly by manufacturers and retailers and are supposed to benefit both parties (Anderson et al. 2016). Lal and Villas-Boas (1998) introduced the model of manufacturers and retailers setting price promotions in a static setting. The current framework would be enriched from the inclusion of manufacturer/retailer interactions that give rise to the decision when to promote the product.

Another avenue for future research is the introduction of strategic consumers into the model. I assumed that all consumers are myopic and purchase the product immediately when they see the price that is below their valuation. If the consumers are strategic, they are aware about the strategies the firms are using and can postpone their purchase if they think the price will be more favorable in the future (Conlisk et al. 1984, Sobel 1991, Su 2007).

Acknowledgements

The author thanks the senior editor, the associate editor, and two anonymous referees for their guidance in improving the paper. The author benefited from the helpful comments of Ryan Kasprzak and Joel Sobel.

References

[1] Anderson, E., B. A. Malin, E. Nakamura, D. Simester, and J. Steinsson. 2016. "Informational Rigidities and the Stickiness of Temporary Sales," NBER Working Paper.

[2] BeautyMark Marketing. 2015. "How to Plan Promotions," http://www.beautymarkmarketing.com/2015-salon-promotion-plan, accessed on June 3, 2015.

[3] Blattberg, R. C., G. D. Eppen, and J. Lieberman. 1981. "A Theoretical and Empirical Evaluation of Price Deals for Consumer Nondurables," *Journal of Marketing*, 45(1), 116-129.

[4] Blattberg, R. C. and S. A. Neslin. 1990. *Sales Promotion: Concepts, Methods, and Strategies*. Englewood Cliffs, NJ: Prentice Hall.

[5] Coibon, O., Y. Gorodnichenko, and G. H. Wong. 2012. "The Cyclicality of Sales, Regular, and Effective Prices: Business Cycle and Policy Implications." *American Economic Review*, 105(3), 993-1029.

[6] Conlisk, J., E. Gerstner, and J. Sobel. 1984. "Cyclic Pricing by a Durable Goods Monopolist," *Quarterly Journal of Economics*, 99(3), 489-505.

[7] Cui, T. H., J. S. Raju, and Z. J. Zhang. 2008. "A Price Discrimination Model of Trade Promotions," *Marketing Science*, 27(5), 779-795.

[8] Diamond, P. A. 1971. "A Model of Price Adjustment," *Journal of Economic Theory*, 3(2), 156-168.

[9] Eichenbaum, M., N. Jaimovich, and S. Rebelo. 2011. "Reference Prices, Costs, and Nominal Rigidities," *American Economic Review*, 101(1), 234-262.

[10] Guimaraes, B. and K. D. Sheedy. 2011. "Sales and Monetary Policy," *American Economic Review*, 101(2), 844-876.

[11] Kehoe, P. and V. Madrigan. 2012. "Prices Are Sticky After All," Working Paper, Federal Reserve Bank of Minneapolis.

[12] Lal, R. and J. M. Villas-Boas. 1998. "Price Promotions and Trade Deals with Multiproduct Retailers," *Management Science*, 44(7), 935-949.

[13] Lazear, E. P. 1986. "Retail Pricing and Clearance Sales," *American Economic Review*, 76(1), 14-32.

[14] Nakamura, E. and J. Steinsson. 2008. "Five Facts about Prices: A Reevaluation of Menu Cost Models," *Quarterly Journal of Economics*, 123(4), 1415-1464.

[15] Narasimhan, C. 2009. "Trade Promotions," *Handbook of Pricing Research in Marketing*. Ed. Vithala R. Rao. Cheltenham: Edward Elgar Publishing, 283-301.

[16] Pesendorfer, M. 2002. "Retail Sales: A Study of Pricing Behavior in Supermarkets," *The Journal of Business*, 75(1), 33-66.

[17] Sinitsyn, M. 2008a. "Characterization of the Support of the Mixed Strategy Price Equilibria in Oligopolies with Heterogeneous Consumers," *Economics Letters*, 99(2), 242-245.

[18] Sinitsyn, M. 2008b. "Technical Note—Price Promotions in Asymmetric Duopolies with Heterogeneous Consumers," *Management Science*, 54(12), 2081-2087.

[19] Sinitsyn, M. 2009. "Price Dispersion in Duopolies with Heterogeneous Consumers," *International Journal of Industrial Organization*, 27(2), 197-205.

[20] Sobel, J. 1984. "The Timing of Sales," *The Review of Economic Studies*, 51(3), 353-368.

