Econ 202a Spring 2000
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Answer Keys for Problem Set 3

1 Delegated control over monetary policy

Output is given by the Lucas supply function

$$y = \bar{y} + b(\pi - \pi^e), \tag{1}$$

social welfare by

$$S = \tilde{\gamma} y - \frac{a}{2}\pi^2, \tag{2}$$

and the central banker's objective function by

$$S^{CB} = c\tilde{\gamma} y - \frac{a}{2}\pi^2, \tag{3}$$

where $c \in R$. The coefficient $\tilde{\gamma}$ is a random variable with mean $\mathbb{E}[\tilde{\gamma}] = \bar{\gamma}$ and variance $\text{Var}(\tilde{\gamma}) = \sigma^2$.

We want to solve the model so that all decisions are 'time-consistent', as macroeconomists like to say. If these macroeconomists also were game theorists, they would probably prefer to call the equilibrium of the game 'subgame perfect' and not 'time-consistent.' The concepts are the same, but let's keep the notion of 'time consistency' for the purpose of this macro exercise. In order to find a time-consistent equilibrium, we need to start with the choice of the *last* agent in the chain of decisions. In our model, there are three stages of decisions. First, the 'public' or the 'private sector' chooses its inflation expectations given its expectations of $\tilde{\gamma}$. Then, on the second stage, nature draws a general preference parameter $\tilde{\gamma}$ and reveals it to the central banker. Thus, the central banker has superior information about the general public's preferences. Finally, on the third stage, the central banker chooses the inflation rate.

1.1 [1a] Central banker's choice

Let's start at the end of the chain. (Game theorists would say: Let's apply backward induction.) The central banker maximizes

$$\max_{\pi} S^{CB} = \max_{\pi} \left(c\gamma\bar{y} + c\gamma b(\pi - \pi^e) - \frac{a}{2}\pi^2 \right), \tag{4}$$

where we have used (1) to express output in terms of the inflation rate π. Note that the central banker knows the realization of γ at the time of her decision. The first-order condition to this problem is

$$c\gamma b - a\pi^* = 0.$$

Thus, the central banker will optimally choose[1]

$$\pi^* = c\frac{\gamma b}{a}. \tag{5}$$

The central banker will choose the higher an inflation rate the more responsive output is (b) and the less inflation-averse the general public is (a).

1.2 [1b] Expectation of inflation π^e

The general public has to make up its mind about expected inflation before nature reveals what realization $\tilde{\gamma}$ takes this time. Their expectations about the impact of $\tilde{\gamma}$ are rational. Therefore,

$$\pi^e = \mathbb{E}[\pi^*] = c\frac{\bar{\gamma} b}{a}. \tag{6}$$

So, again, output will only respond to unexpected and unsystematic deviations of the central banker from her 'rule' $\pi^* = c\gamma b/a$. This can be seen from (1):

$$y = \bar{y} + \frac{cb^2}{a}(\tilde{\gamma} - \bar{\gamma}). \tag{7}$$

So, $\mathbb{E}[y] = \mathbb{E}\left[\bar{y} + \frac{cb^2}{a}(\gamma - \bar{\gamma})\right] = \bar{y}$. Note also that inflation is positive as long as c is, but output does not systematically rise above its reference level $\bar{y}$.

[1] We need not worry about second-order conditions. This problem is concave in π because π^2 is a convex function of π.

The problem here is not even one of possible time-inconsistency. *Irrespective* of what π^e the general public has chosen on the first stage, the central banker will always set $\pi^* = c\gamma b/a$ on the third stage of the game (see (5)). But does the general public want a central banker to choose a positive inflation rate, $\pi^* > 0$? After all, output will not systematically change for $\pi^* > 0$, but inflation is considered bad since $a > 0$.

