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Payoff Kinks in Preferences over Lotteries

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Abstract

This paper identifies two distinct types of payoff kinks that can be exhibited by preference functions over monetary lotteries—"locally separable" vs. "locally nonseparable"—and illustrates their relationship to the payoff and probability derivatives of such functions. Expected utility and Fréchet differentiable preference functions are found to be incapable of exhibiting locally nonseparable payoff kinks; rank-dependent preference functions are incapable of avoiding them.

Keywords: risk, uncertainty, payoff kinks

JEL Classification: D8

The purpose of this paper is to

- identify two distinct types of payoff kinks in preference functions over monetary lotteries, namely "locally nonseparable" versus "locally separable" kinks
- illustrate the relationships between a preference function's (directional) payoff derivatives and its probability derivatives in the presence of these different types of kinks, and
- compare the expected utility model and two important non-expected utility models with respect to the types of payoff kinks that they can, cannot, or must exhibit.

There are several situations where an individual's preferences over lotteries might be expected to exhibit kinks in the payoff levels. The simplest and undoubtedly most pervasive are piecewise-linear income tax schedules, which imply that the individual's utility of *before-tax* income will typically have kinks at the boundaries of the tax brackets. Similar instances include kinks induced by the option of bankruptcy, or the intended purchase of some large indivisible good. Alternatively, payoff kinks may be an inherent part of underlying attitudes toward risk. We also briefly consider another source of kinks, namely the phenomenon of temporal risk.

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Models of preferences over lotteries, like models of preferences elsewhere in economics, should be flexible enough to be able to exhibit kinks in situations where they might be expected to occur, as well as avoid kinks (be globally smooth) in situations where they might not. As it turns out, three important models—(1) *expected utility* risk preferences, (2) *Fréchet differentiable* risk preferences, and (3) *rank-dependent* risk preferences—all exhibit this flexibility with respect to the first type of payoff kink (locally separable). However, none of these models are flexible with respect to the second type: Whereas expected utility and Fréchet differentiable preferences *cannot exhibit* locally nonseparable payoff kinks, probability-smooth rank-dependent preferences *cannot avoid* exhibiting them at every lottery.

As mentioned, another purpose of this paper is to clarify the relationship between payoff kinks, payoff derivatives, and *probability derivatives* of preference functions over lotteries. Probability derivatives have proven useful in generalizing many of the basic concepts and results of expected utility analysis to more general non-expected utility preferences. For the expected utility preference function $V_{EU}(x_1, p_1; \ldots ; x_n, p_n) \equiv \sum_{i=1}^{n} U(x_i) \cdot p_i$, the *probability coefficient* of a payoff level x—the coefficient of $V_{EU}(\cdot)$ with respect to changes in prob(x)—is simply x's utility level $U(x)$. By viewing expected utility results as statements about probability coefficients, researchers have exploited the natural correspondence between coefficients in linear algebra and partial derivatives in calculus to generalize many expected utility results to smooth *non-expected utility* preference functions $V(\cdot)$. As is usual in the linear algebra $\leftrightarrow$ calculus correspondence, such "generalized expected utility" theorems include both local and exact global results.

Early work in generalized expected utility analysis imposed the smoothness property of Fréchet differentiability, which is not satisfied by the rank-dependent form. However, Chew, Karni and Safra (1987) showed that most of its basic results also hold under weaker notions of smoothness, and that many indeed apply to the rank-dependent form. This paper helps clarify the boundaries of this extension, namely that a generalized expected utility result will typically hold for the rank-dependent form *unless* it involves its (full or directional) payoff derivatives, in which case it is usually invalidated by the specific nature of rank-dependent payoff kinks.

This paper does not provide an exhaustive mathematical classification of all types of kinks (nondifferentiabilities) in univariate or multivariate functions. Nor does it provide an axiomatic characterization of smooth versus kinked preference functions over lotteries.[1] Rather, this paper is analytical, and like most analytical work in consumer theory,[2] is directed at both axiomatized and unaxiomatized functional forms, as well as the general unspecified form. As in the standard case, we will find that some functional forms exhibit very specific properties.[3]

Section 1 of this paper provides the background for the analysis by outlining the relationships between payoff derivatives, probability derivatives and Fréchet differentiability. Section 2 identifies the two distinct types of payoff kinks mentioned above. Sections 3 and 4 examine the different properties of expected utility, Fréchet differentiable, and rank-dependent risk preferences with respect to these two types of kinks. Section 5 concludes with brief discussions of modeling implications, the empirical evidence, and induced preferences.

1. The calculus of probabilities and payoffs

We consider the family $\mathscr{L}$ of all finite-outcome *lotteries* **P** over some real interval $[0, M]$.[4] Each such lottery can be uniquely represented by its probability measure $\mu(\cdot)$, or alternatively, by its cumulative distribution function $F(\cdot)$. Such lotteries can also be represented by the notation

$$\mathbf{P} = (x_1, p_1; \ldots ; x_n, p_n) \qquad x_i \in [0, M],\ p_i \in [0, 1],\ \sum_{i=1}^{n} p_i = 1 \tag{1}$$

As the specification (1) allows us to display more specific information about a lottery than either of the general notations $\mu(\cdot)$ or $F(\cdot)$, we adopt it for our analysis. However, since this specification allows two or more of the payoff values $x_1, \ldots, x_n$ to be equal, as well as one or more of the probabilities $p_1, \ldots, p_n$ to be zero, it does not provide a *unique* representation of any given lottery. For example, the following expressions all denote the same lottery **P** in $\mathscr{L}$:

$$(30, \tfrac{1}{2}; 40, \tfrac{1}{2}) \qquad (40, \tfrac{1}{2}; 30, \tfrac{1}{2}) \qquad (30, \tfrac{1}{4}; 30, \tfrac{1}{4}; 40, \tfrac{1}{2}) \qquad (30, \tfrac{1}{2}; 40, \tfrac{1}{2}; 100, 0)$$

In other words, the specification (1) inherently involves the following *identifications*—that is, the identity of expressions that differ only in one or more of the following manners:

$$\begin{array}{lll} (\ldots ; x', p'; \ldots ; x'', p''; \ldots) \text{ and } (\ldots ; x'', p''; \ldots ; x', p'; \ldots) & \text{order of } (x,p) \text{ pairs} \\ (\ldots ; x, p'; \ldots ; x, p''; \ldots) \text{ and } (\ldots ; x, p'+p''; \ldots) & \text{equal-outcome } (x,p) \text{ pairs} \\ (x_1, p_1; \ldots ; x_n, p_n) \text{ and } (x_1, p_1; \ldots ; x_n, p_n; x, 0) & \text{zero-probability outcomes} \end{array} \tag{2}$$

We could eliminate the need for these identifications by imposing the additional conditions that all payoff levels in any expression $\mathbf{P} = (x_1, p_1; \ldots ; x_n, p_n)$ be distinct, and that all probabilities be positive. However, the notational approach (1)/(2) will prove best for our analysis, as it does not *require* a change in n each time two payoffs x_i, x_j merge to a common value, or diverge *from* a common value, or some probability p_i becomes 0. Throughout, we assume that the individual's risk preferences can be represented by a real-valued *preference function* $V(\cdot)$ over $\mathscr{L}$, which accordingly assigns the same value to any pair of identified expressions in (2).

1.1. *Payoff changes vs. probability changes*

Ultimately, there is no real difference between changing the *payoffs* of a given lottery and changing its *probabilities*. That is, for any two lotteries $\mathbf{P} = (x_1, p_1; \ldots ; x_n, p_n)$

and $\mathbf{P}^* = (x_1^*, p_1^*; \ldots; x_{n^*}^*, p_{n^*}^*)$, we can represent the change $\mathbf{P} \to \mathbf{P}^*$ as either

a change in the *probabilities* associated with some fixed list of *payoff levels*, or
a change in the *payoff levels* associated with some fixed list of *probabilities*

To represent $\mathbf{P} \to \mathbf{P}^*$ as a change in the probabilities, invoke (2) to write

$$\begin{aligned} \mathbf{P} &= (x_1, p_1; \ldots; x_n, p_n; x_1^*, 0; \ldots; x_{n^*}^*, 0) \\ \mathbf{P}^* &= (x_1, 0; \ldots; x_n, 0; x_1^*, p_1^*; \ldots; x_{n^*}^*, p_{n^*}^*) \end{aligned} \tag{3}$$

so $\mathbf{P} \to \mathbf{P}^*$ is seen as the changes $(-p_1, \ldots, -p_n, +p_1^*, \ldots, +p_{n^*}^*)$ in the probabilities assigned to the respective payoffs $(x_1, \ldots, x_n, x_1^*, \ldots, x_{n^*}^*)$. To represent it as a change in the payoffs, invoke (2) to write

$$\begin{aligned} \mathbf{P} \; &= (x_1, p_1 \cdot p_1^*; \ldots; x_1, p_1 \cdot p_n^*; \ldots; x_n, p_n \cdot p_1^*; \ldots; x_n, p_n \cdot p_n^*) \\ \mathbf{P}^* &= (x_1^*, p_1 \cdot p_1^*; \ldots; x_n^*, p_1 \cdot p_n^*; \ldots; x_1^*, p_n \cdot p_1^*; \ldots; x_n^*, p_n \cdot p_n^*) \end{aligned} \tag{4}$$

so that $\mathbf{P} \to \mathbf{P}^*$ is seen as the changes $(x_1^* - x_1, \ldots, x_{n^*}^* - x_1, \ldots, x_1^* - x_n, \ldots, x_{n^*}^* - x_n)$ in the payoff levels received with the respective probabilities $(p_1 \cdot p_1^*, \ldots, p_1 \cdot p_{n^*}^*, \ldots, p_n \cdot p_1^*, \ldots, p_n \cdot p_{n^*}^*)$.

Since lotteries can be viewed as probability measures over the payoff space $[0, M]$, it might seem most natural to work in terms of changes in the probabilities assigned to the respective outcomes. On the other hand, many economic situations— such as portfolio choice, insurance and contingent production/exchange—involve optimization and/or equilibrium with respect to the payoff levels over some fixed set of states of nature,[5] in which case working with payoff changes would be most natural. A situation where *both* types of changes come into play is Ehrlich and Becker's (1972) analysis of an agent facing both *self insurance* options (activities that can mitigate the magnitude of a potential disaster, though not its likelihood) as well as *self protection* options (activities that can mitigate the likelihood of the disaster, though not its magnitude).[6] The equivalence of (3) and (4) implies that an individual's risk preferences can be completely represented by either their attitudes toward probability changes or their attitudes toward payoff changes—e.g., by either the probability or payoff derivatives of their preference function $V(\cdot)$.

1.2. Payoff derivatives

The effect of differentially changing payoff level x_i in a lottery $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$, that is, of shifting its probability mass p_i to payoff level $x_i + dx_i$, is given by $V(\cdot)$'s *regular payoff derivative*

$$\frac{\partial V(\mathbf{P})}{\partial x_i} = \frac{\partial V(\ldots; x_i, p_i; \ldots)}{\partial x_i} \stackrel{\text{def}}{=} \left. \frac{\partial V(\ldots; x, p_i; \ldots)}{\partial x} \right|_{x=x_i} \tag{5}$$

We can also consider the effect of shifting just a *part* of x_i's probability mass, say some amount $\rho < p_i$, to obtain the *partial-probability payoff derivative*

$$\left.\frac{\partial V(\ldots;x,\rho;x_i,p_i-\rho;\ldots)}{\partial x}\right|_{x=x_i} \tag{6}$$

Finally, when the same payoff level x is given by more than one (x_i, p_i) pair in $\mathbf{P}$, we can also consider the effect of shifting *all* the probability mass assigned to this payoff level—that is, the *whole-probability payoff derivative*

$$\frac{dV(\ldots;x_i,p_i;\ldots)}{dx^W} \stackrel{\text{def}}{=} \frac{d}{dx}V(\ldots;\underbrace{\underbrace{x,p_j};\ldots;\underbrace{x,p_k};\ldots;\underbrace{x,p_l}}_{\text{all pairs yielding payoff level } x};\ldots) \tag{7}$$

One might expect a preference function's regular, partial-probability and whole-probability payoff derivatives to satisfy the following *total derivative relationships*

$$\frac{\partial V(\ldots;x_i,p_i;\ldots)}{\partial x_i} = \left.\frac{\partial V(\ldots;x,\rho;x_i,p_i-\rho;\ldots)}{\partial x}\right|_{x=x_i} + \left.\frac{\partial V(\ldots;x_i,\rho;x,p_i-\rho;\ldots)}{\partial x}\right|_{x=x_i} \tag{8}$$

$$\frac{dV(\ldots;x_i,p_i;\ldots)}{dx^W} = \sum_{x_j=x} \frac{\partial V(\ldots;x_i,p_i;\ldots)}{\partial x_j} \tag{9}$$

$$\left.\frac{dV(\ldots;x_i+\alpha\cdot t,p_i;x_j+\beta\cdot t,p_j;\ldots)}{dt}\right|_{t=0} = \alpha\cdot\frac{\partial V(\ldots;x_i,p_i;x_j,p_j;\ldots)}{\partial x_i} + \beta\cdot\frac{\partial V(\ldots;x_i,p_i;x_j,p_j;\ldots)}{\partial x_j} \tag{10}$$

and the payoff derivatives of any smooth *expected utility* preference function $V_{EU}(\cdot)$, given in (20) and (21) below, do satisfy them. But for a *non-expected utility* preference function $V(\cdot)$, even if (2) holds and even if $V(\cdot)$'s payoff derivatives (5)–(7) all exist, they will not necessarily satisfy (8)–(10) unless *additional* smoothness is imposed on $V(\cdot)$. The reason is that the payoff derivatives (5)–(7) represent movements along *three different paths* in the underlying space of measures over $[0, M]$,[7] so they will not satisfy (8)–(10) without additional smoothness on $V(\cdot)$ that links its responses to movements along these distinct paths. Section 1.4 presents such a smoothness condition (Fréchet differentiability), Section 3 examines preferences that satisfy this smoothness condition and hence the above total derivative relationships, and Section 4 examines an important example of risk preferences that do not (and can not) satisfy this smoothness condition, and whose payoff derivatives generally violate these total derivative relationships.

1.3. Probability derivatives and local utility functions

Since the probabilities in a lottery $(x_1, p_1; \ldots; x_n, p_n)$ must sum to one, there is no behavioral meaning to an individual's attitude toward changes in a *single* probability p_i. Nor is it mathematically appropriate to define a probability derivative $\partial V(x_1, p_1; \ldots; x_n, p_n)/\partial p_i$ by the standard formula

$$\lim_{\Delta p_i \to 0} \frac{V(x_1, p_1; \ldots; x_i, p_i + \Delta p_i; \ldots; x_n, p_n) - V(x_1, p_1; \ldots; x_i, p_i; \ldots; x_n, p_n)}{\Delta p_i} \tag{11}$$

since the "object" $(x_1, p_1; \ldots; x_i, p_i + \Delta p_i; \ldots; x_n, p_n)$ lies outside the set of lotteries $\mathscr{L}$, and hence outside the domain of $V(\cdot)$.

Rather, since *probability changes* can only occur jointly (at least two at a time) and always sum to zero, *probability derivatives* can only be defined, evaluated or applied with respect to such zero-sum change vectors $(\Delta p_1, \ldots, \Delta p_n)$. We thus define $V(x_1, p_1; \ldots; x_n, p_n)$'s derivatives with respect to the variables $p_1, \ldots, p_n$ as any *set* of values $\{\partial V(x_1, p_1; \ldots; x_n, p_n)/\partial p_i | i = 1, \ldots, n\}$ that satisfies

$$\lim_{t \to 0} \frac{V(x_1, p_1 + t \cdot \Delta p_1; \ldots; x_n, p_n + t \cdot \Delta p_n) - V(x_1, p_1; \ldots; x_n, p_n)}{t} = \sum_{i=1}^{n} \frac{\partial V(x_1, p_1; \ldots; x_n, p_n)}{\partial p_i} \cdot \Delta p_i \tag{12}$$

for all change vectors $\{\Delta p_1, \ldots, \Delta p_n\}$ such that $\sum_{i=1}^{n} \Delta p_i = 0$.[8] Observe that if a set of values $\{\partial V(x_1, p_1; \ldots; x_n, p_n)/\partial p_i | i = 1, \ldots, n\}$ satisfies this property, so will any other set of the form $\{\partial V(x_1, p_1; \ldots; x_n, p_n)/\partial p_i + k | i = 1, \ldots, n\}$ for any constant k. To include probability changes $\Delta p_{n+1} \geq 0, \ldots, \Delta p_m \geq 0$ for one or more payoff levels $x_{n+1}, \ldots, x_m$ outside of $\{x_1, \ldots, x_n\}$, we define $V(\cdot)$'s *probability derivative function* at $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$ as any function $\partial V(\mathbf{P})/\partial \text{prob}(x)$ satisfying

$$\lim_{t \to 0} \frac{V(\overbrace{\ldots; x_i, p_i + t \cdot \Delta p_i; \ldots}^{i=1,\ldots,n} \overbrace{\ldots; x_j, t \cdot \Delta p_j; \ldots}^{j=n+1,\ldots,m}) - V(x_1, p_1; \ldots; x_n, p_n)}{t} = \sum_{k=1}^{m} \frac{\partial V(\mathbf{P})}{\partial \text{prob}(x_k)} \cdot \Delta p_k \tag{13}$$

for all zero-sum change vectors $\{\Delta p_1, \ldots, \Delta p_n, \Delta p_{n+1}, \ldots, \Delta p_m\}$, and observe that the function $\partial V(\mathbf{P})/\partial \text{prob}(x)$ is similarly invariant to any transformation of the form $\partial V(\mathbf{P})/\partial \text{prob}(x) + k$.

