

Economics 245 — Fall 2012

International Trade

## Problem Set 2

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**Due:** Thu, November 8, 2012  
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### 1 Properties of the Pareto and Fréchet Distributions

Consider a Pareto distributed variable  $\phi \sim \mathcal{P}(\phi_0, \theta)$ , where  $\phi_0$  is called the location parameter and  $\theta$  the shape parameter. The Pareto distribution function is  $F(Z \leq \phi) = 1 - (\phi_0/\phi)^\theta$ .

1. Show that the Pareto density function is  $\mu(\phi|\phi_0, \theta) = \theta(\phi_0)^\theta/(\phi)^{\theta+1}$ .
2. For any  $\phi^* > \phi_0$ , show that the conditional distribution function is:  $F(\phi|\phi \geq \phi^*) = 1 - (\phi^*/\phi)^\theta$ , also a Pareto distribution function.
3. Consider a transformed random variable  $A(\phi)^B$  with  $A, B > 0$ . Show that the transformed variable is Pareto distributed with location parameter  $A(\phi_0)^B$  and shape parameter  $\theta/B$ .
4. Show that the mean of a Pareto distributed variable  $\phi$  is  $\mathbb{E}[\phi|\phi_0, \theta] = \theta\phi_0/(\theta - 1)$  if  $\theta > 1$ .
5. Consider the Fréchet distribution  $G(Z \leq z) = \exp\{-Tz^{-\theta}\}$ . Show that the Fréchet distribution approaches the Pareto distribution “in the right tail”, that is show that

$$\lim_{z \rightarrow \infty} G(Z \leq z) = 1 - Tz^{-\theta}.$$

(Hint: Use L'Hôpital's rule and the fact that  $\lim_{z \rightarrow \infty} z^{-\theta} = 0$  for  $\theta > 0$ .)

### 2 Chaney (AER 2008) and Gravity

There are  $N$  countries. A country carries a subscript  $s$  if it is the source of exports, and a subscript  $d$  if it is a destination ( $s, d = 1, \dots, N$ ). In each country lives a measure of  $L_d$  consumers, who inelastically supply one unit of labor and own the shares of domestic firms. Firms choose to enter their respective home market  $s$  and any export destination  $d$ . There are source-destination iceberg

transportation costs  $\tau_{sd}$  between countries, and there is a source-destination specific fixed cost of entry  $F_{sd}$ .

The  $L_d$  representative consumers have identical CES preferences over a continuum of varieties with

$$U_d = \left[ \sum_{s=1}^N \int_{\omega \in \Omega_{sd}} c(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}} \quad \text{for } \varepsilon = \sigma > 1.$$

$\Omega$  is the (fixed) measure of available varieties, given a predetermined measure of firms  $J_s$  in every country that sells varieties.

Each firm  $\omega$ 's production technology is constant returns to scale but firms from country  $s$  differ in productivity  $\phi$ , which they draw from a Pareto distribution  $F(\phi) = 1 - (b_s/\phi)^\theta$ . It will be convenient to call any firm  $\omega$  with a given productivity level  $\phi$  the firm  $\phi$ .

1. Show that consumer demand for a firm  $\phi$ 's variety  $c_{sd}(\phi)$  is

$$c_{sd}(\phi) = \frac{(p_{sd})^{-\sigma}}{(P_d)^{1-\sigma}} y_d L_d$$

for the ideal price index

$$P_d \equiv \left( \sum_{s=1}^N \int_{\phi \in \Omega_{sd}} p_{sd}^{1-\sigma} d\phi \right)^{\frac{1}{1-\sigma}}.$$

2. Upon entry, each firm maximizes operational profits

$$\pi_{sd}(\phi) = \left( p_{sd} - \tau_{sd} \frac{w_s}{\phi} \right) \left( \frac{p_{sd}}{P_d} \right)^{-\sigma} \frac{y_d L_d}{P_d} - F_{sd}.$$

Show that optimal price is a constant markup over unit production cost with

$$p_{sd}(\phi) = \eta \frac{\tau_{sd} w_s}{\phi}, \quad \eta = \frac{\sigma}{\sigma - 1}.$$