1.3 [1c] Expected Social Welfare

To see what kind of central banker the general public would love to employ, let's maximize expected social welfare. Using (7) in (2) and taking expectations of both sides, we find expected social welfare

$$\begin{aligned}
\mathbb{E}[S] &= \mathbb{E}\left[\tilde{\gamma}\bar{y} + \frac{cb^2}{a}\tilde{\gamma}\left(\tilde{\gamma}-\bar{\gamma}\right) - \frac{a}{2}\left(\tilde{\gamma}\frac{cb}{a}\right)^2\right] \\
&= \bar{\gamma}\bar{y} + \frac{cb^2}{a}\left\{\left(1-\frac{c}{2}\right)\mathbb{E}\left[\tilde{\gamma}^2\right] - \bar{\gamma}^2\right\} \\
&= \bar{\gamma}\bar{y} + \frac{cb^2}{a}\left\{\left(1-\frac{c}{2}\right)\sigma^2 - \frac{c}{2}\bar{\gamma}^2\right\} \\
&= \bar{\gamma}\bar{y} + \frac{cb^2}{a}\sigma^2 - \frac{c^2b^2}{2a}\left(\sigma^2+\bar{\gamma}^2\right). \qquad (8)
\end{aligned}$$

1.4 [1d] The best central banker

What is the best central banker? Or, what value of c maximizes expected social welfare? To see that, maximize (8) with respect to c. The first order condition of this problem is[2]

$$\frac{b^2}{a}\sigma^2 - \frac{c^*b^2}{a}\left(\sigma^2+\bar{\gamma}^2\right) = 0.$$

Hence,

$$c^* = \frac{\sigma^2}{\sigma^2+\bar{\gamma}^2}. \qquad (9)$$

[2] Again, we need not concern ourselves with second-order conditions. This problem is concave in c because c^2 is a convex function of c.

So, the general public should specify a contract with the central banker. That contract should state: We will pay you exactly $c^*\tilde{\gamma}y - \frac{a}{2}\pi^2$ each period. Now you choose π. The optimal contract induces the central banker to be more conservative than the general public. Since $c^* \in (0,1]$, the central banker will always put less weight on output stabilization and more weight on low inflation than the general public. The optimal contract makes the central banker more conservative than the public sector is.

We can say even more about the optimal contract. Whenever the variance of preferences is extremely high, that is whenever σ^2 is large, c^* is close to one. Thus, the public wants a very responsive central banker whenever it knows that there is a lot of change in fundamentals. However, the more responsive the central banker gets, the higher equilibrium inflation will be, too. Hence, there is a trade-off. How strong the trade-off becomes depends on the expected value of γ. Why? The higher the expected value of γ, the higher expected equilibrium inflation will be for any given c: $\pi^e = \bar{\gamma}cb/a$. Thus, the expected welfare loss from high inflation will weigh more if $\bar{\gamma}$ is high. Therefore, c^* is falling in $\bar{\gamma}$. In fact, it is close to zero when $\bar{\gamma}$ far exceeds σ. The optimal contract for the central banker incorporates this trade-off. A responsive central banker is nice, but a too responsive central banker causes too high an equilibrium inflation rate.

2 A simple model of overshooting

Our simple Dornbusch model of overshooting in discrete time consists of three equations:

$$\bar{m} - p_t = \phi y_t^d - \eta(e_{t+1} - e_t), \tag{10}$$

$$y_t^d = \delta(e_t - p_t) \qquad \delta \in (0, \frac{1}{\phi}), \tag{11}$$

$$p_{t+1} - p_t = \pi(y_t^d - \bar{y}). \tag{12}$$

All variables are in logs. p_t is the aggregate price level, y_t^d is aggregate demand, e_t is the nominal exchange rate (denoted in dollars per foreign currency), $\bar{m}$ is fixed money supply, and $\bar{y}$ the full-employment level of output.

Note that this model is Keynesian in style; we accept that output can systematically deviate from the full-employment level for several periods until price adjusts in a manner determined by (12). Our Dornbusch model is in fact a dynamic system in two variables, e_t and p_t, and two equations. To see this most clearly, plug (11) into (10) and (12) to obtain

$$e_{t+1} - e_t = \frac{\phi\delta}{\eta} e_t + \left(1 - \frac{\phi\delta}{\eta}\right) p_t - \frac{1}{\eta}\bar{m}$$

and

$$p_{t+1} - p_t = \pi\delta(e_t - p_t) - \pi\bar{y}.$$

But there are more insightful ways than this method of brute force to find out about the dynamic properties of our version of the Dornbusch model.