Given a specific formula for $V(\cdot)$, it is usually possible to determine its probability derivative function $\partial V(\mathbf{P})/\partial \text{prob}(x)$ by direct inspection. When this is not the case, or when a formal derivation is desired, a function $\partial V(\mathbf{P})/\partial \text{prob}(x)$ satisfying (13) at a given $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$ can always be derived, by selecting any of $\mathbf{P}$'s positive-probability outcomes x_{i^*}, and defining

$$\frac{\partial V\mathbf{P}}{\partial \text{prob}(x)} \equiv \lim_{\Delta p \to 0} \big(V(x_1, p_1; \ldots; x_{i^*}, p_{i^*} - \Delta p; \ldots; x_n, p_n; x, \Delta p) - V(x_1, p_1; \ldots; x_n, p_n)\big)\big/\Delta p \qquad (14)$$

for all $x \in [0, M]$, in which case $\partial V(\mathbf{P})/\partial \text{prob}(x_{i^*}) = 0$. To obtain a function $\partial V(\mathbf{P})/\partial \text{prob}(x)$ satisfying the uniform normalization $\partial V(\mathbf{P})/\partial \text{prob}(\underline{x}) \equiv 0$ for all $\mathbf{P} \in \mathscr{L}$, replace (14) by

$$\lim_{\Delta p \to 0} \big(V(x_1, p_1; \ldots; x_{i^*}, p_{i^*} - \Delta p; \ldots; x_n, p_n; x, \Delta p) - V(x_1, p_1; \ldots; x_{i^*}, p_{i^*} - \Delta p; \ldots; x_n, p_n; \underline{x}, \Delta p)\big)\big/\Delta p \qquad (14)'$$

for any fixed $\underline{x}$. On the understanding that they must be formally evaluated by joint-change formulas such as (14) or (14)′ rather than a single-change formula such as (11), we heretofore suppress the x_{i^*} and/or $\underline{x}$ terms, and express $V(\cdot)$'s probability derivatives by simpler notation

$$\frac{\partial V(\mathbf{P})}{\partial \text{prob}(x)} = \frac{\partial V(x_1, p_1; \ldots; x_n, p_n)}{\partial \text{prob}(x)} = \left.\frac{\partial V(x_1, p_1; \ldots; x_n, p_n; x, \rho)}{\partial \rho}\right|_{\rho=0} \qquad (15)$$

The identifications (2) imply that if $x_i = x_j = x$ in some lottery $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$, then

$$\frac{\partial V(\mathbf{P})}{\partial \text{prob}(x)} = \frac{\partial V(\ldots; x, p_i; x, p_j; \ldots)}{\partial p_i} = \frac{\partial V(\ldots; x, p_i; x, p_j; \ldots)}{\partial p_j} = \left.\frac{\partial V(\ldots; x, p; \ldots)}{\partial p}\right|_{p=p_i+p_j} \qquad (16)$$

That is, if the same payoff value x appears in both the pairs (x, p_i), (x, p_j), the effect of raising its overall probability does not depend on whether this is done by raising p_i in the pair (x, p_i), raising p_j in the pair (x, p_j), or raising $p_i + p_j$ in a notationally combined pair $(x, p_i + p_j)$. In other words, the probability derivative $\partial V(\mathbf{P})/\partial \text{prob}(x)$ is independent of alternative representations (2) of any lottery $\mathbf{P}$ in $\mathscr{L}$. In situations when the probabilities p_i, p_j in the pairs (x, p_i), (x, p_j) *both* change, then if *any* of the six derivative terms in the following two total derivative relationships exist, (2) implies that they *all* exist, and will satisfy both relationships, namely

$$\left.\frac{dV(\ldots; x, p_i + \rho; x, p_j + \rho; \ldots)}{d\rho}\right|_{\rho=0} = \frac{\partial V(\ldots; x, p_i; x, p_j; \ldots)}{\partial p_i} + \frac{\partial V(\ldots; x, p_i; x, p_j; \ldots)}{\partial p_j} \qquad (17)$$

and more generally

$$\left.\frac{dV(\ldots;x,p_i+\alpha\cdot t;x,p_j+\beta\cdot t;\ldots)}{dt}\right|_{t=0}=\alpha\cdot\frac{\partial V(\ldots;x,p_i;x,p_j;\ldots)}{\partial p_i}+\beta\cdot\frac{\partial V(\ldots;x,p_i;x,p_j;\ldots)}{\partial p_j} \tag{18}$$

The reason for this is that, in contrast with the payoff derivatives (5)–(7), the six individual derivative terms in (17) and (18) all represent the effect of moving, though at different speeds, along the *same* path in the underlying space of measures over $[0, M]$.[9]

As noted in the Introduction, the analytical value of a smooth $V(\cdot)$'s probability derivatives $\partial V(\mathbf{P})/\partial\text{prob}(x)$ stems from their correspondence to the *probability coefficients* $U(x)$ of the expected utility preference function $V_{EU}(\cdot)$. To highlight this correspondence we adopt the notation

$$U(x;\mathbf{P})=\frac{\partial V(\mathbf{P})}{\partial\text{prob}(x)}\qquad\text{all } x,\mathbf{P} \tag{19}$$

and refer to $U(\cdot;\mathbf{P})$ as the *local utility function* of $V(\cdot)$ at $\mathbf{P}$.[10] The simplest example of such a correspondence involves the property of *global first order stochastic dominance (FSD) preference*, which for an expected utility $V_{EU}(\cdot)$ is characterized by its utility function $U(\cdot)$ being nondecreasing, and for a smooth non-expected $V(\cdot)$ is characterized by its *local* utility functions $\{U(\cdot;\mathbf{P})|\mathbf{P}\in\mathscr{L}\}$ being nondecreasing at each lottery $\mathbf{P}$.[11] A similar correspondence holds for the property of *global risk aversion* (global aversion to mean preserving increases in risk), which for expected utility is characterized $U(\cdot)$ being concave, and for a smooth non-expected utility $V(\cdot)$ is characterized by its *local* utility functions $\{U(\cdot;\mathbf{P})|\mathbf{P}\in\mathscr{L}\}$ being concave at each $\mathbf{P}$. This use of local utility functions (probability derivatives), termed *generalized expected utility analysis*, has been applied to generalize additional results of expected utility analysis—including aspects of the Arrow-Pratt characterization of comparative risk aversion, Rothschild-Stiglitz comparative statics of risk, insurance theory and state-dependent preferences—to probability-smooth non-expected utility preference functions.[12]

1.4. Fréchet differentiability and the link between payoff and probability derivatives

Provided $U(\cdot)$ is differentiable at a payoff level $x = x_i$, the regular, partial-probability and whole-probability payoff derivatives of an expected utility preference function $V_{EU}(\cdot)$ are given by

$$\frac{\partial V_{EU}(\ldots;x_i,p_i;\ldots)}{\partial x_i}=\left.\frac{d[\sum_{j\neq i}U(x_j)\cdot p_j+U(x)\cdot p_i]}{dx}\right|_{x=x_i}=p_i\cdot U'(x_i) \tag{20}$$

$$\left.\frac{\partial V_{EU}(\ldots; x, \rho; x_i, p_i - \rho; \ldots)}{\partial x}\right|_{x=x_i} = \rho \cdot U'(x_i)$$
$$\frac{dV_{EU}(\ldots; x_i, p_i; \ldots)}{dx^W} = \left(\sum_{x_j = x} p_j\right) \cdot U'(x) \tag{21}$$

These formulas holds even if $p_i = 0$, and even if the lottery $(\ldots; x_i, p_i; \ldots)$ contains other (x, p) pairs with the same payoff level x_i. In each case, the effect of a differential shift of probability mass from a given payoff level is seen to be proportional to, or additive in, the amount(s) of probability mass shifted. These formulas are also seen to satisfy the total derivative relationships (8)–(10).

Provided a *non*-expected utility preference function $V(\cdot)$ is sufficiently "smooth," its payoff derivatives satisfy corresponding formulas in terms of its *local* utility functions. A smoothness property that suffices for this is *Fréchet differentiability*. Although it can be defined more generally,[13] Fréchet differentiability is typically defined with respect to the L^1 norm over these lotteries' cumulative distribution functions $F(\cdot)$ on $[0, M]$[14]:

$$\|F^*(\cdot) - F(\cdot)\| \equiv \int_0^M |F^*(x) - F(x)| \cdot dx \tag{22}$$

For finite-outcome lotteries, this norm implies convergence in each of the following instances

$$\begin{aligned}
&(x_1^*, p_1; \ldots; x_n^*, p_n) \to (x_1, p_1; \ldots; x_n, p_n) \quad \text{as} \quad (x_1^*, \ldots, x_n^*) \to (x_1, \ldots, x_n) \\
&(x_1, p_1^*; \ldots; x_n, p_n^*) \to (x_1, p_1; \ldots; x_n, p_n) \quad \text{as} \quad (p_1^*, \ldots, p_n^*) \to (p_1, \ldots, p_n) \\
&(\ldots; x', p'; x'', p''; \ldots) \to (\ldots; x, p' + p''; \ldots) \quad \text{as} \quad x', x'' \to x
\end{aligned} \tag{23}$$

A preference function $V_{FR}(\cdot)$ is said to be *Fréchet differentiable* at a general probability distribution $F(\cdot)$ over $[0, M]$ if there exists a continuous (in $\|\cdot\|$) linear function $\psi(\cdot; F)$ such that

$$V_{FR}(F^*(\cdot)) - V_{FR}(F(\cdot)) = \psi(F^*(\cdot) - F(\cdot); F) + o(\|F^*(\cdot) - F(\cdot)\|) \tag{24}$$

where $o(\cdot)$ denotes a function that is 0 at 0 and of higher order than its argument. This can be shown to imply the standard first order approximation formula

$$\begin{aligned}
V_{FR}(F^*(\cdot)) - V_{FR}(F(\cdot)) = &\int_0^M U(x; F) \cdot (dF^*(x) - dF(x)) \\
&+ o(\|F^*(\cdot) - F(\cdot)\|)
\end{aligned} \tag{25}$$

and hence, for finite-outcome lotteries $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$ and $\mathbf{P}^* = (x_1^*, p_1^*; \ldots; x_{n^*}^*, p_{n^*}^*)$,

$$V_{FR}(\mathbf{P}^*) - V_{FR}(\mathbf{P}) = \left[\sum_{i=1}^{n^*} U(x_i^*; \mathbf{P}) \cdot p_i^* - \sum_{i=1}^{n} U(x_i; \mathbf{P}) \cdot p_i\right] + o(\|\mathbf{P}^* - \mathbf{P}\|) \tag{26}$$

for some *absolutely continuous*[15] (and hence almost everywhere differentiable) function $U(\cdot; F)$ or $U(\cdot; \mathbf{P})$. Differentiating (26) as in $(14)/(14)'$ and invoking the properties of $o(\cdot)$ yields that the function $U(\cdot; \mathbf{P})$ in (26) is indeed the probability derivative/local utility function of $V_{FR}(\cdot)$. Note that Fréchet differentiability is stronger than just "differentiability in the probabilities," in that it implies convergence to the first order terms in (25), (26) for any "sideways" approach of $F^*(\cdot)$ to $F(\cdot)$ or $\mathbf{P}^*$ to $\mathbf{P}$—that is, for convergence in the *payoff values* as in the first or third lines of (23).

Although Fréchet differentiability is stronger than differentiability in the probabilities, and it implies *continuity* in the payoff levels, it does not necessarily imply *differentiability* in the payoff levels. To see this, consider any expected utility function $V_{EU}(\cdot)$ with an absolutely continuous but kinked $U(\cdot)$. Since $V_{EU}(\cdot)$ satisfies (25) and (26) with no error term at all, it is Fréchet differentiable at every lottery. But any kink in $U(\cdot)$ will of course imply payoff kinks in $V_{EU}(\cdot)$.

However, if a Fréchet differentiable $V_{FR}(\cdot)$'s local utility function $U(\cdot; \mathbf{P})$ is differentiable at a payoff level $x = x_i$, then its regular, partial-probability and whole-probability payoff derivatives exist, and are given by the exact analogues of the expected utility formulas (20)/(21), namely

$$\frac{\partial V_{FR}(\ldots; x_i, p_i; \ldots)}{\partial x_i} = p_i \cdot U'(x_i; \mathbf{P}) \tag{27}$$

$$\left.\frac{\partial V_{FR}(\ldots; x, \rho; x_i, p_i - \rho; \ldots)}{\partial x}\right|_{x=x_i} = \rho \cdot U'(x_i; \mathbf{P})$$
$$\frac{dV_{FR}(\ldots; x_i, p_i; \ldots)}{dx^W} = \left(\sum_{x_j=x} p_j\right) \cdot U'(x; \mathbf{P}) \tag{28}$$

Thus as under expected utility, the effect on $V_{FR}(\cdot)$ of a differential shift of probability mass from any payoff level will be proportional to, or additive in, the amount(s) of mass shifted. These payoff derivatives are also linked to each other by the total derivative formulas (8)–(10), i.e.[16]

$$\frac{\partial V_{FR}(\ldots; x_i, p_i; \ldots)}{\partial x_i} = \left.\frac{\partial V_{FR}(\ldots; x, \rho; x_i, p_i - \rho; \ldots)}{\partial x}\right|_{x=x_i} + \left.\frac{\partial V_{FR}(\ldots; x_i, \rho; x, p_i - \rho; \ldots)}{\partial x}\right|_{x=x_i} \tag{29}$$

$$\frac{dV_{FR}(\ldots; x_i, p_i; \ldots)}{dx^W} = \sum_{x_j=x} \frac{\partial V_{FR}(\ldots; x_i, p_i; \ldots)}{\partial x_j} \tag{30}$$

$$\left.\frac{dV_{FR}(\ldots; x_i + \alpha \cdot t, p_i; x_j + \beta \cdot t, p_j; \ldots)}{dt}\right|_{t=0} = \alpha \cdot \frac{\partial V_{FR}(\ldots; x_i, p_i; x_j, p_j; \ldots)}{\partial x_i} + \beta \cdot \frac{\partial V_{FR}(\ldots; x_i, p_i; x_j, p_j; \ldots)}{\partial x_j} \tag{31}$$

To see how Fréchet differentiability implies these relationships, let $V_{FR}(\cdot)$'s local utility function $U(\cdot;\mathbf{P})$ at $\mathbf{P}=(\ldots;x_i,p_i;\ldots)$ be differentiable at $x=x_i$. Eqs. (22), (26) and the $o(\cdot)$ property yield

$$\begin{aligned}\frac{\partial V_{FR}(\ldots;x_i,p_i;\ldots)}{\partial x_i} &= \frac{d}{dt}\big[V_{FR}(\ldots;x_i+t,p_i;\ldots)-V_{FR}(\ldots;x_i,p_i;\ldots)\big]\Big|_{t=0}\\ &= \frac{d}{dt}\big[U(x_i+t;\mathbf{P})\cdot p_i - U(x_i;\mathbf{P})\cdot p_i\big]\Big|_{t=0}\\ &\quad + \frac{d}{dt}o\big(\big\|(\ldots;x_i+t,p_i;\ldots)-(\ldots;x_i,p_i;\ldots)\big\|\big)\Big|_{t=0}\\ &= U'(x_i;\mathbf{P})\cdot p_i + \frac{d}{dt}o\big(|t\cdot p_i|\big)\Big|_{t=0} = U'(x_i;\mathbf{P})\cdot p_i \end{aligned} \tag{32}$$

which is (27). Since this derivation does not require x_i to be distinct from any other outcome x_j, the identifications (2) in turn yield the derivative formulas (28), and thus also (29) and (30).[17]

For an example of generalized expected utility analysis using payoff derivatives, recall that under expected utility with differentiable $U(\cdot)$, the marginal effect of a constant ("risk-free") addition t to a lottery $\mathbf{P}=(x_1,p_1;\ldots;x_n,p_n)$ is given by the formula $\sum_{i=1}^{n} U'(x_i)\cdot p_i$, that is, by expected marginal (von Neumann-Morgenstern) utility. For a Fréchet differentiable non-expected utility $V_{FR}(\cdot)$ with differentiable local utility functions, a derivation similar to that of (31)[18] yields that this effect is given by expected marginal *local* utility

$$\left.\frac{dV_{FR}(x_1+t,p_1;\ldots;x_n+t,p_n)}{dt}\right|_{t=0} = \sum_{i=1}^{n} U'(x_i;\mathbf{P})\cdot p_i \tag{33}$$

Similarly, consider an individual with total investible funds w, facing a riskless asset with net return r and a risky asset whose net return $\tilde{x}$ has distribution $(x_1,p_1;\ldots;x_n,p_n)$. If α is the amount invested in the risky asset, the marginal effect of a rise in α on expected utility is given by the standard formula $\sum_{i=1}^{n}(x_i-r)\cdot U'(w\cdot(1+r)+\alpha\cdot(x_i-r))\cdot p_i$, and the marginal effect on a Fréchet differentiable $V_{FR}(\cdot)$ will be given by the corresponding formula

$$\begin{aligned}\frac{dV_{FR}(\mathbf{P}_\alpha)}{d\alpha} &= \frac{dV_{FR}(\ldots;w\cdot(1+r)+\alpha\cdot(x_i-r),p_i;\ldots)}{d\alpha}\\ &= \sum_{i=1}^{n}(x_i-r)\cdot U'(w\cdot(1+r)+\alpha\cdot(x_i-r);\mathbf{P}_\alpha)\cdot p_i \end{aligned} \tag{34}$$

where $\mathbf{P}_\alpha=(\ldots;w\cdot(1+r)+\alpha\cdot(x_i-r),p_i;\ldots)$ is the distribution of random wealth $w\cdot(1+r)+\alpha\cdot(\tilde{x}-r)$.

2. Locally separable vs. locally nonseparable kinks

It is well-known that calculus can also be used for the exact analysis of *nondifferentiable* functions, as long as they are not *too* nondifferentiable. Consider the Fundamental Theorem of Calculus, which gives conditions under which a function $f(\cdot) : R^1 \to R^1$ can be completely and exactly characterized in terms of its derivatives, via the formula $f(z) \equiv f(0) + \int_0^z f'(\omega) \cdot d\omega$. Global differentiability is *not* required for this result: A continuous $f(\cdot)$ can have a finite or even countably infinite number of isolated kinks and the formula will still exactly hold: we simply "integrate over" such kinks. More generally, a function $f(\cdot)$ will satisfy the Fundamental Theorem of Calculus as long as it is absolutely continuous over the interval in question. Provided such a *multivariate* function $f(\cdot, \ldots, \cdot)$ also only has a finite or countable number of kinks, the Fundamental Theorem similarly links global changes in $f(\cdot, \ldots, \cdot)$ to its line and path integrals.

However, there also exist mathematical functions that are simply "too nondifferentiable" to admit this type of analysis. The most notorious example is the well-known *Cantor function*[19] $C(\cdot)$ over [0,1], which is continuous, nondecreasing, satisfies $C(0) = 0$ and $C(1) = 1$ and is differentiable almost everywhere on [0,1], yet has derivative $C'(\cdot) = 0$ almost everywhere, so it does not satisfy the Fundamental Theorem. Except on regions over which it is constant, this function is completely unamenable to calculus.

As mentioned, this paper does not consider all types of kinks (nondifferentiabilities), and in particular, there is little theoretical, empirical or intuitive reason to expect that agents exhibit "Cantor-type" preferences over monetary lotteries.[20] Rather, we consider functions whose various types of kinks still admit of first order approximations (albeit kinked ones), and consider their amenability/unamenability to standard multivariate calculus, including implications for choice under uncertainty. We can exemplify the key distinction examined in this paper—the two types of multivariate kinks mentioned in the Introduction—by the two functions[21]

$$\begin{aligned} &S(z_1, z_2) \equiv \min\{z_1, 1\} + \min\{z_2, 1\} \qquad \text{vs.} \\ &L(z_1, z_2) \equiv \min\{z_1, z_2\} + 1 \qquad z_1, z_2 \geq 0 \end{aligned} \tag{35}$$

and in particular, by their properties about the point $(z_1, z_2) = (1, 1)$. The *common* feature of these two functions' kinks at (1,1)—and of all the kinks examined in this paper—is that they each admit of a "local piecewise-linear approximation," that is, a set of tangent hyperplanes which together serve as a first order approximation to the function about the point (1,1). Section 2.1 formally describes this property. The *distinct* features of the two functions' kinks at (1,1) (and elsewhere), with their respective implications for the applicability of calculus, are laid out in Section 2.2. Section 2.3 compares the ordinal implications of these two types of kinks.