[21] Sobel, J. 1991. "Durable Goods Monopoly with Entry of New Consumers," *Econometrica*, 59(5), 1455-1485.

[22] Su, X. 2007. "Intertemporal Pricing with Strategic Customer Behavior," *Management Science*, 53(5), 726-741.

[23] Varian, H. R. 1980. "A Model of Sales," *American Economic Review*, 70(4), 651-659.

[24] Villas-Boas, S. B. and J. M. Villas-Boas. 2008. "Learning, Forgetting, and Sales," *Management Science*, 54(11), 1951-1960.

Appendix

Proof of Lemma 1. Assume that both firms hold a sale in period s with certainty. Then, their payoff in that period is $p\left(H+\frac{sL}{2}\right)$. The firms are splitting the low-valuations consumers. If one of the firms deviates to holding a sale in the previous period, it would get all of the low-valuation consumers in that period, and its payoff would be $p\left(H+(s-1)L\right)$. The deviation is unprofitable when $p\left(H+(s-1)L\right) \leq p\left(H+\frac{sL}{2}\right)$, which simplifies to $s \leq 2$. Therefore, a pure-strategy equilibrium exists only when a sale is held in the first or in the second period.

If a sale is held in the first period by both firms, each gets a payoff of $p\left(H+\frac{L}{2}\right)$. If, instead, one of the firms deviates and schedules its sale for the second period, it would obtain a payoff of $p(H+L) > p\left(H+\frac{L}{2}\right)$. This means that there is no pure-strategy equilibrium with both firms scheduling their sales for the first period.

If a sale is held in the second period by both firms, each gets a payoff of $p\left(H+L\right)$. If $T>3$, then there is a profitable deviation to rescheduling a sale for the fourth period and obtaining a higher payoff of $p\left(H+2L\right)$ in that period. Thus, a pure-strategy equilibrium exists only for $T=2$ or $T=3$. We also need to check that the firms prefer to hold a sale in the second period instead of charging the reservation price. The firms obtain a higher profit from charging the sale price in the second period if $p\left(H+L\right) \geq H$, from where $p \geq \frac{H}{H+L}$.

■

Proof of Lemma 2. The expected profit from holding a sale in period s is calculated as follows. If the rival held a sale in some period i before period s (which happens with probability δ_i), then there are $(s-i)L$ low-valuation consumers accumulated in period s. If the rival holds a sale in the same period s, then sL low-valuation consumers are split equally between the firms. Finally, if the rival holds a sale in some period i after period s (which happens with probability $1-\delta_t-\delta_{t+1}-\ldots-\delta_s$), the focal firm sells to sL low-valuation consumers. Taken together, the focal firm's expected profit from holding a sale in period s is

$$\begin{aligned}\pi_s &= p(\delta_t(H+(s-t)L)+\delta_{t+1}(H+(s-t-1)L)+\ldots+\delta_{s-1}(H+L)+ \\ &\quad +\delta_s\,(H+sL/2)+(1-\delta_t-\delta_{t+1}-\ldots-\delta_{s-1}-\delta_s)(H+sL)).\end{aligned} \tag{11}$$

Similarly, the focal firm's expected profit from holding a sale in period $s+1$ is

$$\begin{aligned}\pi_{s+1} &= p(\delta_t(H+(s-t+1)L)+\ldots+\delta_{s-1}(H+2L)+\delta_s(H+kL)+ \\ &\quad +\delta_{s+1}\,(H+(s+1)L/2)+(1-\delta_t-\delta_{t+1}-\ldots-\delta_s-\delta_{s+1})(H+(s+1)L)).\end{aligned} \tag{12}$$

The expression for π_s from (11) simplifies to $\pi_s = p(H+sL-\delta_t tL-\delta_{t+1}(t+1)L-\ldots-\delta_{s-1}(s-1)L-\delta_s\frac{sL}{2})$. Similarly, π_{s+1} from (12) simplifies to $\pi_{s+1} = p(H+(s+1)L-\delta_t tL-\delta_{t+1}(t+1)L-\ldots-\delta_{s-1}(s-1)L-\delta_s sL-\delta_{s+1}\frac{(s+1)L}{2})$. After equating the expressions for π_s and π_{s+1}, we obtain $\delta_s\frac{sL}{2}+\delta_{s+1}\frac{(s+1)L}{2}=L$, from where $\delta_{s+1}=\frac{2}{s+1}-\delta_s\frac{s}{s+1}$. ■

Proof of Lemma 3. This is a proof by induction. In period t, $\delta_t=\delta_t\frac{t}{t}$. In period $t+1$, $\delta_{t+1}=\frac{2}{t+1}-\delta_t\frac{t}{t+1}$ from (1). Thus, the initial conditions are satisfied. Now, we need to show that if in period $s-1$, the sale probability satisfies (2), then the sale probability in s also satisfies (2). Consider two cases.