2.1 [2a] The steady state

A very good point of departure usually is the steady state. We can derive it by setting all endogenous variables equal to hypothesized steady state values. So, we simply try whether such a steady state exists or not. In steady state, $e_{t+1} = e_t = \bar{e}$ and $p_{t+1} = p_t = \bar{p}$. Therefore, $e_{t+1} - e_t = 0$ and $p_{t+1} - p_t = 0$. Using $p_{t+1} - p_t = 0$ in (12), we find that $y_t^d = \bar{y}$ in a steady state. Using $y_t^d = \bar{y}$ and $e_{t+1} - e_t = 0$ in (10), we find that

$$\bar{p} = \bar{m} - \phi\bar{y}. \tag{13}$$

Finally, using that very fact in (11), we obtain

$$\bar{e} = \frac{1}{\delta}\bar{y} + \bar{p} = \bar{m} + \frac{1 - \phi\delta}{\delta}\bar{y}. \tag{14}$$

Thus, a steady state exists. We have just found it by trial. Whether this steady state is stable or not still has to be investigated. It will turn out that it is unstable in one dimension, but stable in another. More on this below. For now, let's only make one more observation. The requirement that $\delta < \frac{1}{\phi}$ is obviously crucial to let the steady state output have a positive relationship with the steady state nominal exchange rate. For $\delta < \frac{1}{\phi}$ it must be the case that $\phi\delta < 1$. The key term $1 - \phi\delta$ will show up several more times.

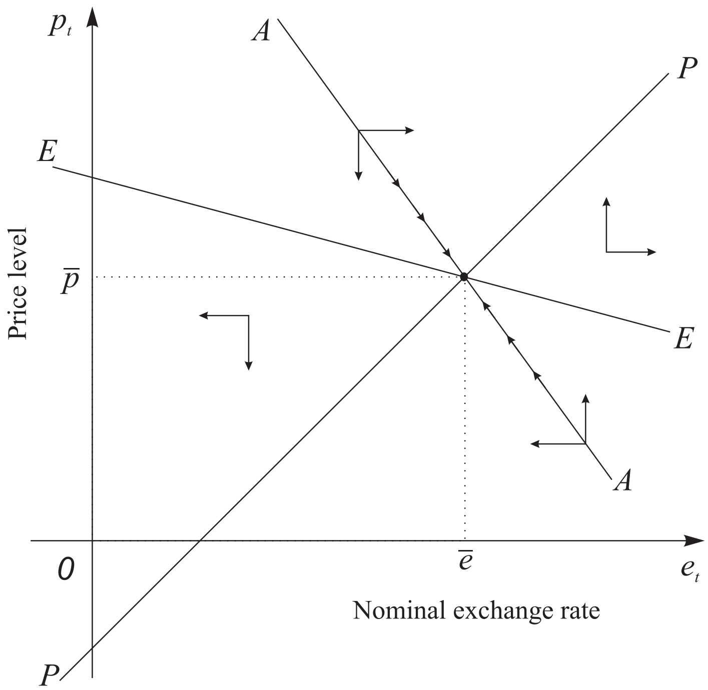

Figure 1: **Phase Diagram for initial steady state**

2.2 [2b] General dynamics of the system

In order to find out more about the dynamic behavior of the system, we want to know under what conditions the nominal exchange rate is increasing (depreciating), and under what conditions the price level is increasing. Start with the nominal exchange rate. Using (10) and (11), the change in the exchange rate over one period can be related to previous levels and exogenous variables:

$$\begin{aligned} e_{t+1} - e_t &= -\frac{\bar{m} - p_t}{\eta} + \frac{\phi}{\eta} y_t^d = -\frac{\bar{m} - p_t}{\eta} + \frac{\phi}{\eta} \delta (e_t - p_t) \\ &= \frac{1 - \phi\delta}{\eta} p_t + \frac{\phi\delta}{\eta} e_t - \frac{\bar{m}}{\eta} \\ &= \frac{1 - \phi\delta}{\eta} (p_t - \bar{p}) + \frac{\phi\delta}{\eta} (e_t - \bar{e}), \end{aligned} \tag{15}$$

