Economics 232c — Spring 2003

International Macroeconomics

# Problem Set 2

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**Due:** **Tue, May 13, 2003**
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## 1 Optimal Consumption with Complete and Incomplete Asset Markets

Consider a two-period model of consumption by a representative agent, who faces a market interest rate $r$ for riskless loans $B_2$. Labor earnings today are $Y_1$. There are $\mathcal{S}$ states of nature tomorrow, and earnings realizations $Y_2(s)$ differ across states. Each state $s$ occurs with a probability $\pi(s)$. The representative consumer maximizes expected life-time utility

$$U_1 = C_1 - \frac{a_0}{2}(C_1)^2 + \beta \mathbb{E}_1 \left[ C_2 - \frac{a_0}{2}(C_2)^2 \right],$$

where period utility is quadratic and $a_0 > 0$. The consumer's time preference parameter $\beta$ is such that $\beta = \frac{1}{1+r}$. You may assume that $Y_1$ and all $Y_2(s)$ levels are small enough so that the marginal utility of period consumption $1 - a_0 C$ is strictly positive for all consumption levels.

When asset markets are incomplete, the relevant constraints can be written as

$$\begin{aligned} B_2 &= (1+r)B_1 + Y_1 - C_1 \\ C_2(s) &= (1+r)B_1 + Y_2(s) \qquad \forall s \in \{1, \ldots, \mathcal{S}\} \end{aligned}$$

Initial bond holdings $B_1$ are given and $B_2$ denotes bonds accumulated through the end of period 1.

1. Show that the preceding constraints imply the $\mathcal{S}$ intertemporal budget constraints
$$C_1 + \frac{C_2(s)}{1+r} = (1+r)B_1 + Y_1 + \frac{Y_2(s)}{1+r}$$
for all states $s \in \{1, \ldots, \mathcal{S}\}$.

2. Temporarily ignore the nonnegativity constraints $C_2(s) \geq 0$ for states $s$ of nature tomorrow. Compute the optimal level of consumption $C_1$ today. What are the implied values of $C_2(s)$. What would the optimal level of $C_1$ be for an infinitely-lived agent and output uncertainty in each future period?

3. Consider the nonnegativity constraint on $C_2$. Relabel the states of nature such that $Y_2(1) = min_s\{Y_2(s)\}$. Show that if

$$(1+r)B_1 + Y_1 + \frac{2+r}{1+r}Y_2(1) \geq \mathbb{E}_1\left[Y_2\right]$$

then the $C_1$ computed in part 2 (for the two-period case) is still valid. What is the intuition? Suppose the preceding inequality fails to hold. Show that optimal consumption $C_1$ today is lower and equals

$$C_1 = (1+r)B_1 + Y_1 + \frac{Y_2(1)}{1+r}$$

This is a precautionary savings effect. Explain why it arises. Does the bond Euler equation hold in this case?

[*Hint*: Apply the Kuhn-Tucker theorem to derive optimal consumption $C_1$.]

4. Assume the consumer faces *complete* global asset markets with $p(s)/(1+r)$, the state $s$ Arrow-Debreu security price, equal to $\pi(s)/(1+r)$. Explain why these are called actuarially fair prices. Find the optimal values of $C_1$ and $C_2(s)$. Why can nonnegativity constraints be disregarded in the case of complete asset markets?

# 2 Consumption-based CAPM model (from Lucas, Econometrica 1978)

Consider a representative agent and a production process, in which a random and exogenous amount of perishable output $y_t$ falls from one fruit tree each period. There is no other output. The fruit output follows the stochastic process

$$\ln y_t = \ln y_{t-1} + \epsilon_t, \tag{2-1}$$

where $\epsilon_t$ is an unanticipated Gaussian shock with $\mathbb{E}_{t-1}\left[\epsilon_t\right] = 0$ and $\mathcal{N}(0, \sigma_2)$. So, $y_t$ has a lognormal distribution. There is no investment, that is there is no way to grow more fruit trees.

The agent's life-time utility function takes the particular form

$$U_t = \mathbb{E}_t\left[\sum_{s=t}^{\infty} e^{-\theta(s-t)}u(c_s)\right],$$

where $\theta > 0$ is the rate of time preference. Assume there is a competitive stock market, in which people can trade shares in the fruit tree. The price of a share is denoted $p_t$ and ex-dividend. So, if the agent buys a share on date $t$, she gets her first dividend payment only on date $t+1$.