If $s-1-t$ is even and $\delta_{s-1}=\delta_t\frac{t}{s-1}$, then using (1), $\delta_s=\frac{2}{s}-\delta_{s-1}\frac{s-1}{s}=\frac{2}{s}-\delta_t\frac{t}{s-1}\frac{s-1}{s}=\frac{2}{s}-\delta_t\frac{t}{s}$.

If $s-1-t$ is odd and $\delta_{s-1}=\frac{2}{s-1}-\delta_t\frac{t}{s-1}$, then using (1), $\delta_s=\frac{2}{s}-\delta_{s-1}\frac{s-1}{s}=\frac{2}{s}-\frac{2}{s-1}\frac{s-1}{s}+\delta_t\frac{t}{s-1}\frac{s-1}{s}=\frac{2}{s}-\frac{2}{s}+\delta_t\frac{t}{s}=\delta_t\frac{t}{s}$. ■

Proof of Lemma 4. For any period s, if $s-t$ is even, we know from (2) that $\delta_s=\delta_t\frac{t}{s}$. Therefore, $\delta_t\geq 0$ guarantees that $\delta_s\geq 0$. If $s-t$ is odd, then we know from (2) that $\delta_s=\frac{2}{s}-\delta_t\frac{t}{s}=\frac{2-\delta_t t}{s}$. Then, $\delta_s\geq 0$ if $2-\delta_t t\geq 0$, which is equivalent to $\delta_t\leq 2/t$.

If a firm considers deviating and holding a sale in some period before t, it should schedule one for period $t-1$ in order to maximize the number of accumulated low-valuation consumers. Take the rival's strategy as fixed. The rival uses strategy $P_{t,T}$ with the earliest possible sale occurring in period t with probability δ_t. If the focal firm deviates and holds a sale in period $t-1$, it gets all low-valuation consumers, and its profit is $\pi_{t-1}=p(H+(t-1)L)$. If the focal firm follows the strategy $P_{t,T}$ and holds its sale in period t, its expected profit is $\pi_t = p\left(\delta_t\left(H + \frac{tL}{2}\right) + (1-\delta_t)(H+tL)\right) = p\left(H + tL - \delta_t\frac{tL}{2}\right)$. The deviation is unprofitable if $\pi_{t-1} \le \pi_t$. This inequality is equivalent to $(t-1)L \le tL - \delta_t\frac{tL}{2}$, which simplifies to $\delta_t \le 2/t$. ■

Proof of Lemma 5. The probabilities within the planning horizon have to sum up to one. Thus, we can substitute the formulas for δ_s from (2) into $\delta_t + \delta_{t+1} + \ldots + \delta_T = 1$. The resulting expression depends on whether $T-t$ is even or odd. If $T-t$ is odd, then $1 = \delta_t + \frac{2}{t+1} - \delta_t\frac{t}{t+1} + \delta_t\frac{t}{t+2} + \ldots + \delta_t\frac{t}{T-1} + \frac{2}{T} - \delta_t\frac{t}{T}$. We combine all the terms containing δ_t to obtain

$$1 = \delta_t t\left(\left(\frac{1}{t} - \frac{1}{t+1}\right) + \left(\frac{1}{t+2} - \frac{1}{t+3}\right) + \ldots + \left(\frac{1}{T-1} - \frac{1}{T}\right)\right) + 2\left(\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T}\right).$$

Since $\frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$, the above equality simplifies to

$1 = \delta_t t\left(\frac{1}{t(t+1)} + \frac{1}{(t+2)(t+3)} + \ldots + \frac{1}{(T-1)T}\right) + 2\left(\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T}\right)$, from where

$$\delta_t = \frac{1-2\left(\frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T}\right)}{t\left(\frac{1}{t(t+1)}+\frac{1}{(t+2)(t+3)}+\ldots+\frac{1}{(T-1)T}\right)}.$$

Similarly, if $T-t$ is even, then $1 = \delta_t + \frac{2}{t+1} - \delta_t\frac{t}{t+1} + \delta_t\frac{t}{t+2} + \ldots + \frac{2}{T-1} - \delta_t\frac{t}{T-1} + \delta_t\frac{t}{T} =$

