Economics 246 — Spring 2008

International Macroeconomics

Problem Set 2

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1 Optimal Consumption with Complete and Incomplete Asset Markets

Consider a two-period model of consumption by a representative agent, who faces a market interest rate r for riskless loans B_2. Labor earnings today are Y_1. There are $\mathcal{S}$ states of nature tomorrow, and earnings realizations $Y_2(s)$ differ across states. Each state s occurs with a probability $\pi(s)$. The representative consumer maximizes expected life-time utility

$$U_1 = C_1 - \tfrac{a_0}{2}(C_1)^2 + \beta \mathbb{E}_1 \left[C_2 - \tfrac{a_0}{2}(C_2)^2 \right],$$

where period utility is quadratic and $a_0 > 0$. The consumer's time preference parameter β is such that $\beta = \frac{1}{1+r}$. You may assume that Y_1 and all $Y_2(s)$ levels are small enough so that the marginal utility of period consumption $1 - a_0 C$ is strictly positive for all consumption levels.

When asset markets are incomplete, the relevant constraints can be written as

$$\begin{aligned} B_2 &= (1+r)B_1 + Y_1 - C_1 \\ C_2(s) &= (1+r)B_2 + Y_2(s) \qquad \forall s \in \{1, \ldots, \mathcal{S}\} \end{aligned}$$

Initial bond holdings B_1 are given and B_2 denotes bonds accumulated through the end of period 1.

1. Show that the preceding constraints imply the $\mathcal{S}$ intertemporal budget constraints
$$C_1 + \frac{C_2(s)}{1+r} = (1+r)B_1 + Y_1 + \frac{Y_2(s)}{1+r}$$
for all states $s \in \{1, \ldots, \mathcal{S}\}$.

2. Temporarily ignore the nonnegativity constraints $C_2(s) \geq 0$ for states s of nature tomorrow. Compute the optimal level of consumption C_1 today. What are the implied values of $C_2(s)$. What would the optimal level of C_1 be for an infinitely-lived agent and output uncertainty in each future period?

3. Consider the nonnegativity constraint on C_2. Relabel the states of nature such that $Y_2(1) = \min_s\{Y_2(s)\}$. Show that if

$$(1 + r)B_1 + Y_1 + \tfrac{2+r}{1+r} Y_2(1) \geq \mathbb{E}_1 [Y_2]$$

then the C_1 computed in part 2 (for the two-period case) is still valid. What is the intuition? Suppose the preceding inequality fails to hold. Show that optimal consumption C_1 today is lower and equals

$$C_1 = (1 + r)B_1 + Y_1 + \tfrac{Y_2(1)}{1+r}.$$

This is a precautionary savings effect. Explain why it arises. Does the bond Euler equation hold in this case?

[*Hint*: Apply the Kuhn-Tucker theorem to derive optimal consumption C_1.]

4. Assume the consumer faces *complete* global asset markets with $p(s)/(1+r)$, the state s Arrow-Debreu security price, equal to $\pi(s)/(1 + r)$. Explain why these are called actuarially fair prices. Find the optimal values of C_1 and $C_2(s)$. Why can nonnegativity constraints be disregarded in the case of complete asset markets?

2 Consumption-based CAPM model (Lucas, *ECMA* 1978)

Consider a representative agent and a production process, in which a random and exogenous amount of perishable output y_t falls from one fruit tree each period. There is no other output. The fruit output follows the stochastic process

$$\ln y_t = \ln y_{t-1} + \epsilon_t, \tag{2-1}$$

where ϵ_t is an unanticipated Gaussian shock with $\mathbb{E}_{t-1} [\epsilon_t] = 0$ and $\mathcal{N}(0, \sigma_2)$. So, y_t has a lognormal distribution. There is no investment, that is there is no way to grow more fruit trees.

The agent's life-time utility function takes the particular form

$$U_t = \mathbb{E}_t \left[\sum_{s=t}^{\infty} e^{-\theta(s-t)} u(c_s) \right],$$

where $\theta > 0$ is the rate of time preference. Assume there is a competitive stock market, in which people can trade shares in the fruit tree. The price of a share is denoted p_t and ex-dividend. So, if the agent buys a share on date t, she gets her first dividend payment only on date $t + 1$.