1. Argue that a share holder's individual budget constraint is $c_s + p_{s+1}x_{s+1} \leq (y_s + p_s)x_s$, where $x_s$ denotes the shares in the tree that the agent holds at the end of period $s-1$. Setup up the Bellman equation for the representative agent's intertemporal consumption and share-holding choice. Show that the optimal consumption path satisfies the Euler equations

$$p_s u'(c_s) = e^{-\theta}\mathbb{E}_t\left[(p_{s+1} + y_{s+1})u'(c_{s+1})\right] \tag{2-2}$$

for all $s \geq t$.

2. Show that, in equilibrium, the fundamental bubble-free price path of the shares in the tree is

$$p_t^* = \mathbb{E}_t\left[\sum_{s=t+1}^{\infty} e^{-\theta(s-t)}\frac{u'(y_s)y_s}{u'(y_t)}\right]$$

Interpret this formula in terms of expected payoffs and a "risk premium." What is the sign of the risk premium on the tree, and why?

3. Let agents have the iso-elastic period utility function

$$u(c) = \frac{c^{1-\rho} - 1}{1-\rho},$$

where $\rho > 0$. Show that this utility function, together with the output process (2-1) implies

$$\mathbb{E}_t\left[(y_s)^{1-\rho}\right] = (y_t)^{1-\rho}e^{\sigma^2(1-\rho)^2(s-t)2}.$$

[*Hint*: You may find it helpful to use the moment generating function (mgf) for a normal probability distribution. The mgf of a normally distributed random variable $X$ (with $X \sim N(\mu, \sigma^2)$) is $M_X(t) \equiv \mathbb{E}\left[e^{-tX}\right] = e^{\mu t + \frac{\sigma^2}{2}t^2}$.]

4. Deduce from your finding in part 3 that if $\theta > \sigma^2(1-\rho)^2/2$, then $p_t^* = \chi y_t$ for

$$\chi \equiv \frac{1}{e^{[\theta - \sigma^2(1-\rho)^2/2]} - 1}.$$

5. Return to general strictly concave period utility function $u(c)$. Let $b_t$ be the random variable $A(y_t)^\lambda / u'(y_t)$, where $\lambda \equiv \sqrt{2\theta/\sigma^2}$ and $A$ is an arbitrary constant. Show that $p_t^* + b_t$ will satisfy the individual's Euler equation (recall part *a*) in equilibrium. So, $b_t$ is an asset price bubble. Show that $p_t = p_t^* + b_t$ violates the transversality condition

$$\lim_{T\to\infty} e^{-\theta(T-t)}\mathbb{E}_t\left[u'(y_{t+T})p_{t+T}\right] = 0. \tag{2-3}$$

6. Together with with the equilibrium Euler equations

$$p_s u'(c_s) = e^{-\theta}\mathbb{E}_t\left[(p_{s+1}+y_{s+1})u'(c_{s+1})\right]$$

from (2-2), the transversality condition (2-3) is *sufficient* for a stochastic price path $\{p_s\}_{s=1}^{\infty}$ to be an equilibrium. You are asked to show here that (2-3) is also a *necessary* condition. for an equilibrium. Iterate (2-2) forward to derive

$$p_t u'(y_t) = \mathbb{E}_t\left[\sum_{s=t+1}^{\infty} e^{-\theta(s-t)}u'(y_s)y_s\right] + \lim_{T\to\infty} e^{-\theta(T-t)}\mathbb{E}_t\left[u'(y_{t+T})p_{t+T}\right].$$

Argue that if the limit above is strictly negative, everyone would want to buy more of the tree and never sell it. Then argue conversely that it cannot be an equilibrium either if the limit is strictly positive. Why? [*Hint*: Look for a strategy that raises lifetime utility infinitely through the sale of a tiny fruit tree share today that is never repurchased.] Why does the Euler equation (2-2) not suffice to rule out an asset price bubble?

# 3 Indexed Debt Contracts in Lieu of Insurance

Consider a small-country sovereign-debt model with two periods (today and tomorrow), where consumption takes place only tomorrow.[1] The country maximizes $U_1 = \mathbb{E}\left[u(C_2)\right]$. Tomorrow's output $Y_2$ is uncertain and given by $Y_2 = \overline{Y} + \epsilon$, where $\epsilon$ is a mean-zero shock to output and can take any of $\mathcal{S}$ different (finite) values. The probability of a shock-realization $\epsilon = \epsilon_s$ is $\pi(\epsilon_s)$ ($\sum_{s=1}^{\mathcal{S}} \pi(\epsilon_s) = 1$) so that $\mathbb{E}\left[Y_2\right] = \overline{Y}$. The country has no initial endowment except for the expected endowment of $\overline{Y}$ tomorrow.

Debt contracts can only be enforced up to a share $\eta$ of income that can be seized. Contrary to the model presented in class, do not allow the country to write insurance contracts.