where the last step makes use of the steady state levels of e_t and p_t. Thus,

$$\begin{aligned} e_{t+1} - e_t \geq 0 \quad &\Leftrightarrow \quad p_t \geq -\frac{\phi\delta}{1-\phi\delta} e_t + \frac{\bar{m}}{1-\phi\delta} \\ &\Leftrightarrow \quad p_t - \bar{p} \geq -\frac{\phi\delta}{1-\phi\delta} \left(e_t - \bar{e} \right). \end{aligned} \tag{16}$$

Using (12) along with (11) and also applying the steady state definitions, we can derive the change in price levels, too.

$$p_{t+1} - p_t = \pi (y_t^d - \bar{y}) = \pi\delta \left(e_t - \bar{e} \right) - \pi\delta \left(p_t - \bar{p} \right). \tag{17}$$

Hence,

$$\begin{aligned} p_{t+1} - p_t \geq 0 \quad &\Leftrightarrow \quad p_t - \bar{p} \leq e_t - \bar{e} \\ &\Leftrightarrow \quad p_t \leq e_t - \frac{\bar{y}}{\delta}. \end{aligned} \tag{18}$$

The second equivalence follows since $p_t \leq e_t - (\bar{e} - \bar{p}) = e_t - \frac{\bar{y}}{\delta}$. Last, we have a no-arbitrage relationship

$$p_{t+1} - \bar{p} = -\hat{\theta} \left(e_{t+1} - \bar{e} \right) \tag{19}$$

to which the economy must always adhere.

We can put this information together in a phase diagram. First, we want to know the curves where the dynamic forces on either the exchange rate changes or the price level are absent. That is, we want to draw the lines where $e_{t+1} - e_t = 0$ and where $p_{t+1} - p_t = 0$. From (16) and (18) we know that these curves are straight lines with $p_t - \bar{p} = -\frac{\phi\delta}{1-\phi\delta} \left(e_t - \bar{e} \right)$ and $p_t - \bar{p} = e_t - \bar{e}$. Hence, both of them pass through the steady state. And they better do because the steady state is certainly a place where no dynamic forces are present at all. Figure 1 depicts these two lines as EE for $p_t - \bar{p} = -\frac{\phi\delta}{1-\phi\delta} \left(e_t - \bar{e} \right)$ (from (16)) and PP for $p_t - \bar{p} = e_t - \bar{e}$ (from (18)). Second, how does the system behave off these lines? For that, we can use (16) and (18) again. From (16) we can infer that the nominal exchange rate increases (depreciates) whenever we are above the EE line in our phase diagram; it decreases (appreciates) otherwise. We indicate this fact with arrows pointing East above the EE line, and pointing West below the EE line. Similarly, from (18) we can infer that the price level increases whenever we are below the PP line and decreases otherwise. This gives rise to an interesting pattern. Before we turn to that, let's draw the so-called 'no-arbitrage' line (19) into the diagram, too. Call it AA. Here we were careful to make the AA line steeper than the EE. The reason for that will become clear shortly.

2.3 [2c] Stability of the steady state

Is the steady state stable? Well, as both lawyers and economists usually like to say: It depends. The arrows—the dynamic forces of the system—*possibly* take us back to the steady state whenever we are either to the Northwest or to the Southeast of the steady state. In this dimension the steady state can be stable indeed. However, the forces take us away from the steady state whenever our economy initially finds itself either to the Northeast or to the Southwest of the steady state. In that dimension the system is unstable. Hence, there must be a saddle path that prevents the economy from exploding. This saddle path must run from Northwest to Southeast, and it must go through the steady state.