1. Argue that a share holder's individual budget constraint is $c_s + p_s x_{s+1} \leq (y_s + p_s)x_s$, where x_s denotes the shares in the tree that the agent holds

at the end of period $s-1$. Setup up the Bellman equation for the representative agent's intertemporal consumption and share-holding choice. Show that the optimal consumption path satisfies the Euler equations

$$p_s u'(c_s) = e^{-\theta}\mathbb{E}_t\left[(p_{s+1}+y_{s+1})u'(c_{s+1})\right] \tag{2-2}$$

for all $s \geq t$.

2. Show that, in equilibrium, the fundamental bubble-free price path of the shares in the tree is

$$p_t^* = \mathbb{E}_t\left[\sum_{s=t+1}^{\infty} e^{-\theta(s-t)}\frac{u'(y_s)y_s}{u'(y_t)}\right].$$

Interpret this formula in terms of expected payoffs and a "risk premium" by rewriting it in CAPM-style. What is the sign of the risk premium on the tree, and why?

3. Let agents have the iso-elastic period utility function

$$u(c) = \frac{c^{1-\rho}-1}{1-\rho},$$

where $\rho > 0$. Show that this utility function, together with the output process (2-1) implies

$$\mathbb{E}_t\left[(y_s)^{1-\rho}\right] = (y_t)^{1-\rho}e^{\sigma^2(1-\rho)^2(s-t)/2}.$$

[*Hint*: You may find it helpful to use the moment generating function (mgf) for a normal probability distribution. The mgf of a normally distributed random variable X (with $X \sim N(\mu,\sigma^2)$) is $M_X(t) \equiv \mathbb{E}\left[e^{-tX}\right] = e^{\mu t+\sigma^2 t^2/2}$. Also recall that the notation $\mathbb{E}_t[\cdot]$ is short-hand for conditional expectations $\mathbb{E}_t[(y_s)^{1-\rho}|y_t,\cdot].$]

4. Deduce from your finding in part 3 that if $\theta > \sigma^2(1-\rho)^2/2$, then $p_t^* = \chi y_t$ for

$$\chi \equiv \frac{1}{e^{[\theta-\sigma^2(1-\rho)^2/2]}-1}.$$

5. Return to a general, strictly concave period utility function $u(c)$. Let b_t be the random variable $b_t = A(y_t)^\lambda/u'(y_t)$, where $\lambda \equiv \sqrt{2\theta/\sigma^2}$ and A is an arbitrary constant. Show that $p_t^* + b_t$ will satisfy the individual's Euler equation (recall part *a*) in equilibrium. So, b_t is an asset price bubble. Show that $p_t = p_t^* + b_t$ violates the transversality condition

$$\lim_{T\to\infty} e^{-\theta(T-t)}\mathbb{E}_t\left[u'(y_{t+T})p_{t+T}\right] = 0. \tag{2-3}$$

6. Together with the equilibrium Euler equations

$$p_s u'(c_s) = e^{-\theta}\mathbb{E}_t\left[(p_{s+1} + y_{s+1})u'(c_{s+1})\right]$$

from (2-2), the transversality condition (2-3) is *sufficient* for a stochastic price path $\{p_s\}_{s=1}^{\infty}$ to be an equilibrium. You are asked to show here that (2-3) is also a *necessary* condition for an equilibrium. Iterate (2-2) forward to derive

$$p_t u'(y_t) = \mathbb{E}_t\left[\sum_{s=t+1}^{\infty} e^{-\theta(s-t)}u'(y_s)y_s\right] + \lim_{T\to\infty} e^{-\theta(T-t)}\mathbb{E}_t\left[u'(y_{t+T})p_{t+T}\right].$$

Argue that if the limit above is strictly negative, everyone would want to buy more of the tree and never sell it. Then argue conversely that it cannot be an equilibrium either if the limit is strictly positive. Why? [*Hint*: Look for a strategy that raises lifetime utility infinitely through the sale of a tiny fruit tree share today that is never repurchased.] Why does the Euler equation (2-2) not suffice to rule out an asset price bubble?

3 Speculative Hyperinflations (Cagan 1956)

Consider the following money-in-the-utility-function model in continuous time. Agents maximize

$$\int_t^{\infty} e^{-\rho(s-t)}\left[u\left(c(s)\right) + v\left(m(s)\right)\right]\mathrm{d}s.$$

1. There are two types of assets, money M and a nominal bond B. Total nominal asset holdings A are $A = M + B$. The representative agent accumulates nominal assets at a rate $\dot{A} = P(y - c) + iB$. Assume the Fisher parity holds and $i = r + \pi$, where $\pi = \dot{P}/P$ is the inflation rate. Show that the intertemporal budget constraint can be written in real terms as

$$\dot{a} = (y - c) + ra - im,$$

where $a \equiv A/P$, $m \equiv M/P$ are real holdings of nominal assets.