3. Derive a firm  $\phi$ 's total sales  $p_{sd}(\phi)q_{sd}(\phi)$  and show that optimal operational profits satisfy  $\pi_{sd}(\phi) = p_{sd}(\phi)q_{sd}(\phi)/\sigma$ . Using results from Question 1 show that total sales  $p_{sd}(\phi)q_{sd}(\phi)$  are Pareto distributed.
4. Show that the least productive firm from country  $s$  with a productivity

$$\phi_{sd}^* = \left( \frac{\sigma F_{sd}}{y_d L_d} \right)^{\frac{1}{\sigma-1}} \frac{\eta \tau_{sd} w_s}{P_d}.$$

just breaks even in destination market  $d$ . Explain the intuition.

5. Using results from Question 1, derive aggregate exports

$$X_{sd} = \int_{\phi_{sd}^*}^{\infty} p_{sd}(\phi) q_{sd}(\phi) \mu(\phi | \phi_{sd}^*, \theta) d\phi$$

from source country  $s$  to destination  $d$ , where  $\mu(\phi | \phi_{sd}^*, \theta)$  is the conditional Pareto density for firms from  $s$  active at  $d$ . State  $X_{sd}$  as a function of  $\tau_{sd}$  and  $\phi_{sd}^*$ . Using the result for  $\phi_{sd}^*$  from above, simplify further.

6. Derive the elasticity of aggregate trade with respect to variable trade costs  $\partial \log X_{sd} / \partial \log \tau_{sd}$ , using the expression of  $X_{sd}$  as a function of  $\tau_{sd}$  and  $\phi_{sd}^*(\tau_{sd})$ . Interpret the two terms.
7. Derive the elasticity of aggregate trade with respect to fixed trade costs  $\partial \log X_{sd} / \partial \log F_{sd}$ .

### 3 Arkolakis & Muendler (2011) and Product Entry

As in the question before, there are  $N$  countries. A country carries a subscript  $s$  if it is the source of exports, and a subscript  $d$  if it is a destination ( $s, d = 1, \dots, N$ ). In each country lives a measure of  $L_d$  consumers, who inelastically supply one unit of labor and own the shares of domestic firms. Firms choose to enter their respective home market  $s$  and any export destination  $d$ . There are source-destination iceberg transportation costs  $\tau_{sd}$  between countries, and there is a source-destination specific fixed cost of entry  $F_{sd}(G_{sd})$  that depends on a firm's exporter scope (number of products)  $G_{sd}$  at a destination.

The  $L_d$  representative consumers have identical CES preferences over a continuum of firms *and* products. The lower tier of the utility aggregate is a firm's unique product mix ("variety")

$$U_d = \left[ \sum_{s=1}^N \int_{\omega \in \Omega_{sd}} \left( \int_1^{G_{sd}(\omega)} q_{sd}(g, \omega)^{\frac{\varepsilon-1}{\varepsilon}} dg \right)^{\frac{\varepsilon}{\varepsilon-1} \frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}} \quad \text{with } \varepsilon \neq \sigma \text{ and } \varepsilon, \sigma > 1.$$

In its product mix, a firm has a continuum of  $g \in [1, G_{sd}]$  products at destination market  $d$ .

Each firm  $\omega$ 's production technology is constant returns to scale but firms from country  $s$  differ in productivity  $\phi$ , which they draw from a Pareto distribution  $F(\phi) = 1 - (b_s/\phi)^\theta$ . It will be convenient to call all firms  $\omega$  with a given productivity level the firms  $\phi$ . For each destination market  $d$ , a firm chooses its specific exporter scope  $G_{sd}(\phi)$ . In production, a firm faces constant marginal cost for each product that it adopts.