In fact, we just drew such a line. We know that the economy must always obey the 'no-arbitrage' relationship. In order to not be explosive, the economy must also always be on the saddle path which has to go through the steady state, too. Since both of these relationships always have to be satisfied, they can't correspond to two separate lines. If they corresponded to two separate lines, they would only be simultaneously satisfied in the steady state. But both the 'no-arbitrage' condition and the 'non-explosiveness' condition have to be satisfied in and out of steady state. Thus, the two corresponding lines must coincide. The no-arbitrage condition is nothing but the saddle path. If we had happened to draw the no-arbitrage line flatter than the EE line, after this reasoning we would have to go back to the phase diagram in figure 1 and draw the AA line steeper than EE. The AA line simultaneously represents the no-arbitrage condition and the saddle path.

In the Dornbusch model, we have to require that our economy be non-explosive. Somehow, this assumption comes out of the blue in the Dornbusch model. And it will have to come out of the blue in any Keynesian style model because unfortunately we don't have any optimizing agents here who could tell us what they would optimally choose to do. Maybe they really love explosive economies. As it turns out in models with micro-foundations, however, agents don't like explosive paths in general. So the requirement of a non-explosive economy is reasonable, although not rigorously justifiable in

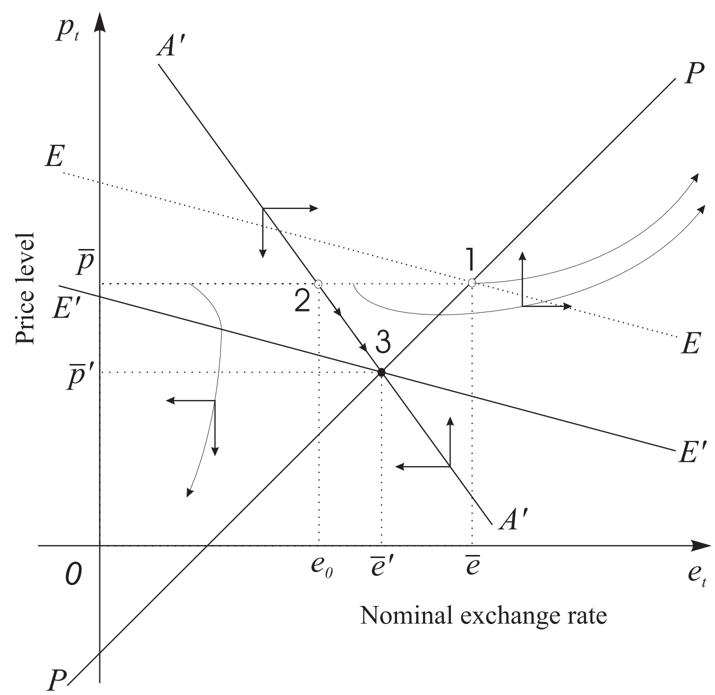

Figure 2: **Phase Diagram for change in money supply**

Keynesian models.

2.4 [2d] A monetary contraction at home

We have setup the model, and we have derived its main properties. Now let's play with the model. Suppose the steady monetary supply is unexpectedly reduced from $\bar{m}$ to $\bar{m}' < \bar{m}$ at time $t = 0$. What happens to our phase diagram? As (18) shows, the PP curve is unaffected because $p_t = e_t - (\bar{e} - \bar{p}) = e_t - \frac{\bar{y}}{\delta}$. The EE curve is shifted downward, however. Following (16), the underlying relationship has changed from $p_t = -\frac{\phi\delta}{1-\phi\delta}e_t + \frac{\bar{m}}{1-\phi\delta}$ to $p_t = -\frac{\phi\delta}{1-\phi\delta}e_t + \frac{\bar{m}'}{1-\phi\delta}$. The no-arbitrage condition must hold again, and therefore the AA curve must now pass through the new steady state.