2.1. Piecewise-linearity and local piecewise-linearity

A function $\widehat{H}(\cdot, \ldots, \cdot)$ over R^n is said to be *piecewise-linear about the origin* (0,…,0) if there exists a finite partition of R^n into convex cones $\{E_1, \ldots, E_J\}$,[22] and linear functions

$\widehat{H}_1(\cdot, \dots, \cdot), \dots, \widehat{H}_J(\cdot, \dots, \cdot)$ on each of these cones, such that

$$\widehat{H}(z_1, \dots, z_n) \equiv \widehat{H}_j(z_1, \dots, z_n) \qquad (z_1, \dots, z_n) \in E_j, \quad j = 1, \dots, J \tag{36}$$

More generally, we say $\widehat{H}(\cdot, \dots, \cdot)$ is *piecewise-linear about the point* $\mathbf{Z}_0 = (\bar{z}_1, \dots, \bar{z}_n)$ if it satisfies

$$\widehat{H}(z_1, \dots, z_n) \equiv \widehat{H}(\bar{z}_1, \dots, \bar{z}_n) + \widehat{H}_0(z_1 - \bar{z}_1, \dots, z_n - \bar{z}_n) \tag{37}$$

for some $\widehat{H}_0(\cdot, \dots, \cdot)$ that is piecewise-linear about the origin. Although it might be more accurate to describe $\widehat{H}(\cdot, \dots, \cdot)$ in (37) as "piecewise-affine," we retain the slight abuse of terminology to conform with standard usage. A function $H(\cdot, \dots, \cdot)$ has a *piecewise-linear kink* at $\mathbf{Z}_0$ if it is piecewise-linear about $\mathbf{Z}_0$ but not linear there, or else is identically equal to such a function over some open neighborhood of $\mathbf{Z}_0$.

Although linearity of each $\widehat{H}_1(\cdot, \dots, \cdot), \dots, \widehat{H}_J(\cdot, \dots, \cdot)$ implies that the function $\widehat{H}(\cdot, \dots, \cdot)$ in (36) is continuous at $(0, \dots, 0)$, it does *not* imply that $\widehat{H}(\cdot, \dots, \cdot)$ is globally continuous, or even that it is continuous over any open neighborhood of $(0, \dots, 0)$. For example, the function $\widehat{H}(z_1, z_2) \equiv z_1 \cdot \text{sgn}[z_2]$ is piecewise-linear about (0,0), with cones $\{E_1, E_2, E_3\}$ = {upper half-plane, horizontal axis, lower half-plane} and corresponding linear functions $\{\widehat{H}_1(z_1, z_2), \widehat{H}_2(z_1, z_2), \widehat{H}_3(z_1, z_2)\} \equiv \{z_1, 0, -z_1\}$. However, $\widehat{H}(\cdot, \cdot)$ is discontinuous at each horizontal axis point $(z_1, 0)$ except $(0, 0)$. Thus, whenever global continuity of such a function is desired, it must be separately established or imposed. Similar remarks apply to the piecewise-linear function $\widehat{H}(\cdot, \dots, \cdot)$ in (37).

Figures 1a and 1b illustrate the piecewise-linear structures of the functions $S(\cdot, \cdot)$ and $L(\cdot, \cdot)$ about the point $(1, 1)$, by indicating the formulas they take over different regions in their domain R^2_+. In addition to their piecewise-linear kinks at $(1, 1)$, $S(\cdot, \cdot)$ is seen to have piecewise-linear kinks at each point on the horizontal and vertical dashed lines (whenever z_1 or z_2 equals 1), and $L(\cdot, \cdot)$ has piecewise-linear kinks at each point on the 45° line (wherever $z_1 = z_2$).

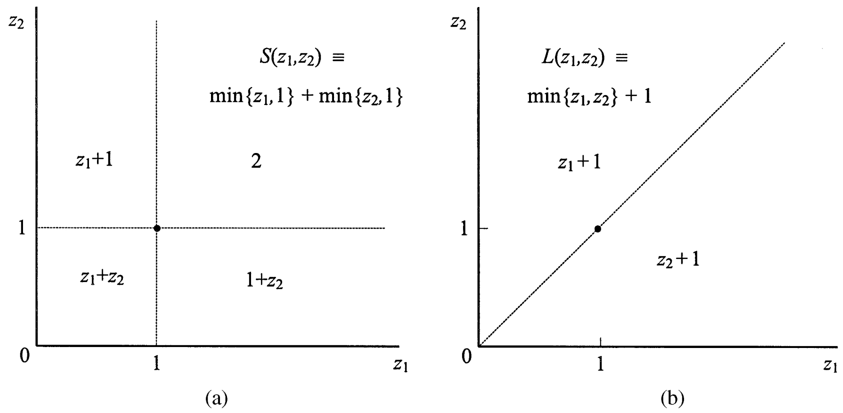

Figure 1. (a and b) Piecewise linear structure of the functions $S(\cdot, \cdot)$ and $L(\cdot, \cdot)$.

In standard usage, calling a function *locally linear* about a point $\mathbf{Z}_0 = (\bar{z}_1, \ldots, \bar{z}_n)$ does not denote that it is *exactly* linear over any open neighborhood of $\mathbf{Z}_0$, but rather, that it has a (continuous) linear first-order approximation there. By analogy, we say that $H(\cdot, \ldots, \cdot)$ is *locally piecewise-linear* about $\mathbf{Z}_0$ if there exists some globally continuous $\widehat{H}(\cdot, \ldots \cdot; \mathbf{Z}_0)$ that is piecewise-linear about the origin, such that

$$H(z_1, \ldots, z_n) - H(\bar{z}_1, \ldots, \bar{z}_n) = \widehat{H}(z_1 - \bar{z}_1, \ldots, z_n - \bar{z}_n; \mathbf{Z}_0) + o(\|(z_1, \ldots, z_n) - (\bar{z}_1, \ldots, \bar{z}_n)\|) \tag{38}$$

where $\|\cdot\|$ is the Euclidean norm over R^n. If $\widehat{H}(\cdot, \ldots, \cdot; \mathbf{Z}_0)$ is piecewise-linear but not linear, $H(\cdot, \ldots, \cdot)$ is said to have a *locally piecewise-linear kink* at $\mathbf{Z}_0$. Geometrically, its first order approximation at $(\bar{z}_1, \ldots, \bar{z}_n)$ consists of the J tangent hyperplanes generated by $\widehat{H}(\cdot, \ldots, \cdot; \mathbf{Z}_0)$ over each of its convex cones $\{E_1, \ldots, E_J\}$. It is straightforward to verify that for any differentiable function $\rho(\cdot)$, the function $H^*(\cdot, \ldots, \cdot) \equiv \rho(H(\cdot, \ldots, \cdot))$ is also locally piecewise-linear about $\mathbf{Z}_0$, with piecewise-linear first order approximation function $\widehat{H}^*(\cdot, \ldots, \cdot; \mathbf{Z}_0) = \rho'(H(\bar{z}_1, \ldots, \bar{z}_n)) \cdot \widehat{H}(\cdot, \ldots, \cdot; \mathbf{Z}_0)$.

As mentioned, this paper is restricted to functions with such locally-piecewise linear kinks. Nevertheless, it is worth noting that not all kinks in multivariate (or even univariate) functions take this form: For example, neither the kink in the function $H(z) \equiv \max\{[z-1]^{1/3}, 0\}$ at $z = 1$, nor the kink in the Cantor function $C(\cdot)$ at $z = 1/3$, has a local piecewise-linear approximation as in (38), even though both functions are continuous and nondecreasing about these points. Although we do not provide an axiomatic analysis such as in Debreu (1972), we will ultimately concentrate on the same type of preference functions as in his analysis, namely those whose first order approximations (in our case, either linear or piecewise-linear) are strictly increasing. For preference functions over lotteries, this will be taken to mean that $V(\cdot)$'s first order approximations are increasing with respect to first order stochastically dominating shifts.

2.2. Local separability vs. local-nonseparability

The functions $S(z_1, z_2) \equiv \min\{z_1, 1\} + \min\{z_2, 1\}$and $L(z_1, z_2) \equiv \min\{z_1, z_2\} + 1$ from (35) take the same value at their common kink point $(1, 1)$, have identical left and right partial derivatives with respect to both z_1 and z_2 there,[23] and in fact, take identical values at all points on the horizontal and vertical dashed lines in Figure 1a. That is, they respond identically to both *local and global* changes in z_1 or z_2 from $(1, 1)$, so long as only one of these variables changes. Nonetheless, the two functions have qualitatively distinct properties about $(1, 1)$ and about each of their other kink points, due to the way they respond to *joint* changes in z_1 and z_2. In particular, while $S(\cdot, \cdot)$'s kinks are amenable to the local and global calculus of directional derivatives, $L(\cdot, \cdot)$'s are not.

To see this, consider the effect of moving from the point $(1, 1)$ in arbitrary direction (k_1, k_2) or its opposite—that is, of moving along the line $(1 + k_1 \cdot t, 1 + k_2 \cdot t)$ as t rises/falls from 0. The effect on any *differentiable* function $H(\cdot, \cdot)$ is given by the standard total derivative formula

$$\left.\frac{dH(1+k_1\cdot t, 1+k_2\cdot t)}{dt}\right|_{t=0} = k_1\cdot H_1(1,1) + k_2\cdot H_2(1,1) \tag{39}$$

where $H_1(\cdot,\cdot)$ and $H_2(\cdot,\cdot)$ denote the respective partial derivatives of $H(\cdot,\cdot)$. The natural analogues of this relationship link the left/right *directional total derivatives* of the kinked function $S(\cdot,\cdot)$ at $(1,1)$ to its left/right *directional partial derivatives* $S_1^L(\cdot,\cdot)$, $S_1^R(\cdot,\cdot)$, $S_2^L(\cdot,\cdot)$ and $S_2^R(\cdot,\cdot)$ there, namely

$$\begin{aligned}
\left.\frac{dS(1+k_1\cdot t, 1+k_2\cdot t)}{dt^L}\right|_{t=0} &= k_1\cdot S_1^{\substack{L \text{ if } k_1>0\\ R \text{ if } k_1<0}}(1,1) + k_2\cdot S_2^{\substack{L \text{ if } k_2>0\\ R \text{ if } k_2<0}}(1,1)\\
\left.\frac{dS(1+k_1\cdot t, 1+k_2\cdot t)}{dt^R}\right|_{t=0} &= k_1\cdot S_1^{\substack{R \text{ if } k_1>0\\ L \text{ if } k_1<0}}(1,1) + k_2\cdot S_2^{\substack{R \text{ if } k_2>0\\ L \text{ if } k_2<0}}(1,1)
\end{aligned} \tag{40}$$

Both (39) and (40) can be generalized to movements along any *differentiable path* $(1+\kappa_1(t), 1+\kappa_2(t))$ through (1,1), by replacing k_1 and k_2 in their right sides by $\kappa_1'(0)$ and $\kappa_2'(0)$. Thus standard multivariate marginal analysis—in the sense that the marginal effects of (directional) changes in the variables can be added when these changes occur jointly—still holds for the function $S(\cdot,\cdot)$. In addition, as long as we account for directions, the Fundamental Theorem of Calculus still applies to all of $S(\cdot,\cdot)$'s line and path integrals, even integrals along its horizontal and vertical lines of kink points in Figure 1a. That is, given arbitrary $\bar{z}_1 > \underline{z}_1$, *or* $\bar{z}_2 > \underline{z}_2$, we have

$$\begin{aligned}
S(\bar{z}_1,1) - S(\underline{z}_1,1) &= \int_{\underline{z}_1}^{\bar{z}_1} S_1^L(z_1,1)\cdot dz_1 = \int_{\underline{z}_1}^{\bar{z}_1} S_1^R(z_1,1)\cdot dz_1\\
S(1,\bar{z}_2) - S(1,\underline{z}_2) &= \int_{\underline{z}_2}^{\bar{z}_2} S_2^L(1,z_2)\cdot dz_2 = \int_{\underline{z}_2}^{\bar{z}_2} S_2^R(1,z_2)\cdot dz_2
\end{aligned} \tag{41}$$

On the other hand, the directional total and directional partial derivatives of the function $L(\cdot,\cdot)$ are *not* linked by linear relationships like (40). Even for the simple case of $k_1 = k_2 = 1$ we have

$$\begin{aligned}
\left.\frac{dL(1+t,1+t)}{dt^L}\right|_{t=0} &= 1 \neq 2 = \left.\frac{dL(1+t,1)}{dt^L}\right|_{t=0} + \left.\frac{dL(1,1+t)}{dt^L}\right|_{t=0}\\
\left.\frac{dL(1+t,1+t)}{dt^R}\right|_{t=0} &= 1 \neq 0 = \left.\frac{dL(1+t,1)}{dt^R}\right|_{t=0} + \left.\frac{dL(1,1+t)}{dt^R}\right|_{t=0}
\end{aligned} \tag{42}$$

In other words, standard additive marginal analysis fails at (1,1), as well as at every other kink point of $L(\cdot,\cdot)$.[24] This in turn yields a failure of the standard line integral formula along the line of $L(\cdot,\cdot)$'s kink points (the 45° line). That is, for arbitrary $\bar{z} > \underline{z}$ we have the pair of failures[25]

$$\begin{aligned}
L(\bar{z},\bar{z}) - L(\underline{z},\underline{z}) &= \bar{z} - \underline{z} \neq 2\cdot(\bar{z} - \underline{z}) = \int_{\underline{z}}^{\bar{z}} [L_1^L(z,z) + L_2^L(z,z)]\cdot dz\\
L(\bar{z},\bar{z}) - L(\underline{z},\underline{z}) &= \bar{z} - \underline{z} \neq 0 = \int_{\underline{z}}^{\bar{z}} [L_1^R(z,z) + L_2^R(z,z)]\cdot dz
\end{aligned} \tag{43}$$

The distinction between $S(\cdot,\cdot)$ and $L(\cdot,\cdot)$ that leads to these different amenabilities to calculus is not the presence versus absence of piecewise-linearity, since both functions are piecewise-linear about (1,1) and over open neighborhoods of each of their kink points in R^2_+. Nor is it the presence versus absence of additive separability—although $S(\cdot,\cdot)$ is additively separable and $L(\cdot,\cdot)$ is not, corresponding distinctions will also hold between the two nonseparable functions

$$S^*(z_1,z_2) \equiv \rho(S(z_1,z_2)) \qquad \text{vs.} \qquad L^*(z_1,z_2) \equiv \rho(L(z_1,z_2)) \tag{44}$$

for any smooth $\rho(\cdot)$ with $\rho'(\cdot) \neq 0$. Rather, the key property that distinguishes $S(\cdot,\cdot)$ from $L(\cdot,\cdot)$, and also $S^*(\cdot,\cdot)$ from $L^*(\cdot,\cdot)$, is the separability/nonseparability of their *local* structure about (1,1).

Formally, we say that a locally-piecewise linear function $H(\cdot,\ldots,\cdot)$ is *locally separable* about a point $\mathbf{Z}_0 = (\bar{z}_1,\ldots,\bar{z}_n)$ if there exists a set of univariate functions $\{\widehat{H}_1(\cdot;\mathbf{Z}_0),\ldots,\widehat{H}_n(\cdot;\mathbf{Z}_0)\}$ such that

$$\begin{aligned} H(z_1,\ldots,z_n) &- H(\bar{z}_1,\ldots,\bar{z}_n) \\ &= \widehat{H}(z_1-\bar{z}_1,\ldots,z_n-\bar{z}_n;\mathbf{Z}_0) + o\big(\|(z_1,\ldots,z_n)-(\bar{z}_1,\ldots,\bar{z}_n)\|\big) \\ &= \sum_{i=1}^{n} \widehat{H}_i(z_i-\bar{z}_i;\mathbf{Z}_0) + o\big(\|(z_1,\ldots,z_n)-(\bar{z}_1,\ldots,\bar{z}_n)\|\big) \end{aligned} \tag{45}$$

that is, if its piecewise-linear first order approximation function $\widehat{H}(\cdot,\ldots,\cdot;\mathbf{Z}_0)$ from (38) is additively separable. By piecewise-linearity of $\widehat{H}(\cdot,\ldots,\cdot;\mathbf{Z}_0)$, and hence of each $\widehat{H}_i(\cdot;\mathbf{Z}_0)$, this takes the form

$$\begin{aligned} &= \sum_{i=1}^{n} \widehat{H}_i'^{L}(0;\mathbf{Z}_0)\cdot(z_i-\bar{z}_i)^- + \sum_{i=1}^{n} \widehat{H}_i'^{R}(0;\mathbf{Z}_0)\cdot(z_i-\bar{z}_i)^+ \\ &\qquad + o\big(\|(z_1,\ldots,z_n)-(\bar{z}_1,\ldots,\bar{z}_n)\|\big) \end{aligned} \tag{45$'$}$$

where $(z_i-\bar{z}_i)^- = \min\{z_i-\bar{z}_i, 0\}$ and $(z_i-\bar{z}_i)^+ = \max\{z_i-\bar{z}_i, 0\}$, and for the $2\cdot n$ directional partial derivatives $\widehat{H}_1'^{L}(0;\mathbf{Z}_0),\ldots,\widehat{H}_n'^{L}(0;\mathbf{Z}_0), \widehat{H}_1'^{R}(0;\mathbf{Z}_0),\ldots,\widehat{H}_n'^{R}(0;\mathbf{Z}_0)$. It is this property of local additivity in the signed variable changes—not shared by *locally nonseparable* forms such as $\min\{z_1,z_2\}$, $\max\{z_1,z_2\}$, $\rho(\min\{z_1,z_2\})$, etc.—that makes locally separable functions amenable to the calculus of directional derivatives.