2. Assume that

$$v(m) = m^{\gamma}/\gamma,$$

where $\gamma \in (-\infty, 1)$. [For $\gamma = 0$, $v(m) = \ln m$.] Show that, in this model, the elasticity of money demand with respect to the nominal interest rate i is

$$-\frac{\mathrm{d}\ln m}{\mathrm{d}\ln i} = \frac{1}{1-\gamma}.$$

[You may but need not use a Hamiltonian. An argument about marginal utility equalization to infer the first-order condition is fine.]

3. Argue that in equilibrium $\rho = r$ and $c = y$. Suppose nominal money supply M grows at a rate $\frac{\dot{M}}{M} = \mu$. Derive the equation of motion for real money holdings

$$\frac{\dot{m}}{m} = \mu + \rho - \frac{m^{\gamma-1}}{u'(y)}.$$

4. Keep assuming that $v(m) = m^\gamma/\gamma$, where $\gamma \in (-\infty, 1)$. Show that, in equilibrium and under constant money supply, speculative hyperinflations such that $P \to \infty$ can arise only if the interest elasticity of money demand exceeds unity. [*Hint:* Speculative hyperinflations result in $m = 0$ in the limit. This is possible iff $\lim_{m\to 0} mv'(m) = 0$. Why?]

5. Cagan thought it more plausible that the interest elasticity of money, rather than being constant, rises as expected inflation rises. Assume that

$$v(m) = \frac{m}{\gamma}\left[1 - \ln\left(\frac{m}{\kappa}\right)\right]. \tag{3-4}$$

for some $\kappa > 0$. Normalize output so that $u'(y) = 1$. Show that $v'(m) > 0$ for $m < \kappa$ and that $v'' < 0$. Show that money demand is given by

$$m = \kappa e^{-\gamma i},$$

the so-called Cagan equation.

Verify that, for this equation, the interest elasticity of money demand is γi, which tends to infinity as $i \to \infty$.

6. Show that in the Cagan version of utility from money holdings (3-4)

$$\lim_{m\to 0} mv'(m) = 0$$

so that speculative hyperinflations are possible. [*Hint*: Invoke L'Hôspital's Rule.]

7. Assume there is a fixed flow of government spending g, which is not tax funded but financed by money creation instead. The central bank monetizes the government deficit $\dot{d}_G^{CB}$. So, the government's budget constraint is

$$g = \dot{d}_G^{CB} = \frac{\dot{M}(s)}{P(s)} = \frac{\dot{M}(s)}{M(s)}m(s) = \mu(s)m(s),$$

where the rate of nominal money supply growth $\mu(s)$ is an *endogenous* variable. Assume money demand is of the Cagan form

$$m = \kappa e^{-\gamma i},$$

where $i(s) = \rho + \pi(s)$. Using the equilibrium condition that the inflation rate equals money supply growth less money demand growth,

$$\pi(s) = \mu(s) - \frac{\dot{m}(s)}{m(s)},$$

derive a differential equation of the form $\dot{\pi}(s) = f(\pi(s), g)$ that characterizes the equilibrium. Graph $\dot{\pi}$ on the vertical axis against π on the horizontal axis and show that there can be *two* different steady-state inflation rates. Also show that the low-inflation steady state is dynamically unstable and that the high-inflation steady state is dynamically stable.

4 Exchange Rate Overshooting (Dornbusch, *JPE* 1976)

Suppose money demand takes the Cagan-like form

$$m_t^d - p_t = \phi y_t^d - \eta\, i_{s+1} \qquad \text{with } \phi, \eta > 0$$

where m_t^d denotes the natural logarithm of nominal money demand, p_t the log price level, y_t^d log aggregate demand and i_{s+1} the nominal interest rate ($\ln(1+i_{s+1})$). Money supply is constant $m_t^s = \overline{m}$.

Suppose the uncovered interest parity condition holds and $i_{s+1} = i_{s+1}^* + e_{s+1} - e_t$, where e_t is the log nominal exchange rate. By definition, the real exchange rate is $q_t = e_t + p_t^* - p_t$.