By assumption, we are in a world of sticky prices. Whereas all variables are assumed to adjust immediately, p_t remains at its given level for one period. What happens to e_t right after a reduction in the monetary base from $\bar{m}$ to $\bar{m}'$? Since prices respond too little (not at all), it is not so surprising that the other leading variable in our system, the nominal exchange rate must make up for the slack in prices. It will overshoot. In which way? Note that at the time of the unexpected change in money supply, the economy is still in the old steady state at point 1 in figure 2. Now everything is allowed to change except for prices. So the economy is restricted to jump out of the old steady state along a horizontal line through point 1. Where will it jump? The only possible point to jump to is 2. From anywhere else, the economy would have to explode subsequently. Some sample paths off the saddle path are drawn in figure 2. They are all explosive. Even remaining at the old steady state would make the economy explode. (Note that the dynamic forces, the arrows, are now all relative to the new steady state 3.) So, at date $t = 0$, when the money supply is changed, the economy jumps to 2. From then on, it simply obeys the dynamics of the new system. These dynamics take the economy gradually from point 2 to point 3 in the phase diagram. The economy converges to the new steady state at 3. The nominal exchange rate e_t and the price level p_t move jointly along the saddle path from $t \geq 1$ on.

The nominal exchange rate overshoots. It initially appreciates to a level e_0 beyond the future steady state level $\bar{e}'$. Subsequently, the nominal exchange rate *depreciates* to the new steady-state level $\bar{e}'$ despite the initial monetary tightening at home. Yet, the over-all effect of the monetary contraction will be an appreciation of the exchange rate from $\bar{e}$ to $\bar{e}'$, as we should reasonably expect after a reduction of the money base at home.

3 Dynamics around the steady state

Let's make the graphical analysis of our Dornbusch model from section 2 rigorous. We will derive the saddle path formally. The first three sub-questions prepare for the derivation.

3.1 [3a] Output deviations from steady state

Using (11) and the steady states of $\bar{e}$ and $\bar{p}$ from section 2, the output deviation from steady state can be expressed as

$$\begin{aligned} y_t^d - \bar{y} &= \delta(e_t - p_t) - \bar{y} \\ &= \delta(e_t - \bar{e}) + \delta\bar{e} - \delta(p_t - \bar{p}) - \delta\bar{p} - \bar{y} \\ &= \delta(e_t - \bar{e}) - \delta(p_t - \bar{p}). \end{aligned} \tag{20}$$

3.2 [3b] Exchange rate deviations from steady state

Similarly, using (10) along with the results in section 2.1 and section 3.1, exchange rate deviations from steady state are

$$\begin{aligned} e_{t+1} - \bar{e} &= \frac{1}{\eta}\left(\phi y_t^d + p_t - \bar{m}\right) + e_t - \bar{e} \\ &= \frac{1}{\eta}\left[\phi\delta(e_t - \bar{e}) - \phi\delta(p_t - \bar{p}) + \phi\bar{y} + p_t - \bar{m}\right] + e_t - \bar{e} \\ &= \frac{\eta + \phi\delta}{\eta}(e_t - \bar{e}) + \frac{1 - \phi\delta}{\eta}(p_t - \bar{p}). \end{aligned} \tag{21}$$

3.3 [3c] Price deviations from steady state

Finally, using (11) and (12) along with the results in section 2.1, the price deviation from steady state becomes a weighted sum of $e_t - \bar{e}$ and $p_t - \bar{p}$:

$$\begin{aligned} p_{t+1} - \bar{p} &= p_t - \bar{p} + \pi\delta(e_t - p_t) - \pi\bar{y} \\ &= \pi\delta(e_t - \bar{e}) + (1 - \pi\delta)(p_t - \bar{p}). \end{aligned} \tag{22}$$

3.4 [3d] The eigenvalues of the dynamic system

With results (20), (21), and (22) at hand, we can rewrite the entire Dornbusch model in terms of deviations from steady state. This is particularly useful because we obtain a system of two first-order difference equations for which we know the general solution. Writing (21) and (22) in matrix form, we find

$$\begin{pmatrix} e_{t+1} - \bar{e} \\ p_{t+1} - \bar{p} \end{pmatrix} = \begin{pmatrix} \frac{\eta+\phi\delta}{\eta} & \frac{1-\phi\delta}{\eta} \\ \pi\delta & 1 - \pi\delta \end{pmatrix} \begin{pmatrix} e_t - \bar{e} \\ p_t - \bar{p} \end{pmatrix}. \tag{23}$$