It is clear from (45)/(45)′ that if a locally separable $H(\cdot,\ldots,\cdot)$ is kinked at $\mathbf{Z}_0 = (\bar{z}_1,\ldots,\bar{z}_n)$, it must have at least one *univariate kink* of the form $\partial H(\bar{z}_1,\ldots,\bar{z}_n)/\partial z_i^L \neq \partial H(\bar{z}_1,\ldots,\bar{z}_n)/\partial z_i^R$. In such a case, $H(\cdot,\ldots,\cdot)$ will also have *directional kinks* of the form $dH(\bar{z}_1+k_1\cdot t,\ldots,\bar{z}_n+k_n\cdot t)/dt^L|_{t=0} \neq dH(\bar{z}_1+k_1\cdot t,\ldots,\bar{z}_n+k_n\cdot t)/dt^R|_{t=0}$ in at least some more general linear directions $(k_1,\ldots,k_n)$. However, the *locally* *nonseparable* function

$$H(z_1, z_2) = \frac{1}{2} \cdot \big[\mathrm{sgn}[z_1] + \mathrm{sgn}[z_2]\big] \cdot \min\{|z_1|, |z_2|\}$$
$$= \begin{cases} \min\{z_1, z_2\} & \text{for } z_1, z_2 > 0 \\ \max\{z_1, z_2\} & \text{for } z_1, z_2 < 0 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

has no univariate *or* directional kinks through (0, 0), yet it is still kinked at (0,0) since it has no first order linear approximation there, and exhibits the same type of multivariate calculus failures as (42), namely

$$\left.\frac{dH(t,t)}{dt}\right|_{t=0} = 1 \neq 0 = \left.\frac{dH(t,0)}{dt}\right|_{t=0} + \left.\frac{dH(0,t)}{dt}\right|_{t=0} \tag{47}$$

This example underscores the important point that for multivariate functions, the distinction between "smooth" and "kinked" is an *inherently multivariate concept* rather than simply a univariate or directional one, and that a function can be locally nonseparably kinked at a point—and thus unamenable to the calculus of directional derivatives—*in spite of* having smooth left = right derivatives in each individual variable and in every linear direction from that point. In fact, for locally nonseparable functions, smooth univariate and directional derivatives can mean very little indeed: The locally nonseparable function $H(z_1, z_2) \equiv \{1 \text{ if } 0 < z_2 < z_1{}^2; 0 \text{ otherwise}\}$ satisfies $dH(k_1 \cdot t, k_2 \cdot t)/dt|_{t=0} = 0$ in every direction (k_1, k_2) from (0, 0), including along each axis, yet it is not smooth, locally piecewise-linear, or even *continuous* at (0, 0).

2.3. *Ordinal implications of locally separable vs. locally nonseparable kinks*

Preference functions over lotteries, like elsewhere in consumer theory, represent an individual's preferences over the objects of choice by mapping them to unobservable "preference levels." Although researchers occasionally derive results directly from the underlying preference relation, most analyses of maximizing behaviour and its comparative statics operate on the preference function, especially when it is posited to have some specific functional form.

However, not all properties of a preference function are empirically meaningful. Properties like "the *level* of $V(\mathbf{P})$ is positive for all $\mathbf{P} \in \mathscr{L}$" or "the *derivative* $\partial V(\mathbf{P})/\partial \mathrm{prob}(x)$ is less than 2 whenever $\mathrm{prob}(x) = \frac{1}{2}$ and $x = 8$" have no observable implications. For a mathematical property of a preference function to have empirical significance, it must have implications for its indifference curves, Engel curves, certainty equivalent function or some other such observable construct. But since local separability/local nonseparability are precisely properties of a function's level and /or derivatives, such ordinal implications must be established.

The link between a function's *first order (linear or piecewise-linear) approximation* about a point and its *ranking* about that point are not exact, even for the simplest of properties. Consider:

- $H(z_1, z_2) = z_1 + (z_2 - 1)^3$, whose first order linear approximation about (1, 1) is $1 \cdot \Delta z_1 + 0 \cdot \Delta z_2$, even though $H(\cdot, \cdot)$ is strictly increasing in both z_1 and z_2 about (1, 1) (and globally)
- $H(z_1, z_2) = (2 \cdot z_1 + z_2 - 3)^3$, whose first order linear approximation about (1, 1) is $0 \cdot \Delta z_1 + 0 \cdot \Delta z_2$, even though $H(\cdot, \cdot)$'s marginal rate of substitution is exactly -2 about (1, 1) (and globally)

For smoothness/kink properties, such "cardinal/ordinal disconnects" can arise from either:

- *kinked labeling of indifference curves:* As with $H(z_1, z_2) = \min\{2 + z_1 + z_2, 2 \cdot (z_1 + z_2)\}$, whose first order approximation about (1, 1) is the locally nonseparably kinked function $\min\{\Delta z_1 + \Delta z_2, 2 \cdot (\Delta z_1 + \Delta z_2)\}$, even though its indifference curves are all parallel straight lines.
- *"locally-horizontal"(zero-derivative) labeling of indifference curves:* As with $H(z_1, z_2) = (\min\{z_1, z_2\} - 1)^3$ which is differentiable at (1, 1) with first order linear approximation $0 \cdot \Delta z_1 + 0 \cdot \Delta z_2$, even though its (Leontief) preferences are locally nonseparably kinked there.

The formal links between the smoothness/kink properties of a function and of its ordinal preferences that *do* exist can take two forms: one-way implications from local properties of the function to local properties of its ranking; and two-way correspondences between local/global properties of the function and local/global properties of its observable "valuation functions" (e.g., certainty or probability equivalent functions), which hold as long as the function has been "normalized" to eliminate the above type of kinked or locally-horizontal labeling at the appropriate locations. We consider each type of link in turn:

Local rankings. The successively stronger properties of (i) local piecewise-linearity (which allows locally nonseparable kinks in $H(\cdot)$), (ii) local piecewise-linearity + local separability (which allows only locally separable kinks), and (iii) local linearity (no kinks) ensure successively stronger conditions on $H(\cdot)$'s rankings about a point $\mathbf{Z}_0$. It is straightforward to show that these local properties of $H(\cdot)$ at $\mathbf{Z}_0$—that is properties of its piecewise-linear first order approximation $\widehat{H}(\cdot; \mathbf{Z}_0)$ from (38)—have the following successively stronger implications for $H(\cdot)$'s ranking about $\mathbf{Z}_0$, interpretable as properties of its indifference surfaces about the point $\mathbf{Z}_0$ [26]:

For $H(\cdot)$ *locally piecewise linear about* $\mathbf{Z}_0$ with respect to the convex cones $\{E_1, \ldots, E_J\}$:

$$\underbrace{\widehat{H}(\Delta\mathbf{Z}^A; \mathbf{Z}_0) > \widehat{H}(\Delta\mathbf{Z}^B; \mathbf{Z}_0)}_{\Delta\mathbf{Z}^A, \Delta\mathbf{Z}^B \text{ in same cone } E_j}$$

$$\Rightarrow \underbrace{H(\mathbf{Z}_0 + t \cdot (\Delta\mathbf{Z}^A + \Delta\mathbf{Z})) > H(\mathbf{Z}_0 + t \cdot (\Delta\mathbf{Z}^B + \Delta\mathbf{Z}))}_{\text{all } \Delta\mathbf{Z} \text{ with } \Delta\mathbf{Z}^A + \Delta\mathbf{Z},\ \Delta\mathbf{Z}^B + \Delta\mathbf{Z} \text{ in } E_j, \text{ all sufficiently small } t > 0} \tag{48}$$

For $H(\cdot)$ locally piecewise linear + locally separable about $\mathbf{Z}_0$:

$$\underbrace{\widehat{H}(\Delta\mathbf{Z}^A;\mathbf{Z}_0) > \widehat{H}(\Delta\mathbf{Z}^B;\mathbf{Z}_0)}_{\text{arbitrary } \Delta\mathbf{Z}^A,\Delta\mathbf{Z}^B} \Rightarrow \underbrace{H(\mathbf{Z}_0 + t\cdot(\Delta\mathbf{Z}^A+\Delta\mathbf{Z})) > H(\mathbf{Z}_0 + t\cdot(\Delta\mathbf{Z}^B+\Delta\mathbf{Z}))}_{\substack{\text{all } \Delta\mathbf{Z} \text{ with } \Delta z_i=0 \text{ if } \mathrm{sgn}[\Delta z_i^A]\neq\mathrm{sgn}[\Delta z_i^B],\\ \Delta\mathbf{Z}^A+\Delta\mathbf{Z} \text{ in same orthant as } \Delta\mathbf{Z}^A,\\ \Delta\mathbf{Z}^B+\Delta\mathbf{Z} \text{ in same orthant as } \Delta\mathbf{Z}^B,\\ \text{all sufficiently small } t>0}} \quad (49)$$

For $H(\cdot)$ locally linear about $\mathbf{Z}_0$:

$$\underbrace{\widehat{H}(\Delta\mathbf{Z}^A;\mathbf{Z}_0) > \widehat{H}(\Delta\mathbf{Z}^B;\mathbf{Z}_0)}_{\text{arbitrary } \Delta\mathbf{Z}^A,\Delta\mathbf{Z}^B} \Rightarrow \underbrace{H(\mathbf{Z}_0 + t\cdot(\Delta\mathbf{Z}^A+\Delta\mathbf{Z})) > H(\mathbf{Z}_0 + t\cdot(\Delta\mathbf{Z}^B+\Delta\mathbf{Z}))}_{\text{all } \Delta\mathbf{Z},\ \text{all sufficiently small } t>0} \quad (50)$$

Although these local ranking properties are *implied by* the respective local properties of $H(\cdot)$, they do not necessarily imply them, as seen by the above kinked-labeling example $H(z_1, z_2) = \min\{2 + z_1 + z_2, 2\cdot(z_1 + z_2)\}$, which is not differentiable about $\mathbf{Z}_0 = (1, 1)$ yet does satisfy (50) there.

Local and global valuation functions. In addition to the local ranking properties (48)–(50), there are also straightforward conditions under which the smoothness/kink properties of a preference function directly "pass through" to an important class of its observable constructs, namely its *valuation functions*. Given a function $H(\cdot)$ over some compact subset of R^n, let $\{\mathbf{Z}_\alpha = (z_1(\alpha), \ldots, z_n(\alpha)) | \alpha \in [\underline{\alpha}, \bar{\alpha}]\}$ be any smooth path from $H(\cdot)$'s lowest-valued point to its highest-valued point, such that $H(\mathbf{Z}_\alpha)$ is increasing in α. $H(\cdot)$'s *α-equivalent function* $\alpha(\cdot)$, defined by $H(\mathbf{Z}_{\alpha(\mathbf{Z})}) \equiv_{\mathbf{z}} H(\mathbf{Z})$, is an observable function which represents $H(\cdot)$'s global ranking. For any point $\mathbf{Z}$ on or off the path, if $dH(\mathbf{Z}_\alpha)/d\alpha$ exists and is positive at $\alpha = \alpha(\mathbf{Z})$, the implicit function theorem[27] implies that the first order (linear or piecewise-linear) approximation of $\alpha(\cdot)$ about $\mathbf{Z}$ will differ from the first order approximation of $H(\cdot)$ about $\mathbf{Z}$ only by the positive multiplicative term $[dH(\mathbf{Z}_\alpha)/d\alpha|_{\alpha=\alpha(\mathbf{Z})}]^{-1}$. This implies that $H(\cdot)$'s smoothness/kink structure about any point $\mathbf{Z}$—including its local separability/nonseparability properties—passes directly through to the observable function $\alpha(\cdot)$ about that point.

Of course, the smoothness/kink properties of $H(\cdot)$ generally *do not* pass through to $\alpha(\cdot)$ at any point $\mathbf{Z}$ where the positive derivative condition $dH(\mathbf{Z}_\alpha)/d\alpha|_{\alpha=\alpha(\mathbf{Z})} > 0$ fails, either because $H(\cdot)$ has a kinked labeling along the path $\{\mathbf{Z}_\alpha | \alpha \in [\underline{\alpha}, \overline{\alpha}]\}$ at $\alpha = \alpha(\mathbf{Z})$, or else because it has a locally-horizontal labeling there. Furthermore, for given $H(\cdot)$, some paths and their implied valuation functions may satisfy the positive derivative condition at a given $\mathbf{Z}$, while other paths and valuation functions may not. However, for any given path $\{\mathbf{Z}_\alpha | \alpha \in [\underline{\alpha}, \overline{\alpha}]\}$ with implied valuation function $\alpha(\cdot)$, as long as $H(\mathbf{Z}_\alpha)$ is at least increasing in α, $H(\cdot)$ will have an ordinally equivalent "normalization" (increasing transformation) $H^*(\cdot)$ that satisfies $dH^*(\mathbf{Z}_\alpha)/d\alpha > 0$ along the entire path, such as the transformation $H^*(\cdot) = \phi^{-1}(H(\cdot))$ for $\phi(\alpha) \equiv_\alpha H(\mathbf{Z}_\alpha)$. Since such an $H^*(\cdot)$ satisfies $dH^*(\mathbf{Z}_\alpha)/d\alpha|_{\alpha=\alpha(\mathbf{Z})} > 0$ for all $\mathbf{Z}$, its local smoothness/kink structure passes through to the observable function $\alpha(\cdot)$ at every point $\mathbf{Z}$ in its domain. Should $H^*(\cdot)$'s local nonseparability along some ridge of kink points yield a breakdown of a *global* line or path integral formula (as in (43)), this global breakdown will generally also pass through to the observable function $\alpha(\cdot)$.

A simple example of this for standard utility functions $H(\cdot)$ over commodity bundles $\mathbf{Z} = (z_1, \ldots, z_n)$ consists of selecting $\{\mathbf{Z}_\alpha | \alpha \in [\underline{\alpha}, \overline{\alpha}]\}$ as the consumer's income-consumption locus (expansion path) with respect to income α at given prices, so that $\alpha(\cdot)$ represents the consumer's observable *income-equivalent function (*or *money-metric utility function)* over all commodity bundles $\mathbf{Z}$. Thus, if $H(\cdot)$ satisfies (or has been normalized to satisfy) $dH(\mathbf{Z}_\alpha)/d\alpha > 0$ at all income levels α, the income-equivalent function $\alpha(\cdot)$ inherits $H(\cdot)$'s smoothness/kink structure about each bundle $\mathbf{Z}$. Straightforward extensions yield similar results for the consumer's observable *willingness-to-pay* and *willingness-to-accept* functions for changes from any bundle $\mathbf{Z}$.

In our context of preference functions $V(\cdot)$ over lotteries, this framework takes the form of a smooth path $\{\mathbf{P}_\alpha | \alpha \in [\underline{\alpha}, \overline{\alpha}]\}$ from $V(\cdot)$'s least preferred lottery to its most preferred lottery, with valuation function $\alpha(\cdot)$ defined by $V(\mathbf{P}_{\alpha(\mathbf{P})}) \equiv_{\mathbf{P}} V(\mathbf{P})$ and positive derivative condition $dV(\mathbf{P}_\alpha)/d\alpha|_{\alpha=\alpha(\mathbf{P})} > 0$ at a given $\mathbf{P}$.[28] About any such $\mathbf{P}$, the observable function $\alpha(\cdot)$ again inherits the general smoothness/kink structure of $V(\cdot)$ in any given set of variables (payoffs and/or probabilities), including its local separability/nonseparability properties. Two important examples are:

- *certainty equivalents:* If $\{\mathbf{P}_x = (x, 1) | x \in [0, M]\}$ is the path of *degenerate lotteries* over $[0, M]$, its associated valuation function is $V(\cdot)$'s *certainty equivalent function* $CE(\cdot)$. At any $\mathbf{P}$ where $dV(x, 1)/dx|_{x=CE(\mathbf{P})} > 0$, the local separability/nonseparability properties of $V(\cdot)$ about $\mathbf{P}$ will pass through to its certainty equivalent function $CE(\cdot)$
- *probability equivalents:* If $\{\mathbf{P}_\rho = (0, 1-\rho; M, \rho) | \rho \in [0, 1]\}$ is the path of *basic reference lotteries* over the payoff levels 0 and M, its associated valuation function is $V(\cdot)$'s *probability equivalent function* $PE(\cdot)$. At any $\mathbf{P}$ where $dV(0, 1-\rho; M, \rho)/d\rho|_{\rho=PE(\mathbf{P})} > 0$, the local separability/nonseparability properties of $V(\cdot)$ will again pass through to its function $PE(\cdot)$

Note that if $U(\cdot)$ in the additive expected utility form $V_{EU}(\mathbf{P}) \equiv \sum_{i=1}^n U(x_i) \cdot p_i$ has kinks, this form *fails* the condition $dV_{EU}(x, 1)/dx|_{x=CE(\mathbf{P})} > 0$ at any $\mathbf{P}$ whose certainty equivalent is a kink point of $U(\cdot)$, although $V_{EU}(\cdot)$'s smoothness/kink structure does pass through to $CE_{EU}(\cdot)$ at all other $\mathbf{P}$. On the other hand, assuming $U(M) > U(0)$, $V_{EU}(\cdot)$ satisfies $dV_{EU}(0, 1-\rho; M, \rho)/d\rho|_{\rho=PE(\mathbf{P})} > 0$ at every $\mathbf{P}$, so that all kinks in $V_{EU}(\cdot)$ pass through to its probability equivalent function $PE_{EU}(\cdot)$. For a Fréchet differentiable $V_{FR}(\cdot)$ with local utility function $U(\cdot; \mathbf{P})$, (32) and (19) imply that these positive derivative conditions take the respective forms $U'(x; (CE(\mathbf{P}), 1))|_{x=CE(\mathbf{P})} > 0$ and $[U(M; (0, 1-\rho; M, \rho)) - U(0; (0, 1-\rho; M, \rho))]|_{\rho=PE(\mathbf{P})} > 0$. Although we do not introduce it until Section 4, we note here that if the functions $v(\cdot)$ and $G(\cdot)$ in the "rank-dependent" form $V_{RD}(\cdot)$ are both smooth with positive derivatives (as we shall assume), the conditions $dV_{RD}(x, 1)/dx|_{x=CE(\mathbf{P})} > 0$ and $dV_{RD}(0, 1-\rho; M, \rho)/d\rho|_{\rho=PE(\mathbf{P})} > 0$ hold at all $\mathbf{P}$, so that $V_{RD}(\cdot)$'s smoothness/kink structure, including its prevalence of locally nonseparable payoff kinks, passes through to both its certainty equivalent function and its probability equivalent function about every $\mathbf{P}$ in $\mathscr{L}$.

3. Expected utility and Fréchet differentiable payoff kinks

A preference function $V(\cdot)$ over lotteries is said to be *piecewise-linear in the payoffs* about a lottery $\mathbf{P} \in \mathscr{L}$ if, for each representation $(\bar{x}_1, \bar{p}_1; \ldots; \bar{x}_n, \bar{p}_n)$ of $\mathbf{P}$, $V(x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n)$ is piecewise-linear in the payoff variables $(x_1, \ldots, x_n)$ about the values $(\bar{x}_1, \ldots, \bar{x}_n)$. Similarly, $V(\cdot)$ is *locally piecewise-linear in the payoffs* about $\mathbf{P}$ if $V(x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n)$ is locally piecewise-linear in $(x_1, \ldots, x_n)$ about $(\bar{x}_1, \ldots, \bar{x}_n)$ for every representation $(\bar{x}_1, \bar{p}_1; \ldots; \bar{x}_n, \bar{p}_n)$ of $\mathbf{P}$. This implies that $V(\cdot)$'s regular, partial-probability and whole-probability payoff derivatives

$$\frac{\partial V(\ldots; x_i, p_i; \ldots)}{\partial x_i} \qquad \left.\frac{\partial V(\ldots; x, \rho; x_i, p_i - \rho; \ldots)}{\partial x}\right|_{x=x_i} \qquad \frac{dV(\ldots; x_i, p_i; \ldots)}{dx^W}$$

or at least their left/right directional versions, all exist. A locally piecewise-linear $V(\cdot)$ is said to be *kinked in the payoffs* (or have a *payoff kink*) about $\mathbf{P}$ if $V(x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n)$ is *not* differentiable in $(x_1, \ldots, x_n)$ about $(\bar{x}_1, \ldots, \bar{x}_n)$ for some representation $(\bar{x}_1, \bar{p}_1; \ldots; \bar{x}_n, \bar{p}_n)$. In this and the following section, we examine the nature and prevalence of such payoff kinks in expected utility, Fréchet differentiable and rank-dependent risk preferences.