$= \delta_t t\left(\left(\frac{1}{t} - \frac{1}{t+1}\right) + \left(\frac{1}{t+2} - \frac{1}{t+3}\right) + \ldots + \left(\frac{1}{T-2} - \frac{1}{T-1}\right) + \frac{1}{T}\right) + 2\left(\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T-1}\right) =$

$= \delta_t t\left(\frac{1}{t(t+1)} + \frac{1}{(t+2)(t+3)} + \ldots + \frac{1}{(T-2)(T-1)} + \frac{1}{T}\right) + 2\left(\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T-1}\right)$, from where

$$\delta_t = \frac{1-2\left(\frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T-1}\right)}{t\left(\frac{1}{t(t+1)}+\frac{1}{(t+2)(t+3)}+\ldots+\frac{1}{(T-2)(T-1)}+\frac{1}{T}\right)}.$$ ■

Proof of Lemma 6. When $T-t$ is odd, $\delta_t \ge 0$ is satisfied when the numerator of the corresponding ratio from (3) is nonnegative, i.e., when $\frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T} \le \frac{1}{2}$. The restriction $\delta_t \le \frac{2}{t}$ is satisfied when $\frac{1-2\left(\frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T}\right)}{t\left(\frac{1}{t(t+1)}+\frac{1}{(t+2)(t+3)}+\ldots+\frac{1}{(T-1)T}\right)} \le \frac{2}{t}$ or $1 - 2\left(\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T}\right) \le$

$\le 2\left(\frac{1}{t(t+1)} + \frac{1}{(t+2)(t+3)} + \ldots + \frac{1}{(T-1)T}\right)$. Rearranging the terms, we obtain $1 \le$

$\le 2\left(\left(\frac{1}{t(t+1)} + \frac{1}{t+1}\right) + \left(\frac{1}{(t+2)(t+3)} + \frac{1}{t+3}\right) + \ldots + \left(\frac{1}{(T-1)T} + \frac{1}{T}\right)\right) =$

$= 2\left(\frac{1}{t} + \frac{1}{t+2} + \ldots + \frac{1}{T-1}\right)$. From here, $\frac{1}{t} + \frac{1}{t+2} + \ldots + \frac{1}{T-1} \ge \frac{1}{2}$.

When $T-t$ is even, $\delta_t \ge 0$ is satisfied when the numerator of the corresponding ratio from (3) is nonnegative, i.e., when $\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T-1} \le \frac{1}{2}$. The restriction $\delta_t \le \frac{2}{t}$ is satisfied when $\frac{1-2\left(\frac{1}{t+1}+\frac{1}{t+3}+\ldots+\frac{1}{T-1}\right)}{t\left(\frac{1}{t(t+1)}+\frac{1}{(t+2)(t+3)}+\ldots+\frac{1}{(T-2)(T-1)}+\frac{1}{T}\right)} \le \frac{2}{t}$ or $1 - 2\left(\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T-1}\right) \le$ $2\left(\frac{1}{t(t+1)} + \frac{1}{(t+2)(t+3)} + \ldots + \frac{1}{(T-2)(T-1)} + \frac{1}{T}\right)$. Rearranging the terms, we obtain $1 \le$

$\le 2\left(\left(\frac{1}{t(t+1)} + \frac{1}{t+1}\right) + \left(\frac{1}{(t+2)(t+3)} + \frac{1}{t+3}\right) + \ldots + \left(\frac{1}{(T-2)(T-1)} + \frac{1}{T-1}\right) + \frac{1}{T}\right) =$

$= 2\left(\frac{1}{t} + \frac{1}{t+2} + \ldots + \frac{1}{T-2} + \frac{1}{T}\right)$. From here, $\frac{1}{t} + \frac{1}{t+2} + \ldots + \frac{1}{T-2} + \frac{1}{T} \geq \frac{1}{2}$. ■

Proof of Proposition 1. I will prove this result by illustrating how to find the value of t that satisfies the conditions of Lemma 6. Start with $\frac{1}{T}$ and add to it the terms $\frac{1}{T-2}$, $\frac{1}{T-4}$, etc., until the addition of a new term makes the sum larger than $\frac{1}{2}$. Label the period corresponding to this term as t. Then

$$\frac{1}{t+2} + \ldots + \frac{1}{T-2} + \frac{1}{T} \leq \frac{1}{2}, \tag{13}$$

but

$$\frac{1}{t} + \frac{1}{t+2} + \ldots + \frac{1}{T-2} + \frac{1}{T} \geq \frac{1}{2}. \tag{14}$$

Now, examine the sum $\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T-3} + \frac{1}{T-1}$. If it is less than $\frac{1}{2}$, then together with (14), the second condition of Lemma 6 (for even $T-t$) is satisfied and t serves as the starting period of the planning horizon.