Suppose aggregate demand increases when the real exchange rate depreciates so that

$$y_s^d = \delta q_t \qquad \text{with } \delta \in (0, \tfrac{1}{\phi})$$

The full-employment level of aggregate supply is

$$y_s^s = \bar{y}.$$

Finally, suppose in Keynesian style, that prices are not immediately set to the expected equilibrium level, but adjusted slowly. In particular, prices obey the response function

$$p_{s+1} - p_s = \pi(y_s^d - y_s^s).$$

Standardize all foreign variables to constants $p_s^* = i_{s+1}^* = 0$, and suppose that money markets clear instantaneously: $m_s^d = m_s^s = \bar{m}$.

1. Show that, under these assumption, a Dornbusch model can be built from three equations:

$$\bar{m} - p_s = \phi y_s^d - \eta(e_{s+1} - e_s), \tag{4-1}$$

$$y_s^d = \delta(e_s - p_s) \qquad \delta \in (0, \tfrac{1}{\phi}), \tag{4-2}$$

$$p_{s+1} - p_s = \pi(y_s^d - \bar{y}). \tag{4-3}$$

2. Find the steady-state values of the exchange rate and the price level ($e_{s+1} = e_s = \bar{e}$, $p_{s+1} = p_s = \bar{p}$).

3. Express both $(e_{s+1} - e_s)$ and $(p_{s+1} - p_s)$ as functions of e_s, p_s (substitute exogenous variables in the equations with their relationships to $\bar{e}, \bar{p}$). Find the two functional relationships between p_s and e_s that satisfy $e_{s+1} - e_s = 0$ and $p_{s+1} - p_s = 0$. Draw them in a phase diagram with p_s on the y-axis and e_s on the x-axis. Complete the phase diagram indicating the motion of the system (using the conditions for $e_{s+1} - e_s \geq 0$ and $p_{s+1} - p_s \geq 0$).

   Finally, add a line to the diagram that obeys a 'no-arbitrage condition' as mandated by UIP: $p_s - \bar{p} = -\hat{\theta}\,(e_s - \bar{e})$ for some $\hat{\theta} > \phi\delta/(1-\phi\delta)$. (The $\hat{\theta}$ is not quite the same as in the original Dornbusch model since there is no uncertainty in the present setup.)

4. Is the steady-state stable, saddle-path stable, or unstable? If not, what is the unique stable ("saddle") path given a steady-state of $\bar{p}$ and $\bar{e}$?

5. Suppose all variables except for p_s immediately respond to a monetary shock.

   What is the new steady-state? Draw a 'no-arbitrage' line through it.

   What happens to e_s right after an unanticipated reduction in the monetary base from $\bar{m}$ to $\bar{m}' < \bar{m}$? How do e_{s+s} and p_{s+s} evolve over time?

6. Now consider the dynamics of the model around its steady-state. Using (4-2) and the steady-states of $\bar{e}$ and $\bar{p}$ that you found in part 1, express $y_s^d - \bar{y}$ as a function of $e_s - \bar{e}$ and $p_s - \bar{p}$.

7. Using (4-1) along with the results in parts 1 and 6, express $e_{s+1} - \bar{e}$ as a function of $\frac{1}{\eta}(e_s - \bar{e})$ and $\frac{1}{\eta}(p_s - \bar{p})$.

8. Using (4-2) and (4-3) along with the results in part 1, express $p_{s+1} - \bar{p}$ as a weighted sum of $e_s - \bar{e}$ and $p_s - \bar{p}$.

   From now on, assume that $\pi = \frac{1}{\eta}$ and $\phi = 3$ for simplicity.

9. Write your findings from 7 and 8 into a system of two difference equations that takes the form

$$\begin{pmatrix} e_{s+1} - \bar{e} \\ p_{s+1} - \bar{p} \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} e_s - \bar{e} \\ p_s - \bar{p} \end{pmatrix}. \tag{4-4}$$

   Find the eigenvalues and eigenvectors of the system.

   [*Hint*: An intermediate result is $\mathrm{tr}(\mathbf{A}) = \frac{2(\delta+\eta)}{\eta}$ and $\det(\mathbf{A}) = 1 - \frac{(1-2\eta)\delta}{\eta^2}$.]

10. Is the system stable? That is, do the exchange rate and price levels converge to the steady-state?

    If not, set the coefficient of the unstable root to zero. Then, using the simplified system in part 9 and the results from 6, express p_{s+1} as a function of e_{s+1} and exogenous variables. Draw this function into the phase diagram from part 5 so that it passes through the steady-state. (You can safely pretend that $\bar{y} \neq 0$. Also, remember that $\delta < \frac{1}{\phi}$.)

What did you just find? What is the intuition for the fact that the economy obeys this relationship?