3.1. Local separability of expected utility payoff kinks

It is clear that an expected utility preference function $V_{EU}(\mathbf{P}) \equiv \sum_{i=1}^{n} U(x_i) \cdot p_i$ will exhibit payoff kinks if and only if its von Neumann-Morgenstern utility function $U(\cdot)$ is kinked at one or more payoff levels. As noted in the Introduction, this may be due to kinked tax schedules, indivisibilities, bankruptcy, or it may be an inherent property of the individual's underlying preferences over wealth or consumption lotteries. Since $V_{EU}(\cdot)$ is globally separable in the payoffs regardless of the shape of $U(\cdot)$, expected utility payoff kinks must all be locally separable—i.e., expected utility preferences *cannot* exhibit locally nonseparable payoff kinks.[29]

Figures 2a and 2b illustrate a strictly risk averse von Neumann-Morgenstern utility function $U(\cdot)$ with a kink at payoff level $x = 100$, along with its indifference curves in the Hirshleifer-Yaari diagram[30] for fixed state probabilities $\bar{p}_1, \bar{p}_2$. These curves will be smooth at all points (x_1, x_2) off the horizontal and vertical lines in Figure 2b, with marginal rate of substitution

$$MRS_{EU}(x_1, x_2) = \left.\frac{dx_2}{dx_1}\right|_{V_{EU}(\cdot)=\text{constant}} = -\frac{U'(x_1) \cdot \bar{p}_1}{U'(x_2) \cdot \bar{p}_2} \qquad x_1 \neq 100 \neq x_2. \tag{51}$$

Since $MRS_{EU}(x, x) = -\bar{p}_1/\bar{p}_2$ at all certainty points (x, x) except (100,100), the common slope of the indifference curves along the 45° line or *certainty line* in the figure reveals the individual's subjective *odds ratio* for the two states. The downward sloping line segments are portions of *iso-expected value lines* or *fair odds lines*, with a common

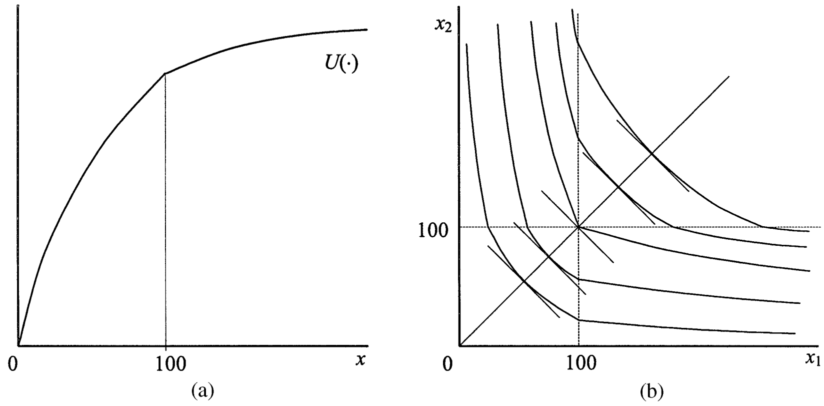

Figure 2. (a and b) Kinked von Neumann-Morgenstern utility function and its indifference curves (indifference curves are kinked along dashed lines).

expected value $x_1 \cdot \bar{p}_1 + x_2 \cdot \bar{p}_2$, which are tangent to the indifference curves at all points on the certainty line expect the kink point (100,100).

$U(\cdot)$'s concave kink at $x = 100$ implies that its indifference curves will have quasi-concave kinks at the certainty point $(x, x) = (100, 100)$ as well as at all other points on the vertical line $x_1 = 100$ and horizontal line $x_2 = 100$ in the figure. At such kink points, $V_{EU}(\cdot)$'s *directional payoff derivatives* are given by the natural analogues of (20), (21), namely

$$\frac{\partial V_{EU}(\ldots; x_i, p_i; \ldots)}{\partial x_i^{L/R}} = p_i \cdot U'_{L/R}(x_i) \tag{52}$$

$$\left.\frac{\partial V_{EU}(\ldots; x, \rho; x_i, p_i - \rho; \ldots)}{\partial x^{L/R}}\right|_{x=x_i} = \rho \cdot U'_{L/R}(x_i)$$

$$\frac{dV_{EU}(\ldots; x_i, p_i; \ldots)}{dx^{W,L/R}} = \left(\sum_{x_j = x} p_j\right) \cdot U'_{L/R}(x) \tag{53}$$

and as seen, they continue to be proportional to/additive in the amount(s) of probability mass shifted in a given direction, although the coefficient of linearity, namely $U'_L(x_i)$ vs. $U'_R(x_i)$, now depends upon the direction of shift (left vs. right). These directional payoff derivatives also satisfy the directional analogues of the total derivative formulas (8)–(10), for example

$$\begin{aligned} \left.\frac{dV_{EU}(x_1 + \alpha \cdot t, \bar{p}_1; x_2 + \beta \cdot t, \bar{p}_2)}{dt^R}\right|_{t=0} &= \alpha \cdot \frac{\partial V_{EU}(x_1, \bar{p}_1; x_2, \bar{p}_2)}{\partial x_1^R} \\ &\quad + \beta \cdot \frac{\partial V_{EU}(x_1, \bar{p}_1; x_2, \bar{p}_2)}{\partial x_2^R} \qquad \alpha, \beta > 0 \\ &= \alpha \cdot U'_R(x_1) \cdot \bar{p}_1 + \beta \cdot U'_R(x_2) \cdot \bar{p}_2. \end{aligned} \tag{54}$$

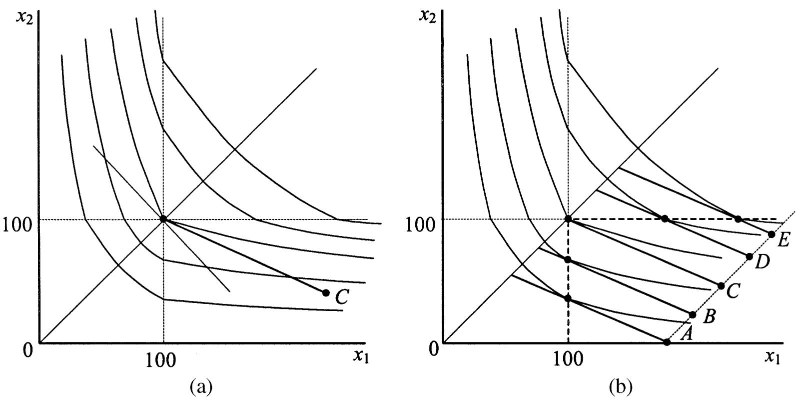

Figure 3. (a and b) Full purchase of actuarially unfair insurance; wealth effects on the demand for coinsurance.

This in turn implies the natural analogue of (51), linking $U(\cdot)$'s directional payoff derivatives and $V_{EU}(\cdot)$'s *left/right marginal rates of substitution*

$$MRS_{EU,L}(x_1, x_2) = -\frac{U'_L(x_1) \cdot \bar{p}_1}{U'_R(x_2) \cdot \bar{p}_2} \qquad MRS_{EU,R}(x_1, x_2) = -\frac{U'_R(x_1) \cdot \bar{p}_1}{U'_L(x_2) \cdot \bar{p}_2} \tag{55}$$

Finally, the fundamental theorem of calculus continues to link global changes in $V_{EU}(\cdot)$ with its directional partial derivatives along any line of kink points, for example

$$\begin{aligned} V_{EU}(150, \bar{p}_1; 100, \bar{p}_2) - V_{EU}(50, \bar{p}_1; 100, \bar{p}_2) &= \int_{50}^{150} \frac{\partial V_{EU}(x_1, \bar{p}_1; 100, \bar{p}_2)}{\partial x_1^L} \cdot dx_1 \\ &= \int_{50}^{150} \frac{\partial V_{EU}(x_1, \bar{p}_1; 100, \bar{p}_2)}{\partial x_1^R} \cdot dx_1 \end{aligned} \tag{56}$$

In these senses, expected utility payoff kinks remain amenable to the local/global calculus of directional payoff derivatives.[31]

Figures 3a and 3b illustrate some implications of expected utility payoff kinks for optimization and comparative statics. Figure 3a is the standard illustration of how an individual at a risky initial point C might purchase full coinsurance, even at an actuarially unfair price.[32] However, if $U(\cdot)$ has a single kink at $x = 100$ this will be a knife-edge phenomenon in the following sense: For any initial point C, there is *exactly one* actuarially unfair load factor that would lead the individual to choose full insurance, namely, the one whose budget line from C leads exactly to the point (100, 100). Any larger or smaller load factor from C will lead to a *partial insurance* optimum, at either a tangency or kink point, located *strictly southeast* of the certainty line.[33]

Figure 3b illustrates a potential comparative statics implication of a kinked $U(\cdot)$ that is *not* knife-edge. The uninsured positions A, B, C, D, E lie along a line of slope $+1$; that is, they differ from each other only in the addition/subtraction of some sure amount of wealth. As such increases in wealth raise the individual from A to E, the amount of insurance purchased first rises at a constant rate, becomes complete at the wealth corresponding to point C, and then falls at a constant rate—that is, the Engel curve for insurance is $\wedge$-shaped. To see that this implication is generally not knife-edge, observe that since the kinks in the figure are strictly quasiconcave, there will be a non-degenerate range of load factors about the one in the figure, each of which leads to a similar $\wedge$-shaped Engel curve for insurance. Each load factor in this range will imply a full insurance optimum at (100,100) from some unique point C' on the line $\overline{AE}$, which for higher load factors will lie above point C on the line, and for lower load factors, below it.

3.2. Local separability of Fréchet differentiable payoff kinks

As with expected utility preference functions, Fréchet differentiable preference functions are also locally payoff-separable. To see this, pick arbitrary $\mathbf{P}_0 = (\bar{x}_1, \bar{p}_1; \ldots; \bar{x}_n, \bar{p}_n)$ and consider alternative lotteries of the form $\mathbf{P} = (x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n)$. Fréchet differentiability (eq. (26)) will imply[34]

$$\begin{aligned} V_{FR}(x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n) &- V_{FR}(\bar{x}_1, \bar{p}_1; \ldots; \bar{x}_n, \bar{p}_n) \\ &= \sum_{i=1}^{n} [U(x_i; \mathbf{P}_0) - U(\bar{x}_i; \mathbf{P}_0)] \cdot \bar{p}_i + o\big(\|(x_1, \ldots, x_n) - (\bar{x}_1, \ldots, \bar{x}_n)\|\big) \\ &= \sum_{i=1}^{n} \widehat{H}_i(x_i - \bar{x}_i; \mathbf{P}_0) + o\big(\|(x_1, \ldots, x_n) - (\bar{x}_1, \ldots, \bar{x}_n)\|\big) \end{aligned} \tag{57}$$

so that $V_{FR}(\cdot)$ exhibits local separability (eq. (45)) with respect to the set of univariate functions

$$\widehat{H}_i(\Delta x_i; \mathbf{P}_0) \equiv [U(\bar{x}_i + \Delta x_i; \mathbf{P}_0) - U(\bar{x}_i; \mathbf{P}_0)] \cdot \bar{p}_i \qquad i = 1, \ldots, n \tag{58}$$

To establish (57), observe from (22) that for small enough $|x_1 - \bar{x}_1|, \ldots, |x_n - \bar{x}_n|$ we will have

$$\begin{aligned} \|\mathbf{P} - \mathbf{P}_0\| \leq \sum_{i=1}^{n} p_i \cdot |x_i - \bar{x}_i| &\leq \max\big\{|x_i - \bar{x}_i| \big| i = 1, \ldots, n\big\} \\ &\leq \|(x_1, \ldots, x_n) - (\bar{x}_1, \ldots, \bar{x}_n)\| \end{aligned} \tag{59}$$

and hence

$$\begin{aligned} &\frac{\big|V_{FR}(\mathbf{P}) - V_{FR}(\mathbf{P}_0) - \big[\sum_{i=1}^{n} U(x_i; \mathbf{P}) \cdot p_i - \sum_{i=1}^{n} U(\bar{x}_i; \mathbf{P}) \cdot p_i\big]\big|}{\|\mathbf{P} - \mathbf{P}_0\|} \\ &\leq \frac{\big|V_{FR}(\mathbf{P}) - V_{FR}(\mathbf{P}_0) - \big[\sum_{i=1}^{n} U(x_i; \mathbf{P}) \cdot p_i - \sum_{i=1}^{n} U(\bar{x}_i; \mathbf{P}) \cdot p_i\big]\big|}{\|(x_1, \ldots, x_n) - (\bar{x}_1, \ldots, \bar{x}_n)\|} \end{aligned} \tag{60}$$

From (59), convergence of $(x_1, \ldots, x_n)$ to $(\bar{x}_1, \ldots, \bar{x}_n)$ in the Euclidean norm implies uniform convergence of $\mathbf{P}$ to $\mathbf{P}_0$ in the lottery norm (22). By (26) this implies that the left side of (60) converges to 0, and thus so does the right side (and does so uniformly in $\|(x_1, \ldots, x_n) - (\bar{x}_1, \ldots, \bar{x}_n)\|$), which establishes (57) and hence the property of local separability in (individual or joint) regular and whole probability payoff changes. Local payoff separability in all *partial-probability* payoff changes, say two-outcome partial-probability changes from x_1, is established by invoking (2) to write $\mathbf{P}_0$ as $(\bar{x}_1, \bar{p}_1 - \rho_a - \rho_b; y_a, \rho_a; y_b, \rho_b; \bar{x}_2, \bar{p}_2; \ldots; \bar{x}_n, \bar{p}_n)|_{y_a = y_b = \bar{x}_1}$ and applying (57) to the $n+2$ payoff variables $(x_1, y_a, y_b, x_2, \ldots, x_n)$ about the values $(\bar{x}_1, \bar{x}_1, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. A similar argument applies for the $n+k$ variables $(x_1, y_1, \ldots, y_k, x_2, \ldots, x_n)$ about $(\bar{x}_1, \bar{x}_1, \ldots, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, etc. Accordingly, all regular, partial-probability and whole-probability payoff kinks in Fréchet differentiable preference functions will be locally separable.

A derivation equivalent to (32) yields that if $U(\cdot; \mathbf{P})$ has directional derivatives at x_i, then $V_{FR}(\cdot)$ has directional payoff derivatives corresponding to the expected utility formulas (52), (53):

$$\frac{\partial V_{FR}(\ldots; x_i, p_i; \ldots)}{\partial x_i^{L/R}} = p_i \cdot U'_{L/R}(x_i; \mathbf{P}) \tag{61}$$

$$\left.\frac{\partial V_{FR}(\ldots; x, \rho; x_i, p_i - \rho; \ldots)}{\partial x^{L/R}}\right|_{x = x_i} = \rho \cdot U'_{L/R}(x_i; \mathbf{P})$$
$$\frac{dV_{FR}(\ldots; x_i, p_i; \ldots)}{dx^{W,L/R}} = \left(\sum_{x_j = x} p_j\right) \cdot U'_{L/R}(x; \mathbf{P}) \tag{62}$$

which are again proportional/additive in the amount(s) of probability mass shifted in a given direction from a given mass point. These formulas imply the natural analogues of (55), linking $V_{FR}(\cdot)$'s *left/right marginal rates of substitution* and $U(\cdot; \mathbf{P})$'s directional payoff derivatives

$$MRS_{FR,L}(x_1, x_2) = \frac{U'_L(x_1; \mathbf{P}) \cdot \bar{p}_1}{U'_R(x_2; \mathbf{P}) \cdot \bar{p}_2} \qquad MRS_{FR,R}(x_1, x_2) = \frac{U'_R(x_1; \mathbf{P}) \cdot \bar{p}_1}{U'_L(x_2; \mathbf{P}) \cdot \bar{p}_2} \tag{63}$$

In addition, the directional payoff derivatives (61), (62) are also seen to satisfy the directional analogues of the total derivative formulas (8)–(10), for example,[35]

$$\left.\frac{dV_{FR}(x_1 + \alpha \cdot t, \bar{p}_1; x_2 + \beta \cdot t, \bar{p}_2)}{dt^R}\right|_{t=0}$$
$$= \alpha \cdot \frac{\partial V_{FR}(x_1, \bar{p}_1; x_2, \bar{p}_2)}{\partial x_1^R} + \beta \cdot \frac{\partial V_{FR}(x_1, \bar{p}_1; x_2, \bar{p}_2)}{\partial x_2^R} \qquad \alpha, \beta > 0$$
$$= \alpha \cdot U'_R(x_1; \mathbf{P}) \cdot \bar{p}_1 + \beta \cdot U'_R(x_2; \mathbf{P}) \cdot \bar{p}_2 \tag{64}$$

That is, the marginal effect of a joint payoff shift equals the sum of the marginal effects of its individual component shifts.

Some specific non-expected utility functional forms for $V(x_1, p_1; \ldots; x_n, p_n)$, and researchers who have studied them, are:

$$\begin{array}{lcl} \text{weighted utility} & \sum_{i=1}^{n} \upsilon(x_i) \cdot p_i \Big/ \sum_{i=1}^{n} \tau(x_i) \cdot p_i & \text{Chew (1983)} \\ \text{moments of utility} & g\Big(\sum_{i=1}^{n} \upsilon(x_i) \cdot p_i, \sum_{i=1}^{n} \upsilon(x_i)^2 \cdot p_i, \sum_{i=1}^{n} \upsilon(x_i)^3 \cdot p_i\Big) & \text{Hagen (1989)} \\ \text{quadratic in the probabilities} & \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa(x_i, x_j) \cdot p_i \cdot p_j & \text{Chew, Epstein and Segal (1991)} \end{array} \quad (65)$$

These forms are similar to expected utility in that they can be used to represent payoff-smooth preferences by choosing smooth component functions $\upsilon(\cdot)$, $\tau(\cdot)$, $g(\cdot,\cdot,\cdot)$ or $\kappa(\cdot,\cdot)$, as well as preferences with kinks at prespecified (e.g., tax-bracket) payoff levels by choosing component functions with kinks at those values.[36] In the latter case, these forms will: continue to have well-defined local utility functions, satisfy the generalized expected utility results of Section 1.3, satisfy the directional payoff derivative formulas (61)–(64), and satisfy the first order and first order conditional risk aversion properties and results of the following section.