1. Show that consumer demand for an individual product  $q_{sd}(g, \phi)$  is

$$q_{sd}(g, \phi) = (p_{sd}(g))^{-\varepsilon} \frac{P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma}}{(P_d)^{1-\sigma}} y_d L_d$$

for the ideal price indexes

$$P_d \equiv \left( \sum_{s=1}^N \int_{\phi \in \Omega_{sd}} P_{sd}^{1-\sigma} d\phi \right)^{\frac{1}{1-\sigma}} \quad \text{and} \quad P_{sd} \equiv \left( \int_1^{G_{sd}(\phi)} p_{sd}(g, \phi)^{1-\varepsilon} dg \right)^{\frac{1}{1-\varepsilon}}.$$

*Hint:* The first-order conditions imply for Marshallian demand of a firm's product-mix

$$X_d \equiv \left( \sum_{s=1}^N \int_{\phi \in \Omega_{sd}} X_{sd}(\phi; G_{sd})^{\frac{\sigma-1}{\sigma}} d\phi \right)^{\frac{\sigma}{\sigma-1}} \quad \text{and} \quad X_{sd}(\phi; G_{sd}) \equiv \left( \int_1^{G_{sd}(\phi)} q_{sd}(g, \omega)^{\frac{\varepsilon-1}{\varepsilon}} dg \right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

2. Show that  $P_{sd}(\phi; G_{sd})$  strictly decreases in exporter scope  $G_{sd}$ .
3. *Cannibalization.* Show that scope diminishes infra-marginal shipments  $q_{sd}(g, \phi)$  and infra-marginal scale  $p_{sd}(g, \phi)q_{sd}(g, \phi)$  if and only if  $\varepsilon > \sigma$ .
4. Upon entry in market  $d$ , each firm maximizes operational profits

$$\pi_{sd}(\phi) = \int_1^{G_{sd}(\phi)} \left( p_{sd}(g) - \frac{\tau_{sd} w_s}{\phi} \right) p_{sd}(g)^{-\varepsilon} \frac{(P_{sd})^{\varepsilon-\sigma}}{(P_d)^{-(\sigma-1)}} y_d L_d dg - F_{sd}(G_{sd}).$$

Show that optimal price is a constant markup over unit production cost with

$$p_{sd}(g, \phi) = \eta \frac{\tau_{sd} w_s}{\phi}, \quad \eta = \frac{\sigma}{\sigma - 1}.$$

*Hint:* For this purpose, maximize the firm's constrained Lagrangian objective function

$$\max_{P_{sd}, \{p_{sd}(g)\}_{g \in [1, G_{sd}]}} \pi_{sd}(\phi) + \lambda \left( P_{sd} - \left[ \int_1^{G_{sd}} p_{sd}(g)^{-(\varepsilon-1)} dg \right]^{-\frac{1}{\varepsilon-1}} \right).$$

5. Consider  $G_{sd}$  as given. Show that optimal product scale is the same for every product  $g$  with

$$p_{sd}(g, \phi) q_{sd}(g, \phi) = (G_{sd})^{-\frac{\varepsilon-\sigma}{\varepsilon-1}} y_d L_d \left( \frac{\phi P_d}{\eta \tau_{sd} w_s} \right)^{\sigma-1}.$$

*Hint:* Use the fact that  $P_{sd}(\phi; G_{sd}) = (G_{sd})^{1/(1-\varepsilon)} p_{sd}(g, \phi)$ . Is constant product scale for every product  $g$  realistic? How can the firm's optimization problem be extended to generate a product scale distribution within the firm?

6. Using optimal product sales, show that the profit function becomes

$$\pi_{sd}(\phi) = \frac{(G_{sd})^{\bar{\sigma}} y_d L_d}{\sigma} \left( \frac{\phi P_d}{\eta \tau_{sd} w_s} \right)^{\sigma-1} - w_d F_d(G_{sd}) \quad \text{where } \bar{\sigma} \equiv \frac{\sigma - 1}{\varepsilon - 1}.$$

From now on, suppose the fixed entry costs take the form  $F_d(G_{sd}) = \kappa_d + \gamma_d (G_{sd})^{\delta+1}/(\delta + 1)$  where  $\gamma_d > 0$ . Fixed costs are paid in destination market wages  $w_d$ .