We know the eigenvalues and eigenvectors of such a system. We even have formulas for them.[3] The eigenvalues of a linear difference equation system in two variables, $\mathbf{x}_{t+1} - \bar{\mathbf{x}} = \mathbf{A}(\mathbf{x}_t - \bar{\mathbf{x}})$ with $\mathbf{x} \in R^2$, are

$$\lambda_{1,2} = \frac{\mathrm{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{\left[\mathrm{tr}(\mathbf{A})\right]^2 - 4\det(\mathbf{A})}.$$

[3] For a derivation, see section 3.2 of the handout "Linear Difference Equations and Autoregressive Processes." (http://socrates.berkeley.edu/~muendler/teach/diffeqn.pdf)

The eigenvectors are

$$\mathbf{e}_i = \begin{pmatrix} \frac{\lambda_i - a_{22}}{a_{21}} \\ 1 \end{pmatrix} \qquad i = 1, 2,$$

and the general solution is

$$\mathbf{x}_t = \begin{pmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{21}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 (\lambda_1)^t \\ c_2 (\lambda_2)^t \end{pmatrix},$$

where the coefficients c_1 and c_2 have to be determined through boundary conditions.

For simplicity, we assume that $\pi = \frac{1}{\eta}$ and $\phi = 3$. Then (23) becomes

$$\begin{pmatrix} e_{t+1} - \bar{e} \\ p_{t+1} - \bar{p} \end{pmatrix} = \frac{1}{\eta} \begin{pmatrix} \eta + 3\delta & 1 - 3\delta \\ \delta & \eta - \delta \end{pmatrix} \begin{pmatrix} e_t - \bar{e} \\ p_t - \bar{p} \end{pmatrix}. \tag{24}$$

And, as stated in the hint, $\text{tr}(\mathbf{A}) = \frac{2(\delta+\eta)}{\eta}$ and $\det(\mathbf{A}) = 1 - \frac{(1-2\eta)\delta}{\eta^2}$. Plugging this into the formulas for the eigenvalues, and simplifying, yields

$$\begin{aligned} \lambda_{1,2} &= \frac{\text{tr}(\mathbf{A})}{2} \pm \frac{1}{2}\sqrt{[\text{tr}(\mathbf{A})]^2 - 4\det(\mathbf{A})} \\ &= \frac{\delta + \eta}{\eta} \pm \sqrt{\frac{4\left(\delta^2 + 2\delta\eta + \eta^2\right) - 4 + 4\delta/\eta^2 - 8\delta/\eta}{4}} \\ &= 1 + \frac{\delta}{\eta} \pm \frac{1}{\eta}\sqrt{\delta\left(1+\delta\right)} \\ &= \begin{cases} 1 - \frac{\sqrt{\delta(1+\delta)} - \delta}{\eta} \\ 1 + \frac{\delta + \sqrt{\delta(1+\delta)}}{\eta} \end{cases}. \end{aligned} \tag{25}$$

Clearly, the lower eigenvalue in (25) exceeds one. Thus, the system will be unstable. The according eigenvectors are

$$\begin{aligned} (\mathbf{e}_1, \mathbf{e}_2) &= \begin{pmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{21}} \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1 - \frac{\sqrt{\delta(1+\delta)} - \delta}{\eta} - 1 + \frac{\delta}{\eta}}{\frac{\delta}{\eta}} & \frac{1 + \frac{\delta + \sqrt{\delta(1+\delta)}}{\eta} - 1 + \frac{\delta}{\eta}}{\frac{\delta}{\eta}} \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 - \sqrt{\frac{1+\delta}{\delta}} & 2 + \sqrt{\frac{1+\delta}{\delta}} \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

With this, we also know the general solution to the Dornbusch model:

$$\begin{pmatrix} e_t - \bar{e} \\ p_t - \bar{p} \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{\frac{1+\delta}{\delta}} & 2 + \sqrt{\frac{1+\delta}{\delta}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \left(1 - \frac{\sqrt{\delta(1+\delta)}-\delta}{\eta}\right)^t \\ c_2 \left(1 + \frac{\delta+\sqrt{\delta(1+\delta)}}{\eta}\right)^t \end{pmatrix}. \tag{26}$$