Instead, assume that the country is only able to borrow in the form of equity or output-indexed debt contracts. In particular, think of the country as borrowing $D$ today and making *nonnegative* payments $P(\epsilon) \geq 0$ tomorrow subject to the zero-profit condition

$$\sum_{s=1}^{\mathcal{S}} \pi(\epsilon_s)P(\epsilon_s)$$

and the incentive-compatibility constraint

$$P(\epsilon_s) \leq \eta\left[\overline{Y} + (1+r)D + \epsilon_s\right] \quad \forall s = 1,\ldots,\mathcal{S}. \tag{3-4}$$

The country also has access to a domestic linear production technology (or banking sector) such that $F(K) = (1+r)K$. So, the country can invest borrowed funds $D$ locally and earn the world-market rate of return.

[1]See lecture 9 notes from April 29, 2003.

1. Note that the country will use foreign borrowing for domestic lending. So, $K = D$. Why?

   Observe that in the present model, creditors pay up front so their credibility is not at issue. Use a no-arbitrage argument to justify the incentive-compatibility constraint in the presence of the domestic linear production technology (or banking sector).

2. Treat $D$ as given. Characterize the optimal incentive-compatible payment schedule $P(\epsilon_s)$ $\forall s = 1, \ldots, \mathcal{S}$. Draw a diagram illustrating your answer. [*Hint*: Order the states of nature so that $\underline{\epsilon} = \epsilon_1, \leq \ldots \leq \epsilon_{\mathcal{S}} = \bar{\epsilon}$ and show that $C(\underline{\epsilon}) = \overline{Y} + \underline{\epsilon} + (1 + r)D$.]

3. How large must $\eta$ be to achieve full insurance in the present model? How large was $\eta$ when the country was allowed to write insurance contracts?

# 4 Cagan's Monetary Model

Consider the following money-in-the-utility-function model in continuous time. Agents maximize

$$\int_t^\infty e^{-\delta(s-t)} \left[u\left(c(s)\right) + v\left(m(s)\right)\right] \mathrm{d}s.$$

1. Assume that
$$v(m) = m^\gamma/\gamma,$$
   where $\gamma \in (-\infty, 1)$. [For $\gamma = 0$, $v(m) = \ln m$.] Show that in this model, the elasticity of money demand with respect to the nominal interest rate $i$ is
$$-\frac{\mathrm{d} \ln m}{\mathrm{d} \ln m} = \frac{1}{1-\gamma}.$$
   [You may but need not use a Hamiltonian.]

2. Keep assuming that $v(m) = m^\gamma/\gamma$, where $\gamma \in (-\infty, 1)$. Show that, in equilibrium and under constant money supply, speculative hyperinflations such that $P \to \infty$ can arise only if the interest elasticity of money demand exceeds unity.

3. P. Cagan thought it more plausible that the interest elasticity of money, rather than being constant, rises as expected inflation rises. Assume that
$$v(m) = \frac{m}{\gamma}\left[1 - \ln\left(\frac{m}{\kappa}\right)\right]. \qquad \text{(4-5)}$$
   for some $\kappa > 0$. Normalize output so that $u'(y) = 1$. Show that $v'(m) > 0$ for $m < \kappa$ and that $v'' < 0$. Show that money demand is given by
$$m = \kappa e^{-\gamma i},$$

the so-called Cagan equation.

Verify that, for this equation, the interest elasticity of money demand is $\gamma i$, which tends to infinity as $i \to \infty$.

4. Show that in the Cagan version of utility from money holdings (4-5)

$$\lim_{m \to 0} m v'(m) = 0$$

so that speculative hyperinflations are possible. [*Hint*: Invoke L'Hôspital's Rule.]

5. Assume there is a fixed flow of government spending $g$, financed by money creation. So, the government's budget constraint is

$$g = \frac{\dot{M}(s)}{P(s)} = \frac{\dot{M}(s)}{M(s)} m(s) = \mu(s) m(s),$$

where the rate of nominal money supply growth $\mu(s)$ necessarily is an endogenous variable. Assume money demand is of the Cagan form

$$m = \kappa e^{-\gamma i},$$

where $i(s) = \delta + \pi(s)$. Using the equilibrium condition that

$$\pi(s) = \mu(s) - \frac{\dot{m}(s)}{m(s)},$$

derive a differential equation of the form $\dot{\pi}(s) = f(\pi(s), g)$ that characterizes the equilibrium. Graph $\dot{\pi}$ on the vertical axis against $\pi$ on the horizontal axis and show that there can be *two* different steady-state inflation rates. Also show that the low-inflation steady state is dynamically unstable and that the high-inflation steady state is dynamically stable.