3.3. *First order and first order conditional risk aversion*

Hilton (1988), Montesano (1985, 1988) and Segal and Spivak (1990) have examined senses in which risk preferences about piecewise-linear payoff kinks can be qualitatively different from smooth preferences: For a given initial wealth level x^* and a nondegenerate zero-mean risk $\tilde{\varepsilon}$, denote the standard *risk premium* for any additive risk $\tilde{\varepsilon}$ by $\pi(\tilde{\varepsilon}; x^*)$, so that the individual is indifferent between the risky wealth $x^* + \tilde{\varepsilon}$ and the sure wealth $x^* - \pi(\tilde{\varepsilon}; x^*)$. It is well known (e.g., Pratt (1964)) that expected utility preferences with a *differentiable* $U(\cdot)$ satisfy $\partial\pi(t \cdot \tilde{\varepsilon}; x^*)/\partial t|_{t=0} = 0$. In the Segal and Spivak formulation, an individual is said to exhibit:

$$\begin{array}{rl} \textit{first order risk aversion} \text{ about } x^* & \text{if } \partial\pi(t \cdot \tilde{\varepsilon}; x^*)/\partial t|_{t\downarrow 0} > 0 \\ \textit{second order risk aversion} \text{ about } x^* & \text{if } \partial\pi(t \cdot \tilde{\varepsilon}; x^*)/\partial t|_{t=0} = 0 \\ & \quad \text{but } \partial^2\pi(t \cdot \tilde{\varepsilon}; x^*)/\partial t^2|_{t\downarrow 0} > 0 \end{array}$$

for every nondegenerate zero-mean $\tilde{\varepsilon}$. Segal and Spivak show that if an individual (expected utility or otherwise) exhibits first order risk aversion about x^*, then for small enough positive k, the individual will strictly prefer x^* over the random wealth $x^* + t \cdot (k + \tilde{\varepsilon})$ for all sufficiently small $t > 0$. They also provide the following expected utility results linking properties of the von Neumann-Morgenstern utility function $U(\cdot)$ to its order of risk aversion about wealth level x^*:

- If a concave utility function $U(\cdot)$ is not differentiable at x^* but has well-defined left and right derivatives there, then the individual exhibits first order risk aversion about x^*
- If a concave utility function $U(\cdot)$ is twice differentiable at x^* with $U''(x^*) \neq 0$, then the individual exhibits second order risk aversion about x^*

This notion is not limited to preferences about certainty. Loomes and Segal (1994) have shown that any risk averse $U(\cdot)$ with a kink at x^* also exhibits *first order conditional risk aversion* about x^* in the following sense: Consider a random wealth of the form $[p$ chance of $x^* + \tilde{\varepsilon} : (1-p)$ chance of $\tilde{x}]$. Such distributions arise in cases of *uninsured events*, such as war or "acts of God," in which wealth is exogenously $\tilde{x}$ and no indemnity is paid. Many insurance contracts explicitly specify such events, and may or may not refund the original contract price if they occur. The individual's risk premium $\pi(\tilde{\varepsilon}; x^*, \tilde{x}, p)$ for contracts that give such refunds will solve

$$p \cdot E[U(x^* + \tilde{\varepsilon})] + (1-p) \cdot E[U(\tilde{x})] = p \cdot U(x^* - \pi(\tilde{\varepsilon}; x^*, \tilde{x}, p)) + (1-p) \cdot E[U(\tilde{x})] \qquad (66)$$

For contracts that do not give such refunds, the final term in (66) takes the form $(1 - p) \cdot E[U(\tilde{x} - \pi(\tilde{\varepsilon}; x^*, \tilde{x}, p))]$. In either case, if $U(\cdot)$ has a kink at x^* we get $\partial\pi(t \cdot \tilde{\varepsilon}; x^*, \tilde{x}, p)/\partial t|_{t\downarrow 0} > 0$.[37]

Segal and Spivak (1990, 1997) have also generalized the above expected utility results to Fréchet differentiable preferences: Given a risk averse $V_{FR}(\cdot)$, if its local utility function $U(\cdot; \mathbf{P}_{x^*})$ at any degenerate lottery $\mathbf{P}_{x^*} = (x^*, 1)$ has a kink at $x = x^*$, $V_{FR}(\cdot)$ will exhibit first order risk aversion at x^*. Similarly, if $V_{FR}(\cdot)$'s local utility functions are twice differentiable and $U(x; \mathbf{P})$, $U'(x; \mathbf{P})$, $U''(x; \mathbf{P})$ are continuous in $\mathbf{P}$, $V(\cdot)$ exhibits second order risk aversion at all sure wealth levels.[38]

4. Rank-dependent payoff kinks

As it turns out, one of the most important non-expected utility preference functions has *probability derivatives* that are amenable to generalized expected utility analysis, but *payoff kinks* that are not (nor to the calculus of directional derivatives). This form, proposed by Quiggin (1982),[39] is known as the "rank-dependent expected utility" or *rank-dependent* form. For general cumulative distribution functions $F(\cdot)$ over $[0, M]$, it has the structure $V_{RD}(F(\cdot)) \equiv \int_0^M \upsilon(x) \cdot d(G(F(x))$ for increasing continuous functions $\upsilon(\cdot)$ and $G(\cdot)$ with normalizations $G(0) = 0$ and $G(1) = 1$.[40] In our setting of finite-outcome lotteries, this implies the form

$$\begin{aligned} V_{RD}(x_1, p_1; \ldots; x_n, p_n) \\ &\equiv \upsilon(\hat{x}_1) \cdot G(\hat{p}_1) \\ &+ \upsilon(\hat{x}_2) \cdot [G(\hat{p}_1 + \hat{p}_2) - G(\hat{p}_1)] \\ &\vdots \\ &+ \upsilon(\hat{x}_i) \cdot [G(\hat{p}_1 + \ldots + \hat{p}_i) - G(\hat{p}_1 + \ldots + \hat{p}_{i-1})] \qquad \hat{x}_1 \leq \ldots \leq \hat{x}_n \\ &\vdots \end{aligned}$$

$$+\,v(\hat{x}_n)\cdot[G(1)-G(\hat{p}_1+\ldots+\hat{p}_{n-1})]$$
$$=\sum_{i=1}^{n} v(\hat{x}_i)\cdot\left[G\left(\sum_{j\leq i}\hat{p}_j\right)-G\left(\sum_{j<i}\hat{p}_j\right)\right] \tag{67}$$

where $\hat{x}_1, \hat{p}_1$ denote the *lowest payoff* of the lottery $\mathbf{P}=(x_1, p_1;\ldots;x_n, p_n)$ and its associated probability, $\hat{x}_2, \hat{p}_2$ denote **P**'s *second lowest payoff* and its probability, etc. (In the lower formula, for $i=1$ the sum $\sum_{j\leq i}\hat{p}_j$ will be vacuous and hence takes value 0.) When two or more of **P**'s payoff values are equal, ties in the definition of $\hat{x}_i, \hat{p}_i$ can be broken in any manner. The structure of (67) ensures that $V_{RD}(\cdot)$ satisfies the identifications (2). When $G(\cdot)$ is linear (so that $G(p)\equiv p$), $V_{RD}(\cdot)$ reduces to the expected utility form $V_{EU}(F(\cdot))\equiv\int_0^M v(x)\cdot dF(x)$ or $V_{EU}(\mathbf{P})\equiv\sum_{i=1}^n v(\hat{x}_i)\cdot\hat{p}_i\equiv\sum_{i=1}^n v(x_i)\cdot p_i$. If $v(\cdot)$ were exogenously kinked at some payoff level x^*, $V_{RD}(\cdot)$ would exhibit the same type of locally separable payoff kinks as illustrated for the expected utility model in Figures 2b, 3a and 3b. Having noted this, and in order to concentrate on the *additional* kinks inherent in the rank dependent form, we henceforth assume that $v(\cdot)$ and $G(\cdot)$ are both smooth (continuously or up to infinitely differentiable) with $v'(\cdot), G'(\cdot)>0$.

4.1. Rank-dependent probability derivatives and local utility functions

Chew, Karni and Safra (1987) have shown that the rank-dependent form $V_{RD}(\cdot)$ does not/cannot satisfy the strong smoothness condition of Fréchet differentiability (eq. (26)), except for its special case $V_{EU}(\cdot)$. However, they have also shown that as long as $G(\cdot)$ is differentiable, the finite-outcome[41] rank-dependent form will nonetheless still be differentiable in the probabilities, so its local utility function $U_{RD}(\cdot;\mathbf{P})$ is well-defined at each distribution $\mathbf{P}\in\mathscr{L}$.

To derive $V_{RD}(\cdot)$'s local utility function[42] at a given $\mathbf{P}=(x_1, p_1;\ldots;x_n, p_n)$, observe that each existing or potential payoff level $x\in[0, M]$ lies in exactly one[43] of the $n+1$ successive intervals

$$I_0=[0,\hat{x}_1),\quad I_1=[\hat{x}_1,\hat{x}_2)\ldots\; I_k=[\hat{x}_k,\hat{x}_{k+1})\ldots\; I_{n-1}=[\hat{x}_{n-1},\hat{x}_n),$$
$$I_n=[\hat{x}_n, M] \tag{68}$$

Given arbitrary payoff level x in arbitrary interval $I_k (k=0,\ldots,n)$, from (67) we can write

$$V_{RD}(x_1,p_1;\ldots;x_n,p_n;x,\rho)=\sum_{1\leq i\leq k} v(\hat{x}_i)\cdot\left[G\left(\sum_{j\leq i}\hat{p}_j\right)-G\left(\sum_{j<i}\hat{p}_j\right)\right]$$
$$+v(x)\cdot\left[G\left(\sum_{j\leq k}\hat{p}_j+\rho\right)-G\left(\sum_{j\leq k}\hat{p}_j\right)\right]\qquad x\in I_k$$
$$+\sum_{k<i\leq n} v(\hat{x}_i)\cdot\left[G\left(\sum_{j\leq i}\hat{p}_j+\rho\right)-G\left(\sum_{j<i}\hat{p}_j+\rho\right)\right] \tag{69}$$

where as before, vacuous sums such as $\sum_{j<1}$, (and if relevant, $\sum_{1\le i\le 0}$ or $\sum_{n<i\le n}$) take value 0. From (15)/(19), the local utility function $U_{RD}(x;\mathbf{P}) = \partial V_{RD}(x_1, p_1; \ldots; x_n, p_n; x, \rho)/\partial\rho|_{\rho=0}$ is thus given by

$$U_{RD}(x;\mathbf{P}) = v(x)\cdot G'\big(\sum_{j\le k}\hat{p}_j\big) + \sum_{k<i\le n} v(\hat{x}_i)\cdot\left[G'\bigg(\sum_{j\le i}\hat{p}_j\bigg) - G'\bigg(\sum_{j<i}\hat{p}_j\bigg)\right] \qquad x\in I_k \tag{70}$$

Formula (70) is seen to have the structure

$$U_{RD}(x;\mathbf{P}) = a_k\cdot v(x) + b_k \qquad x\in I_k \tag{71}$$

for the constants $a_k = G'(\sum_{j\le k}\hat{p}_j)$ and $b_k = \sum_{k<i\le n} v(\hat{x}_i)\cdot[G'(\sum_{j\le i}\hat{p}_j) - G'(\sum_{j<i}\hat{p}_j)]$ on each interval I_k. That is, the local utility function $U_{RD}(\cdot;\mathbf{P})$ of the rank-dependent form consists of different affine transformations of $v(\cdot)$ over each of the successive intervals $I_0, \ldots, I_n$. $U_{RD}(\cdot;\mathbf{P})$ seen to be continuous from one interval to the next, and smooth over the *interior* of each interval, with

$$U'_{RD}(x;\mathbf{P}) = v'(x)\cdot G'\bigg(\sum_{j\le k}\hat{p}_j\bigg) \qquad x\in \text{int}(I_k) \tag{72}$$

But it also follows that $U_{RD}(\cdot;\mathbf{P})$ generally has kinks at the *boundaries* of these intervals, that is, at each of $\mathbf{P}$'s payoff values $\hat{x}_1, \ldots, \hat{x}_n$, with distinct left/right directional derivatives at $\hat{x}_i$ given by

$$U'_{RD,L}(\hat{x}_i;\mathbf{P}) = v'(\hat{x}_i)\cdot G'\bigg(\sum_{\hat{x}_j<\hat{x}_i}\hat{p}_j\bigg) \quad U'_{RD,R}(\hat{x}_i;\mathbf{P}) = v'(\hat{x}_i)\cdot G'\bigg(\sum_{\hat{x}_j\le\hat{x}_i}\hat{p}_j\bigg) \tag{73}$$

Figure 4 illustrates the local utility function of a (risk averse) $V_{RD}(\cdot)$ at the three-outcome distribution $\mathbf{P} = (x_1, p_1; x_2, p_2; x_3, p_3)$, with kinks at each of $\mathbf{P}$'s (ordered) payoff levels $\hat{x}_1, \hat{x}_2, \hat{x}_3$.

The expected utility characterizations of first order stochastic dominance preference, risk aversion, and even some aspects of comparative risk aversion[44] do not require differentiability of the von Neumann-Morgenstern utility function $U(\cdot)$, and this is also true for the generalized expected utility characterizations of these properties in terms of the local utility functions $U(\cdot;\mathbf{P})$. For the rank-dependent form, Chew, Karni and Safra (1987) have shown that in spite of the kinks in its local utility functions, $V_{RD}(\cdot)$ also satisfies Section 1.3's generalized expected utility results linking monotonicity of the local utility functions to global first order stochastic dominance preference and concavity of the local utility functions to global risk aversion, as well as comparative concavity of two individual's local utility functions to aspects of comparative risk aversion. In other words, many of the basic results of generalized expected utility analysis continue to apply to the inherently kinked local utility functions of the rank-dependent form $V_{RD}(\cdot)$. Thus, at least in the above senses, expected utility, Fréchet differentiable, and rank-dependent *probability derivatives* all characterize features of risk preferences in a common manner.

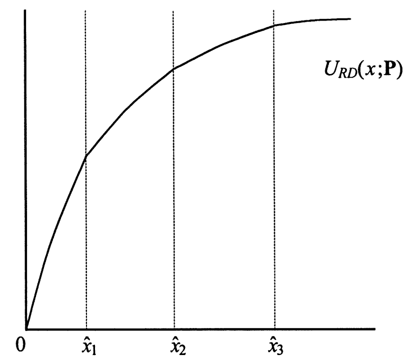

Figure 4. Kinked local utility function of $V_{RD}(\cdot)$ at lottery $\mathbf{P} = (x_1, p_1; x_2, p_2; x_3, p_3)$.

4.2. Local nonseparability of rank-dependent payoff kinks: illustration

Rank-dependent attitudes toward *payoff changes* can be summarized as follows: When $v(\cdot)$ and $G(\cdot)$ are both smooth (up to infinitely differentiable), then the rank-dependent form $V_{RD}(\cdot)$:

(a) is *smooth* with respect to *whole-probability* payoff changes
(b) is generally *locally nonseparably kinked* with respect to *partial-probability* payoff changes

and its payoff derivatives and directional payoff derivatives generally

(c) *are not* proportional to the amount of probability mass shifted from a given payoff level
(d) *do not* satisfy the total derivative formulas (8), (9) linking separate and joint shifts of probability mass from a given payoff level
(e) *do not* satisfy formulas (27), (28) linking the payoff derivatives and local utility function

These features can be illustrated by a simple example involving fixed probabilities and just two free outcome variables. Say a nonlinear $G(\cdot)$ satisfies

$$G\left(\tfrac{1}{2}\right) - G\left(\tfrac{1}{4}\right) \neq G\left(\tfrac{3}{4}\right) - G\left(\tfrac{1}{2}\right) \tag{74}$$

Consider lotteries of the form $\mathbf{P} = (\bar{x}_1, \tfrac{1}{4}; x_2, \tfrac{1}{4}; x_3, \tfrac{1}{4}; \bar{x}_4, \tfrac{1}{4})$, with fixed probabilities all equal to $\tfrac{1}{4}$, fixed lowest and highest payoffs $\bar{x}_1 < \bar{x}_4$, and where the payoffs x_2 and x_3 independently vary over the open interval $(\bar{x}_1, \bar{x}_4)$.[45] That is, we allow the values of the variables x_2 and x_3 to cross each other, but not to cross $\bar{x}_1$ or $\bar{x}_4$. By (67), the additive terms for $\bar{x}_1$ and $\bar{x}_4$ in the rank-dependent formula will remain fixed at $v(\bar{x}_1) \cdot G(\tfrac{1}{4})$ and $v(\bar{x}_4) \cdot [1 - G(\tfrac{3}{4})]$. On the other hand, the additive terms for x_2 and x_3 *do* depend—and in a qualitative manner—on the relative values of these two variables, and appear in the formula for $V_{RD}(\bar{x}_1, \frac{1}{4}; x_2, \frac{1}{4}; x_3, \frac{1}{4}; \bar{x}_4, \frac{1}{4})$ as

$$\begin{array}{ll} \ldots + v(x_2)\cdot[G(\tfrac{1}{2})-G(\tfrac{1}{4})]+v(x_3)\cdot[G(\tfrac{3}{4})-G(\tfrac{1}{2})]+\ldots & \text{for } x_2 \le x_3 \\ \ldots + v(x_2)\cdot[G(\tfrac{3}{4})-G(\tfrac{1}{2})]+v(x_3)\cdot[G(\tfrac{1}{2})-G(\tfrac{1}{4})]+\ldots & \text{for } x_2 \ge x_3 \end{array} \tag{75}$$

We can subsume both branches of formula (75) into the single formula

$$\begin{array}{l} \ldots + v(x_2)\cdot\big[G\big(\tfrac{1}{2}\big) - G\big(\tfrac{1}{4}\big)\big] + v(x_3)\cdot\big[G\big(\tfrac{1}{2}\big) - G\big(\tfrac{1}{4}\big)\big] \\ \quad + v\big(\max\{x_2, x_3\}\big)\cdot\big[G\big(\tfrac{3}{4}\big) - 2\cdot G\big(\tfrac{1}{2}\big) + G\big(\tfrac{1}{4}\big)\big] + \ldots \end{array} \qquad \text{for } x_2 \gtreqless x_3 \tag{76}$$

so the rank-dependent preference function (67) over such lotteries can be fully written out as

$$\begin{array}{l} V_{RD}(\bar{x}_1, \tfrac{1}{4}; x_2, \tfrac{1}{4}; x_3, \tfrac{1}{4}; \bar{x}_4, \tfrac{1}{4}) \\ \quad \equiv v(\bar{x}_1)\cdot G\big(\tfrac{1}{4}\big) + v(x_2)\cdot[G(\tfrac{1}{2}) - G(\tfrac{1}{4})] \\ \qquad + v(x_3)\cdot[G(\tfrac{1}{2}) - G(\tfrac{1}{4})] \\ \qquad + v(\max\{x_2, x_3\})\cdot[G(\tfrac{3}{4}) - 2\cdot G(\tfrac{1}{2}) + G(\tfrac{1}{4})] \\ \qquad + v(\bar{x}_4)\cdot[1 - G(\tfrac{3}{4})] \end{array} \qquad \begin{array}{l} \text{for} \\ x_2, x_3 \in (\bar{x}_1, \bar{x}_4), \\ x_2 \gtreqless x_3 \end{array} \tag{77}$$