7. Show that the optimal exporter scope for a firm from  $s$  shipping to  $d$  is

$$G_{sd}(\phi) = \left[ \frac{\bar{\sigma} y_d L_d}{\sigma w_d \gamma_d} \left( \frac{\phi P_d}{\eta \tau_{sd} w_s} \right)^{\sigma-1} \right]^{\frac{1}{\delta - (\bar{\sigma} - 1)}} \quad \text{for } G_{sd}(\phi) \geq G_d^* \quad \text{and } \delta > \bar{\sigma} - 1.$$

What is the intuition for the condition  $\delta > \bar{\sigma} - 1$ ? Consider the benchmark case of  $\bar{\sigma} = 1$  and relate it to the cannibalization effect. What would optimal scope be if the condition were strictly violated with  $\bar{\sigma} - 1 > \delta$ ?

8. Applying the zero-profit condition  $\pi_{sd}(\phi) = 0$ , show that the minimum optimal scope  $G_d^*$  of any firm in country  $d$  is

$$G_d^* = \left( \frac{(\delta + 1) \bar{\sigma} \kappa_d}{\delta - (\bar{\sigma} - 1) \gamma_d} \right)^{\frac{1}{\delta+1}}.$$

Note that minimum scope is independent of source country characteristics and independent of the destination country's size  $L_d$  and per-capita income  $y_d$ . Is this realistic?

9. Define the productivity threshold for exporting from  $s$  to  $d$

$$\phi_{sd}^* = \left( \frac{y_d L_d}{\sigma} \right)^{-\frac{1}{\sigma-1}} \left( \frac{w_d \gamma_d}{\bar{\sigma}} \right)^{\frac{1}{\sigma-1} \frac{\bar{\sigma}}{\delta+1}} \left( \frac{(\delta + 1) w_d \kappa_d}{\delta - (\bar{\sigma} - 1)} \right)^{\frac{1}{\sigma-1} \frac{\delta - (\bar{\sigma} - 1)}{\delta+1}} \frac{\eta \tau_{sd} w_s}{P_d}.$$

Using  $G_d^*$  as the optimal exporter scope  $G_{sd}(\phi_{sd}^*)$  for a firm at the productivity threshold, verify that the definition is correct.

10. Show that a firm's optimal exporter scope can be expressed as

$$G_{sd}(\phi) = G_d^* \left( \frac{\phi}{\phi_{sd}^*} \right)^{\frac{\sigma-1}{\delta - (\bar{\sigma} - 1)}}.$$

11. Show that a firm's optimal total exports are

$$T_{sd}(\phi) \equiv G_{sd}(\phi) p_{sdg}(\phi) x_{sdg}(\phi) = \frac{(\delta + 1) \sigma w_d \kappa_d}{\delta - (\bar{\sigma} - 1)} \left( \frac{\phi}{\phi_{sd}^*} \right)^{(\delta+1) \frac{\sigma-1}{\delta - (\bar{\sigma} - 1)}}.$$

12. Using results from Question 1 show that  $G_{sd}(\phi)$  and  $T_{sd}(\phi)$  are both Pareto distributed.

## 4 Eaton & Kortum (Econometrica 2002)

Consider the Eaton & Kortum model (Econometrica 2002).

1. Show that the probability for county  $s$  to ship a good at the lowest price to country  $d$  is simply

$$\pi_{sd} = \frac{T_s (c_s d_{sd})^{-\theta}}{\Phi_d}.$$

2. Show that the price of a good that country  $d$  actually imports from county  $s$  has the distribution  $G_d(p) = 1 - \exp\{-\Phi_d p^\theta\}$ .
3. Show that with CES utility, and a Fréchet distribution for the prices actually paid by consumers, the price index takes the form

$$p_d = \gamma \Phi_d^{-1/\theta}, \quad \gamma \equiv \left[ \Gamma \left( \frac{\theta + 1 - \sigma}{\theta} \right) \right]^{\frac{1}{1-\sigma}},$$

where  $\Gamma$  is the Gamma function and the parameter restriction  $\theta > \sigma - 1$  is satisfied for the price index to be well-defined.