If $\frac{1}{t+1} + \frac{1}{t+3} + \ldots + \frac{1}{T-3} + \frac{1}{T-1} \geq \frac{1}{2}$, define $t^* = t+1$. This inequality becomes $\frac{1}{t^*} + \frac{1}{t^*+2} + \ldots + \frac{1}{T-3} + \frac{1}{T-1} \geq \frac{1}{2}$. Together with (13) rewritten as $\frac{1}{t^*+1} + \ldots + \frac{1}{T-2} + \frac{1}{T} \leq \frac{1}{2}$, these inequalities satisfy the first condition of Lemma 6 (for odd $T-t$) and t^* serves as the starting period of the planning horizon. This process illustrates that only one of the conditions of Lemma 6 can be satisfied; therefore, the starting period is unique. ■

Proof of Lemma 7. Consider periods s and $s+1$, where $t \leq s \leq T-1$. In the mixed-strategy equilibrium, the expected profits from holding a sale in these periods must be equal. The expected profit from a sale in period s could be decomposed into four parts. First, if the rival held its sale more than c periods in advance of s, the focal firm will proceed with its sale. The portion of the expected profit corresponding to this case is

$$p\left(\delta_t(H + (s-t)L) + \delta_{t+1}\left(H + (s-t-1)L\right) + \ldots + \delta_{s-c-1}(H + (c+1)L\right). \tag{15}$$

Second, if the rival held its sale within c periods of s, the focal firm would cancel its sale, charge the reservation price, and earn profit H. The portion of the expected profit corresponding to this case is

$$\delta_{s-c}H + \delta_{s-c+1}H + \ldots + \delta_{s-1}H. \tag{16}$$

Third, if the rival also scheduled its sale for period s, both firms hold their sales at the same time and split the low-valuation consumers. The portion of the expected profit corresponding to this case is

$$p\delta_s\left(H+\frac{sL}{2}\right). \tag{17}$$

Finally, if the rival scheduled its sale for some period after s, the focal firm gets all low-valuation consumers in period s. The portion of the expected profit corresponding to this case is

$$p\left(1-\delta_t-\ldots-\delta_s\right)\left(H+sL\right). \tag{18}$$

Combining the expressions from (15), (16), (17), and (18), we receive the following formula for the expected profit from scheduling a sale for period s.

$$\begin{aligned}\pi_s &= p(\delta_t(H+(s-t)L)+\ldots+\delta_{s-c-1}(H+(c+1)L)+ \\ &\quad +\delta_s\left(H+sL/2\right)+(1-\delta_t-\ldots-\delta_s)(H+sL))+\delta_{s-c}H+\ldots+\delta_{s-1}H.\end{aligned} \tag{19}$$

Similarly, the formula for π_{s+1} is

$$\begin{aligned}\pi_{s+1} &= p(\delta_t(H+(s-t+1)L)+\ldots+\delta_{s-c}(H+(c+1)L)+ \\ &\quad +\delta_{s+1}\left(H+(s+1)L/2\right)+(1-\delta_t-\ldots-\delta_{s+1})(H+(s+1)L))+\delta_{s-c+1}H+\ldots+\delta_sH.\end{aligned} \tag{20}$$

In a mixed-strategy equilibrium, π_s and π_{s+1} are equal, so after substituting these values from (19) and (20) into $\pi_{s+1}-\pi_s=0$ and simplifying, we get

$$\begin{aligned}0 &= p(L+\delta_{s-c}(H+cL)-\delta_{s-c+1}L-\delta_{s-c+2}L-\ldots-\delta_{s-1}L- \\ &\quad -\delta_s(H+L+sL/2)-\delta_{s+1}(s+1)L/2)+\delta_sH-\delta_{s-c}H.\end{aligned}$$

From here, $\delta_{s+1}pL\frac{s+1}{2}=pL+\delta_{s-c}(pH+pcL-H)-\delta_{s-c+1}pL-\ldots-\delta_{s-1}pL-\delta_s(pH+pL+pSL/2-H)$ and $\delta_{s+1}=\frac{2}{s+1}\left(1+\delta_{s-c}\left(\frac{H}{L}+c-\frac{H}{pL}\right)-\delta_{s-c+1}-\ldots-\delta_{s-1}-\delta_s\left(\frac{H}{L}+1+\frac{s}{2}-\frac{H}{pL}\right)\right)$. After substituting $A=\frac{H}{pL}-\frac{H}{L}$, we obtain (8). ■