Since (74) implies $\big[G\big(\frac{3}{4}\big) - 2\cdot G\big(\frac{1}{2}\big) + G\big(\frac{1}{4}\big)\big]$ is nonzero, the Leontief component $v(\max\{x_1, x_2\})$ in the above formulas implies that $V_{RD}\big(\bar{x}_1, \frac{1}{4}; x_2, \frac{1}{4}; x_3, \frac{1}{4}; \bar{x}_4, \frac{1}{4}\big)$ is kinked in x_2 and x_3 whenever $x_2 = x_3$. Formula (76) (and hence (77)) can also be written in terms of the minimum function

$$\begin{array}{l} \ldots + v(x_2)\cdot\big[G\big(\tfrac{3}{4}\big) - G\big(\tfrac{1}{2}\big)\big] + v(x_3)\cdot\big[G\big(\tfrac{3}{4}\big) - G\big(\tfrac{1}{2}\big)\big] \\ \quad + v\big(\min\{x_2, x_3\}\big)\cdot\big[-G\big(\tfrac{3}{4}\big) + 2\cdot G\big(\tfrac{1}{2}\big) - G\big(\tfrac{1}{4}\big)\big] + \ldots \end{array} \qquad \text{for } x_2 \gtreqless x_3 \tag{76$'$}$$

By illustrating the way in which the outcome variables x_2 and x_3 enter the rank-dependent formula, this example highlights the characteristic properties of rank-dependent payoff kinks, namely

- such kinks arise from the *nonlinearity* of $G(\cdot)$ (in this example, inequality (74)), even when $G(\cdot)$ and $v(\cdot)$ are both smooth (continuously or even infinitely differentiable)
- because of their Leontief-like structure, such kinks are *locally nonseparable*[46]

The formulas for $V_{RD}(\cdot)$'s x_2 and x_3 payoff derivatives can be derived from any of the formulas (75)–(77). At any such $\mathbf{P} = \big(\bar{x}_1, \frac{1}{4}; x_2, \frac{1}{4}; x_3, \frac{1}{4}; \bar{x}_4, \frac{1}{4}\big)$ where $x_2 \neq x_3$, $V_{RD}(\cdot)$ responds smoothly to marginal changes in either x_2 or x_3 alone, with smooth (left = right) payoff derivatives

$$\begin{aligned}
\frac{\partial V_{RD}\big(\ldots;x_2,\frac{1}{4};x_3,\frac{1}{4};\ldots\big)}{\partial x_2} &= \begin{cases} v'(x_2)\cdot\big[G\big(\frac{1}{2}\big)-G\big(\frac{1}{4}\big)\big] & \text{for } x_2<x_3 \\ v'(x_2)\cdot\big[G\big(\frac{3}{4}\big)-G\big(\frac{1}{2}\big)\big] & \text{for } x_2>x_3 \end{cases} \\
\frac{\partial V_{RD}\big(\ldots;x_2,\frac{1}{4};x_3,\frac{1}{4};\ldots\big)}{\partial x_3} &= \begin{cases} v'(x_3)\cdot\big[G\big(\frac{3}{4}\big)-G\big(\frac{1}{2}\big)\big] & \text{for } x_2<x_3 \\ v'(x_3)\cdot\big[G\big(\frac{1}{2}\big)-G\big(\frac{1}{4}\big)\big] & \text{for } x_2>x_3 \end{cases}
\end{aligned} \tag{78}$$

Provided $x_2 \neq x_3$, $V_{RD}(\cdot)$ also responds smoothly to *joint* marginal changes in x_2 and x_3, with individual and joint responses linked by the standard total derivative formula (10).

Consider now any distribution of the form $\mathbf{P} = \big(\bar{x}_1,\frac{1}{4};x,\frac{1}{4};x,\frac{1}{4};\bar{x}_4,\frac{1}{4}\big)$, where x_2 and x_3 initially take some *common* value x between $\bar{x}_1$ and $\bar{x}_4$. Provided x_2 and x_3 change *jointly and equally*, $V_{RD}(\cdot)$ still responds smoothly, with left = right derivative[47]

$$\frac{dV_{RD}\big(\ldots;x,\frac{1}{4};x,\frac{1}{4};\ldots\big)}{dx} = v'(x)\cdot\big[G\big(\tfrac{3}{4}\big)-G\big(\tfrac{1}{4}\big)\big] \tag{79}$$

But if x_2 and x_3 start out equal to x and then only *one* of them varies, $V_{RD}(\cdot)$ will have a *kinked* response to that variable, with left $\neq$ right directional payoff derivatives

$$\begin{aligned}
\left.\frac{\partial V_{RD}\big(\ldots;x_2,\frac{1}{4};x_3,\frac{1}{4};\ldots\big)}{\partial x_2^L}\right|_{x_2=x_3=x} &= \left.\frac{\partial V_{RD}\big(\ldots;x_2,\frac{1}{4};x_3,\frac{1}{4};\ldots\big)}{\partial x_3^L}\right|_{x_2=x_3=x} \\
&= v'(x)\cdot\big[G\big(\tfrac{1}{2}\big)-G\big(\tfrac{1}{4}\big)\big] \\
\left.\frac{\partial V_{RD}\big(\ldots;x_2,\frac{1}{4};x_3,\frac{1}{4};\ldots\big)}{\partial x_2^R}\right|_{x_2=x_3=x} &= \left.\frac{\partial V_{RD}\big(\ldots;x_2,\frac{1}{4};x_3,\frac{1}{4};\ldots\big)}{\partial x_3^R}\right|_{x_2=x_3=x} \\
&= v'(x)\cdot\big[G\big(\tfrac{3}{4}\big)-G\big(\tfrac{1}{2}\big)\big]
\end{aligned} \tag{80}$$

Thus, the *whole-probability* payoff changes corresponding to (78) and (79) yield smooth (left = right) payoff derivatives, whereas the *partial-probability* payoff changes corresponding to (80) yield kinked (left $\neq$ right) directional payoffs derivatives, illlustrating properties (a) and (b).

Comparison of (79) and (80) also illustrates properties (c) and (d): The nonlinearity of $G(\cdot)$ (inequality (74)) implies $2\cdot[G(\frac{1}{2})-G(\frac{1}{4})] \neq [G(\frac{3}{4})-G(\frac{1}{4})] \neq 2\cdot[G(\frac{3}{4})-G(\frac{1}{2})]$. Starting from any lottery $\mathbf{P} = (\bar{x}_1,\frac{1}{4};x,\frac{1}{4};x,\frac{1}{4};\bar{x}_4,\frac{1}{4})$ with $x \in (\bar{x}_1,\bar{x}_4)$, shifting amount $\frac{1}{4}$ of the total $\frac{1}{2}$ probability mass at x to the right is seen from the bottom line of (80) to have marginal effect $v'(x)\cdot[G(\frac{3}{4})-G(\frac{1}{2})]$, whereas shifting twice as much mass to the right, namely the entire amount $\frac{1}{2}$, is seen from (79) to have marginal effect $v'(x)\cdot[G(\frac{3}{4})-G(\frac{1}{4})]$ which is *not* twice the previous effect, illustrating property (c). Since this rightward shift of the $\frac{1}{2}$ mass at x can also be viewed as a *joint* rightward shift of *both* $\frac{1}{4}$ masses at x, this nonproportionality of marginal effects can also be viewed as a nonadditivity of marginal effects, illustrating property (d). Similar arguments hold for shifting probability mass leftward from x.

To illustrate property (e), namely that $V_{RD}(\cdot)$'s payoff derivatives and local utility function do not generally satisfy the relationship $\partial V(\mathbf{P})/\partial x_i = U'(x_i; \mathbf{P}) \cdot p_i$, recall from (73) (or Figure 4) that even for x_2 and x_3 unequal, say $x_2 < x_3$, $V_{RD}(\cdot)$'s local utility function at the lottery $\mathbf{P} = (\bar{x}_1, \frac{1}{4}; x_2, \frac{1}{4}; x_3, \frac{1}{4}; \bar{x}_4, \frac{1}{4})$, will have kinks at both x_2 and x_3 with left $\neq$ right directional derivatives

$$\begin{matrix} U'_{RD,L}(x_2; \mathbf{P}) = v'(x_2) \cdot G'(\frac{1}{4}) & U'_{RD,R}(x_2; \mathbf{P}) = v'(x_2) \cdot G'(\frac{1}{2}) \\ U'_{RD,L}(x_3; \mathbf{P}) = v'(x_3) \cdot G'(\frac{1}{2}) & U'_{RD,R}(x_3; \mathbf{P}) = v'(x_3) \cdot G'(\frac{3}{4}) \end{matrix} \quad x_2 < x_3 \tag{81}$$

none of which are linked to the payoff derivatives (78), (79) or (80) in the manners (27), (28) or their corresponding directional versions (61), (62).

The properties illustrated in this section are, except for (a), in contrast with those of an expected utility $V_{EU}(\cdot)$ with smooth $U(\cdot)$ or Fréchet differentiable $V_{FR}(\cdot)$ with smooth $U_{FR}(\cdot; \mathbf{P})$'s, whose whole- and partial-probability derivatives are all left = right smooth, proportional to the mass shifted from a given payoff level, additive in joint vs. individual shifts, and linked to their local utility functions via the relationships $\partial V_{EU}(\mathbf{P})/\partial x_i = U'(x_i) \cdot p_i$ and $\partial V_{FR}(\mathbf{P})/\partial x_i = U'_{FR}(x_i; \mathbf{P}) \cdot p_i$.

The Leontief-like structure of the rank-dependent preference function brought out in (76)/(76)$'$ is not readily apparent in its standard formulation (67). Rather, it "lives" in the (usually unexplicated) map from the actual payoff variables $x_1, \ldots, x_n$ to the ordered values $\hat{x}_1 \leq \ldots \leq \hat{x}_n$ that enter the right side of (67). For the case of two variables, we can represent this map as

$$\hat{x}_2(x_1, x_2) = \max\{x_1, x_2\} \qquad \hat{x}_1(x_1, x_2) = \sum_i x_i - \max\{x_1, x_2\} \tag{82}$$

For three variables, it can be written in the similar form[48]

$$\begin{aligned} \hat{x}_3(x_1, x_2, x_3) &= \max\{x_1, x_2, x_3\} \\ \hat{x}_2(x_1, x_2, x_3) &= \sum_{i<j}\sum \max\{x_i, x_j\} - 2 \cdot \max\{x_1, x_2, x_3\} \\ \hat{x}_1(x_1, x_2, x_3) &= \sum_i x_i - \sum_{i<j}\sum \max\{x_i, x_j\} + \max\{x_1, x_2, x_3\} \end{aligned} \tag{83}$$

and for four variables, as:

$$\begin{aligned} \hat{x}_4(x_1, x_2, x_3, x_4) &= \max\{x_1, x_2, x_3, x_4\} \\ \hat{x}_3(x_1, x_2, x_3, x_4) &= \sum_{i<j<k}\sum\sum \max\{x_i, x_j, x_k\} - 3 \cdot \max\{x_1, x_2, x_3, x_4\} \\ \hat{x}_2(x_1, x_2, x_3, x_4) &= \sum_{i<j}\sum \max\{x_i, x_j\} - 2 \cdot \sum_{i<j<k}\sum\sum \max\{x_i, x_j, x_k\} \\ &\quad + 3 \cdot \max\{x_1, x_2, x_3, x_4\} \end{aligned} \tag{84}$$

$$\hat{x}_1(x_1, x_2, x_3, x_4) = \sum_i x_i - \sum_{i<j}\sum \max\{x_i, x_j\} + \sum_{i<j<k}\sum\sum \max\{x_i, x_j, x_k\}$$
$$- \max\{x_1, x_2, x_3, x_4\}$$

For $n > 4$, the formulas for $\hat{x}_1, \ldots, \hat{x}_n$ remain expressible as linear combinations of the full sum $\sum_i x_i$ and the maxima of all the $2, 3, 4, \ldots, n$ element subsets of $\{x_1, \ldots, x_n\}$. For any n, there will be some pair of ordered payoffs values $\hat{x}_k(x_1, \ldots, x_n)$ and $\hat{x}_{k+1}(x_1, \ldots, x_n)$ that exhibit locally nonseparable kinks whenever any two payoff variables x_i, x_j merge to, or depart from, a common value.

4.3. Rank-dependent payoff derivatives and payoff kinks: general formulas and properties

Although the previous illustration captures the characteristic features of rank-dependent payoff kinks, it is just a specific example. This section presents the more general payoff derivative formulas and payoff kink structure of this form. We continue to assume that $\upsilon(\cdot)$ and $G(\cdot)$ are smooth (continuously or even infinitely differentiable) with $\upsilon'(\cdot)$, $G'(\cdot) > 0$.

Except when $G(\cdot)$ is linear (in which case it reduces to expected utility), the rank-dependent form $V_{RD}(\cdot)$ exhibits the five properties listed at the beginning of the previous section, which can be stated more formally as:

(A) If no other (positive probability) outcome x_j has the same value as x_i, then $V_{RD}(\cdot)$ is smooth in x_i with left = right regular payoff derivative $\partial V_{RD}(\ldots; x_i, p_i; \ldots)/\partial x_i$

(B) If some other x_j *does* have the same value as x_i, then $V_{RD}(\cdot)$ generally has distinct directional regular payoff derivatives $\partial V_{RD}(\ldots; x_i, p_i; \ldots)/\partial x_i^L \neq \partial V_{RD}(\ldots; x_i, p_i; \ldots)/\partial x_i^R$, and is locally nonseparably kinked in the variables x_i, x_j

(C) Partial-probability payoff derivatives $\partial V_{RD}(\ldots; x, \rho; x_i, p_i - \rho; \ldots)/\partial x^{L/R}|_{x=x_i}$ are also generally kinked, and not proportional to the amount of probability ρ shifted, even for fixed direction

(D) Whole-probability and partial-probability payoff derivatives are generally not linked by the total derivative relationships (8), (9),[49] or by their directional-derivative analogues

(E) Regular, partial- and whole-probability payoff derivatives are generally not linked to the local utility function in the manners (27), (28), or their directional versions (61), (62)

Although properties (B)–(E) each state that some derivative or smoothness property fails to hold *in general,* they do not formally address the *prevalence* of these failures in the space of lotteries. By way of prevalence, we will show:

(F) If $G(\cdot)$ is smooth but not identically linear over [0, 1], $V_{RD}(\cdot)$ will exhibit locally nonseparable payoff kinks about *every* lottery $\mathbf{P} \in \mathscr{L}$

The general payoff kink structure and payoff derivatives formulas for $V_{RD}(\cdot)$ are derived by an argument similar to Section 4.2: Consider an arbitrary $\mathbf{P} = (x_1, p_1; \ldots; x_n, p_n)$ and an arbitrary one of its outcomes x_i, which may or may not have the same value as any of **P**'s other outcomes. In either case, we can express the total probability mass at payoff level x_i by any of the expressions

$$\sum_{x_j = x_i} p_j = p_i + \sum_{\substack{x_j = x_i; \\ j \neq i}} p_j = \sum_{x_j \leq x_i} p_j - \sum_{x_j < x_i} p_j \tag{85}$$

which reduce to p_i when no other outcome equals x_i. This total mass at payoff level x_i can be represented as entering into the rank-dependent formula via the single additive term

$$\ldots + v(x_i) \cdot \left[G\left(\sum_{x_j \leq x_i} p_j \right) - G\left(\sum_{x_j < x_i} p_j \right) \right] + \ldots \tag{86}$$

Consider shifting amount ρ of the total mass at x_i to some higher or lower level $x_i + t$, where ρ may be less than p_i, equal to p_i, or even greater than p_i if some other outcome also has payoff x_i. If t is small enough so that $x_i + t$ does not cross any of **P**'s other distinct payoff levels $x_j \neq x_i$, the *remaining* mass at x_i, and the mass ρ now at $x_i + t$, now enter via the pair of terms

$$\begin{aligned}
&\ldots + v(x_i) \cdot \left[G\left(\sum_{x_j \leq x_i} p_j - \rho \right) - G\left(\sum_{x_j < x_i} p_j \right) \right] \\
&\qquad + v(x_i + t) \cdot \left[G\left(\sum_{x_j \leq x_i} p_j \right) - G\left(\sum_{x_j \leq x_i} p_j - \rho \right) \right] + \ldots \quad \text{for } t \geq 0 \\
&\ldots + v(x_i) \cdot \left[G\left(\sum_{x_j \leq x_i} p_j \right) - G\left(\sum_{x_j < x_i} p_j + \rho \right) \right] \\
&\qquad + v(x_i + t) \cdot \left[G\left(\sum_{x_j < x_i} p_j + \rho \right) - G\left(\sum_{x_j < x_i} p_j \right) \right] + \ldots \quad \text{for } t \leq 0
\end{aligned} \tag{87}$$

As before, this two-part formula can be represented by the single expression

$$\begin{aligned}
&\ldots + v(x_i) \cdot \left[G\left(\textstyle\sum_{x_j \leq x_i} p_j - \rho \right) - G\left(\textstyle\sum_{x_j < x_i} p_j \right) \right] \\
&\quad + v(x_i + t) \cdot \left[G\left(\textstyle\sum_{x_j < x_i} p_j + \rho \right) - G\left(\textstyle\sum_{x_j < x_i} p_j \right) \right] \\
&\quad + v(\max\{x_i, x_i + t\}) \cdot \left[G\left(\textstyle\sum_{x_j \leq x_i} p_j \right) - G\left(\textstyle\sum_{x_j \leq x_i} p_j - \rho \right) \right. \qquad \text{for } t \gtreqless 0 \\
&\qquad\qquad \left. - G\left(\sum_{x_j < x_i} p_j + \rho \right) + G\left(\sum_{x_j < x_i} p_j \right) \right] + \ldots
\end{aligned} \tag{88}$$

or in terms of the minimum function, as

$$\begin{aligned}\ldots &+ v(x_i)\cdot\Big[G\Big(\sum_{x_j\le x_i}p_j\Big)-G\Big(\sum_{x_j<x_i}p_j+\rho\Big)\Big]\\ &+ v(x_i+t)\cdot\Big[G\Big(\sum_{x_j\le x_i}p_j\Big)-G\Big(\sum_{x_j\le x_i}p_j-\rho\Big)\Big]\\ &+ v(\min\{x_i,x_i+t\})\cdot\Big[-G\Big(\sum_{x_j\le x_i}p_j\Big)+G\Big(\sum_{x_j\le x_i}p_j-\rho\Big)\\ &\qquad +G\Big(\sum_{x_j<x_i}p_j+\rho\Big)-G\Big(\sum_{x_j<x_i}p_j\Big)\Big]+\ldots\end{aligned}\qquad \text{for } t \gtreqless 0 \quad (88)'$$

Formulas (88)/(88)′ show that the type of locally nonseparable payoff kink seen in Section 4.2 arises in general. As long as t is small enough so that $x_i + t$ does not cross any distinct payoff level $x_j \neq x_i$, none of the six square-bracketed weights in (88)/(88)′ depend on either the sign or magnitude of t. The weights on the Leontief terms $v(\max\{x_i, x_i + t\})$ and $v(\min\{x_i, x_i + t\})$ are both second differences of the form $\pm[G(\alpha) - G(\alpha - \rho) - G(\beta + \rho) + G(\beta)]$. For nonlinear $G(\cdot)$ such expressions are generally nonzero, yielding locally nonseparable payoff kinks as in (76)/(76)′.