3.5 [3e] Stability and the saddle path

We have already seen in (25) that the system must be unstable because at least one eigenvalue exceeds one in absolute value. Are they both unstable? We have seen in the graphical derivation in section 2 that the system is stable in one dimension. Therefore let's suppose that the other eigenvalue is less than one in absolute value and try where we get. If things are consistent, we can infer that one of the two eigenvalues must be below one in absolute value. The price and exchange rate will only converge to the steady state if the coefficient of the explosive eigenvalue, c_2 here, is set to zero. Thus, the general solution (26) further simplifies to

$$\begin{pmatrix} e_t - \bar{e} \\ p_t - \bar{p} \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{\frac{1+\delta}{\delta}} & 2 + \sqrt{\frac{1+\delta}{\delta}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \left(1 - \frac{\sqrt{\delta(1+\delta)}-\delta}{\eta}\right)^t \\ 0 \end{pmatrix}, \tag{27}$$

so that

$$e_t - \bar{e} = c_1 \left(2 - \sqrt{\frac{1+\delta}{\delta}}\right) \left(1 - \frac{\sqrt{\delta\,(1+\delta)}-\delta}{\eta}\right)^t \tag{28}$$

and

$$p_t - \bar{p} = c_1 \left(1 - \frac{\sqrt{\delta\,(1+\delta)}-\delta}{\eta}\right)^t. \tag{29}$$

How can we pin down c_1? Well, we generally have some boundary conditions. For example, we know where the undisturbed Dornbsuch model started out in the very beginning of section 2: at $\bar{e}$ and $\bar{p}$ (figure 1, p. 6). Similarly, we know that, after the monetary contraction from $\bar{m}$ to $\bar{m}'$ at home, the system had to start out at e_0 and $\bar{p}$ on the new saddle path (figure 2, p. 9).

These kinds of boundary conditions are enough to completely and rigorously describe the dynamic behavior of the Dornbusch model.

However, for the mere purpose of deriving the saddle path we need not even worry about the proper boundary condition. We can simply plug (29) into (28) and we get, after all:

$$e_t - \bar{e} = \left(2 - \sqrt{\frac{1+\delta}{\delta}}\right)(p_t - \bar{p}). \qquad (30)$$

What does this line look like in the phase diagram? It certainly passes through the steady state. What slope does it have? Recall that $\delta < \frac{1}{\phi} = \frac{1}{3}$. Hence $\frac{1}{\delta} > 3$ and $\sqrt{\frac{1+\delta}{\delta}} = \sqrt{\frac{1}{\delta} + 1} > \sqrt{4} = 2$. The slope must be negative. Beautiful. So, we just found the analytic expression for the saddle path AA in figure 1 (p. 6). If we write (30) as $e_t - \bar{e}' = \left(2 - \sqrt{\frac{1+\delta}{\delta}}\right)(p_t - \bar{p}')$ we have the analytic expression for the saddle path in figure 2 (p. 9) as well.

There are mainly two ways to think about this relationship. One interpretation is, as we saw in lecture, that the economy must obey a no-arbitrage relationship $p_{t+1} - \bar{p} = -\hat{\theta}(e_{t+1} - \bar{e})$. If prices and the exchange rate would not move in phase at any point in time, an investor could construct a costless currency portfolio at that time and make a profit one period later. An other interpretation is that the system must not become explosive. The only schedule along which the economy does not explode is the saddle path, depicted as AA in figures 1 (p. 6) and 2 (p. 9). With this interpretation, we have in fact found $\hat{\theta}$: $\hat{\theta} = \sqrt{\frac{1+\delta}{\delta}} - 2$. Whenever the economy encountered itself off the saddle path, the exchange rate would have to adjusts immediately and to take the whole economy back to a point on the saddle path. Since prices adjust sloppily, the exchange rate will have to overshoot at times.