Since it includes each type of payoff shift as a special case, (87) allows for the derivation of the regular, whole-probability and partial-probability payoff derivatives for the rank-dependent form. $V_{RD}(\cdot)$'s *regular payoff derivatives*, for which ρ in (87) equals p_i, take the directional forms

$$\begin{aligned}\frac{\partial V_{RD}(\ldots;x_i,p_i;\ldots)}{\partial x_i^L} &\equiv v'(x_i)\cdot\Big[G\Big(\sum_{x_j<x_i}p_j+p_i\Big)-G\Big(\sum_{x_j<x_i}p_j\Big)\Big]\\ \frac{\partial V_{RD}(\ldots;x_i,p_i;\ldots)}{\partial x_i^R} &\equiv v'(x_i)\cdot\Big[G\Big(\sum_{x_j\le x_i}p_j\Big)-G\Big(\sum_{x_j\le x_i}p_j-p_i\Big)\Big]\end{aligned}\qquad (89)$$

and when no other x_j equals x_i, they reduce to the smooth (left = right) form

$$\frac{\partial V_{RD}(\ldots;x_i,p_i;\ldots)}{\partial x_i} \equiv v'(x_i)\cdot\Big[G\Big(\sum_{x_j\le x_i}p_j\Big)-G\Big(\sum_{x_j<x_i}p_j\Big)\Big]\qquad (90)$$

$V_{RD}(\cdot)$'s *whole-probability* payoff derivative corresponds to ρ equaling the total probability (85), and takes the smooth form

$$\frac{dV_{RD}(\ldots;x_i,p_i;\ldots)}{dx^W} \equiv v'(x)\cdot\Big[G\Big(\sum_{x_j\le x}p_j\Big)-G\Big(\sum_{x_j<x}p_j\Big)\Big]\qquad (91)$$

Finally, $V_{RD}(\cdot)$'s *partial-probability* payoff derivatives, for which ρ is any amount strictly less than the total probability (85), take the left $\neq$ right directional forms

$$\begin{aligned}\left.\frac{\partial V_{RD}(\ldots;x,\rho;x_i,p_i-\rho;\ldots)}{\partial x^L}\right|_{x=x_i} &\equiv v'(x_i)\cdot\left[G\left(\sum_{x_j<x_i}p_j+\rho\right)-G\left(\sum_{x_j<x_i}p_j\right)\right]\\ \left.\frac{\partial V_{RD}(\ldots;x,\rho;x_i,p_i-\rho;\ldots)}{\partial x^R}\right|_{x=x_i} &\equiv v'(x_i)\cdot\left[G\left(\sum_{x_j\le x_i}p_j\right)-G\left(\sum_{x_j\le x_i}p_j-\rho\right)\right]\end{aligned} \tag{92}$$

The payoff derivatives (89)–(92) are seen to generically exhibit properties (A)–(E) above.

To demonstrate property (F) (locally non-separable payoff kinks about every lottery), pick arbitrary $\mathbf{P}\in\mathscr{L}$ and invoke (2) to express it as $\mathbf{P}=(\hat{x}_1,\hat{p}_1;\ldots;\hat{x}_m,\hat{p}_m)$, with $\hat{p}_i>0$ for each i and $\hat{x}_1<\ldots<\hat{x}_m$. Since $G(\cdot)$ is smooth but not linear over $[0,\sum_{j\le m}\hat{p}_j]=[0,1]$, it must be nonlinear on at least one of the m successive nondegenerate cumulative probability intervals[50]

$$[0,\hat{p}_1],[\hat{p}_1,\hat{p}_1+\hat{p}_2]\ldots\left[\sum_{j<i}\hat{p}_j,\sum_{j\le i}\hat{p}_j\right]\ldots\left[\sum_{j<m-1}\hat{p}_j,\sum_{j\le m-1}\hat{p}_j\right],\left[\sum_{j<m}\hat{p}_j,\sum_{j\le m}\hat{p}_j\right] \tag{93}$$

say the interval $[\sum_{j<i^*}\hat{p}_j,\sum_{j\le i^*}\hat{p}_j]$. Thus, there exist some $\rho_a,\rho_b>0$ with $\rho_a+\rho_b\le\hat{p}_{i^*}$ such that

$$G\left(\sum_{j\le i^*}\hat{p}_j\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_b\right)+G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a-\rho_b\right)\neq 0 \tag{94}$$

Express $\mathbf{P}$ as $(\ldots;\hat{x}_{i^*},\hat{p}_{i^*}-\rho_a-\rho_b;y_a,\rho_a;y_b,\rho_b;\ldots)|_{y_a=y_b=\hat{x}_{i^*}}$, and consider lotteries of the form $(\ldots;\hat{x}_{i^*},\hat{p}_{i^*}-\rho_a-\rho_b;y_a,\rho_a;y_b,\rho_b;\ldots)$, where y_a and y_b independently vary upward from $\hat{x}_{i^*}$ over the interval $[\hat{x}_{i^*},\hat{x}_{i^*+1})$ (or $[\hat{x}_{i^*},M)$ if $i^*=m$).[51] $V_{RD}(\cdot)$'s term for the pair $(\hat{x}_{i^*},\hat{p}_{i^*}-\rho_a-\rho_b)$ stays fixed at $v(\hat{x}_{i^*})\cdot[G(\sum_{j\le i^*}\hat{p}_j-\rho_a-\rho_b)-G(\sum_{j<i^*}\hat{p}_j)]$, its terms for all other $(\hat{x}_i,\hat{p}_i)$ pairs stay fixed as in (67), and its terms for (y_a,ρ_a), (y_b,ρ_b) take the form

$$\begin{aligned}&\ldots+v(y_a)\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j-\rho_b\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a-\rho_b\right)\right]\\ &\quad+v(y_b)\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_b\right)\right]+\ldots\qquad\text{for } y_a\le y_b\\ &\ldots+v(y_a)\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a\right)\right]\\ &\quad+v(y_b)\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a-\rho_b\right)\right]+\ldots\qquad\text{for } y_a\ge y_b\end{aligned} \tag{95}$$

As before, this two-part formula can be represented by a single expression for $y_a \gtreqless y_b$, namely

$$\begin{aligned} \ldots &+ v(y_a)\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j-\rho_b\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a-\rho_b\right)\right] \\ &+ v(y_b)\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a-\rho_b\right)\right] \\ &+ v(\max\{y_a,y_b\})\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a\right)\right. \\ &\qquad\left.-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_b\right)+G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a-\rho_b\right)\right]+\ldots \end{aligned} \tag{96}$$

or in terms of the minimum function, by

$$\begin{aligned} \ldots &+ v(y_a)\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a\right)\right] \\ &+ v(y_b)\cdot\left[G\left(\sum_{j\le i^*}\hat{p}_j\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_b\right)\right] \\ &+ v(\min\{y_a,y_b\})\cdot\left[-G\left(\sum_{j\le i^*}\hat{p}_j\right)+G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a\right)\right. \\ &\qquad\left.+G\left(\sum_{j\le i^*}\hat{p}_j-\rho_b\right)-G\left(\sum_{j\le i^*}\hat{p}_j-\rho_a-\rho_b\right)\right]+\ldots \end{aligned} \tag{96$'$}$$

Since (94) implies that the Leontief terms in (96)/(96)′ are nonzero, $V_{RD}(\cdot)$ is locally nonseparably kinked in the payoff variables y_a and y_b as they vary upward from their common initial value of $\hat{x}_{i^*}$ at the lottery $\mathbf{P}=(\ldots;\hat{x}_{i^*},\hat{p}_{i^*}-\rho_a-\rho_b;y_a,\rho_a;y_b,\rho_b;\ldots)|_{y_a=y_b=\hat{x}_{i^*}}$, yielding property (**F**).[52]

5. Concluding topics

We conclude with remarks on:

- the qualitatively distinct ways in which Fréchet differentiable and rank-dependent preferences model the non-expected utility property of nonseparability in the payoffs
- informal field evidence regarding payoff kinks, in particular from insurance demand
- payoff and probability kinks in induced preferences over delayed-resolution lotteries

5.1. *Modeling departures from separability: two approaches*

A primary motivation for the study of non-expected utility models is the large body of evidence that individuals' lottery preferences depart from the expected utility property of payoff separability—that is, from the property that for given $(\bar{p}_1, \ldots, \bar{p}_n)$, preferences over the lotteries $\mathbf{P} = (x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n)$ are globally separable in the payoff variables $(x_1, \ldots, x_n)$. Payoff separability follows directly from the foundational Independence Axiom of expected utility theory.

The most widely known violation of payoff separability is the Allais (1953) Paradox, which consists of the following frequently observed pair of rankings (where \$1M = \$1,000,000)

$$\begin{aligned} a_1&: (\$1\text{M}, 1) \succ a_2: \left(\$5\text{M}, \frac{10}{100}; \$1\text{M}, \frac{89}{100}; \$0, \frac{1}{100}\right) \\ a_3&: \left(\$5\text{M}, \frac{10}{100}; \$0, \frac{90}{100}\right) \succ a_4: \left(\$1\text{M}, \frac{11}{100}; \$0, \frac{89}{100}\right) \end{aligned} \tag{97}$$

Under the identifications (2), these four lotteries can be represented in following form, for fixed probabilities $(\bar{p}_1, \bar{p}_2, \bar{p}_3) = (.10, .01, .89)$ and payoff vectors (x_1, x_2, x_3) given by

$$\begin{aligned} a_1 &= (\$1\text{M}, \$1\text{M}, \$1\text{M}) \qquad & a_2 &= (\$5\text{M}, \$0, \$1\text{M}) \\ a_3 &= (\$5\text{M}, \$0, \$0) \qquad & a_4 &= (\$1\text{M}, \$1\text{M}, \$0) \end{aligned} \tag{98}$$

Allais-type preferences are thus nonseparable in payoff variables (x_1, x_2, x_3), since they imply $(\$1\text{M}, \$1\text{M}, x_3) \succ (\$5\text{M}, \$1\text{M}, x_3)$ when $x_3 = \$1\text{M}$, but the reverse ordering when $x_3 = \$0$. That is, starting at $(x_1, x_2, x_3) = (\$1\text{M}, \$1\text{M}, x_3)$ individuals' willingness to bear the *mean-increasing* risk implied by the payoff changes $(\Delta x_1, \Delta x_2) = (+\$4\text{M}, -\$1\text{M})$ seems to be inversely related to level of the mutually exclusive variable x_3. This form of departure from payoff-separability has been observed for more general lotteries, and has been termed the *common consequence effect*.[53]

There are two qualitatively distinct approaches to modeling departures from global properties such as linearity or separability. Figures 5a and 5b provide an illustration of these two approaches, as applied to the more basic task of modeling departures from global linearity in preferences over nonstochastic $(x_1, x_2) =$ (apple, banana) commodity bundles. Say some "classical linear theory" hypothesizes a preference function of the form $W_{LIN}(x_1, x_2) \equiv k_1 \cdot x_1 + k_2 \cdot x_2$ for fixed coefficients k_1, k_2, and hence predicts a constant marginal rate of substitution over the commodity space R^2_+. But say the evidence suggests that individuals' *MRS*'s vary systematically, and tend to be flatter toward in southeast (many apples, few bananas) and steeper toward the northwest (few apples, many bananas). *Someplace*, therefore, preferences must be nonlinear.

The two approaches to modeling such linearity can be exemplified by the functions

$$W_{SM}(x_1, x_2) \equiv x_1^{\alpha} \cdot x_2^{\beta} \quad \text{vs.} \quad W_{RL}(x_1, x_2) \equiv \begin{cases} k_1 \cdot x_1 + k_2 \cdot x_2 & \text{for } x_1 \leq x_2 \\ k_1^* \cdot x_1 + k_2^* \cdot x_2 & \text{for } x_1 \geq x_2 \end{cases} \tag{99}$$

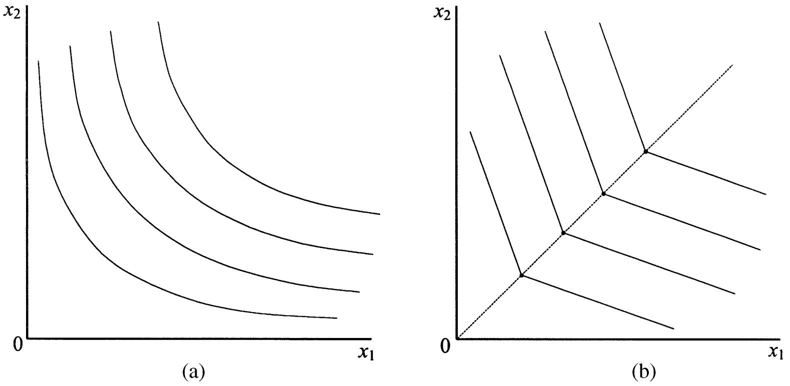

Figure 5. (a and b) Modelling departures from linearity: smooth nonlinearity vs. regionwise-linear preferences [b. Indifference curves for the rank-dependent formula $\hat{x}_1 \cdot G(\hat{p}_1) + \hat{x}_2 \cdot [1 - G(\hat{p}_1)]$].

where we assume $k_1 + k_2 = k_1^* + k_2^*$ to ensure continuity of $W_{RL}(\cdot)$ along the 45° line. While both functions can capture the phenomenon of "flatness in the southeast/steepness in the northwest," they do so in very different ways. Although the smooth function $W_{SM}(\cdot,\cdot)$ is not *exactly linear* over any region, it is everywhere *locally linear*, and its indifference curves can gradually move from flatness to steepness across the domain. On the other hand, the regionwise-linear function $W_{RL}(\cdot,\cdot)$ retains the classical property of exact linearity over two large regions of the domain, and "concentrates" all its nonlinearity along the boundary of these regions, namely the 45° line. It is clear from both Figure 5b and formula (99) that, by concentrating its nonlinearity on this lower-dimensional boundary set, $W_{RL}(\cdot,\cdot)$ must be *locally nonlinear* (i.e., kinked) there.

Figures 6a and 6b illustrate the same pair of approaches, this time applied to departures from global *separability*. In this case, say the "classic theory" hypothesizes the separable form $W_{SEP}(x_1, x_2) \equiv f(x_1) + g(x_2)$ for general $f(\cdot)$ and $g(\cdot)$. But say its predictions of

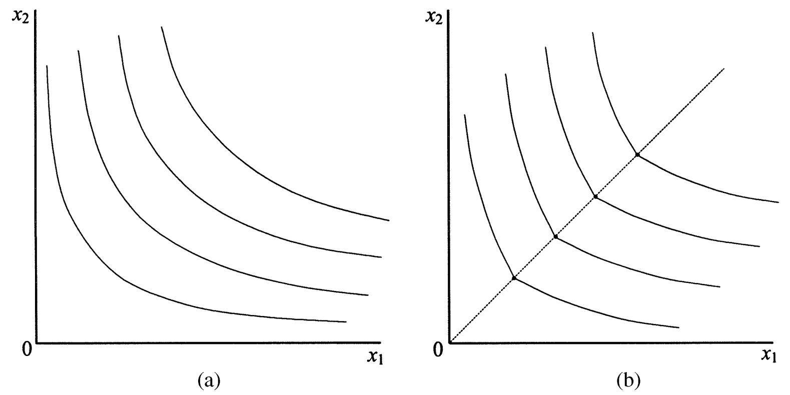

Figure 6. (a and b) Modelling departures from separability: smoothly nonseparable vs. regionwise separable preferences [b. Indifference curves for the rank-dependent function $V_{RD}(x_1, \bar{p}_1; x_2, \bar{p}_2)$].

how $MRS(x_1,x_2)$ varies over the domain[54] were in some way systematically violated by the evidence. As before, we could model such nonseparability in two different ways, exemplified by

$$W_{SM}(x_1,x_2) \equiv x_1^{\alpha} + x_2^{\beta} + \gamma \cdot x_1 \cdot x_2 \qquad \text{vs.}$$
$$W_{RS}(x_1,x_2) \equiv \begin{cases} f(x_1)+g(x_2) & \text{for } x_1 \le x_2 \\ f^*(x_1)+g^*(x_2) & \text{for } x_1 \ge x_2 \end{cases} \tag{100}$$

for smooth $f(\cdot), g(\cdot), f^*(\cdot), g^*(\cdot)$, where we assume $f(x)+g(x) \equiv_x f^*(x)+g^*(x)$ to ensure continuity of $W_{RS}(\cdot)$ along the 45° line. As before, $W_{SM}(\cdot,\cdot)$ and its indifference curves are not *exactly separable* over any open set, but are everywhere *locally separable*, and for different choice of parameters (or different smooth functional form) could exhibit the observed form of departure from separability smoothly and gradually over the domain. On the other hand, the regionwise-separable function $W_{RS}(\cdot,\cdot)$ retains exact separability on both regions of its domain, and concentrates all its nonseparability along their boundary. But as seen in Figure 6b, by concentrating its nonseparability on this lower dimensional set, $W_{RS}(\cdot,\cdot)$ must be *locally nonseparable* there.[55]

The distinction between how rank-dependent and Fréchet differentiable functions model payoff-nonseparability is analogous to the examples of Figures 5a/5b and 6a/6b. For any pair of probabilities $(\bar{p}_1,\bar{p}_2)$, expected utility preferences over the lotteries $\mathbf{P} = (x_1,\bar{p}_1;x_2,\bar{p}_2)$ take the globally payoff-separable form $V_{EU}(x_1,\bar{p}_1;x_2,\bar{p}_2) = U(x_1)\cdot\bar{p}_1 + U(x_2)\cdot\bar{p}_2$. Fréchet differentiable non-expected utility functions $V_{FR}(\cdot)$, such as the smooth cases of the functions in (65), are typically not *exactly* payoff-separable over any region, but have been seen to be everywhere *locally* payoff separable. On the other hand, rank-dependent preferences over such lotteries have the following *regionwise* outcome-separable structure, seen to be a special case of the form $W_{RS}(\cdot,\cdot)$ from (100):

$$V_{RD}(x_1,\bar{p}_1;x_2,\bar{p}_2) \equiv \begin{cases} \upsilon(x_1)\cdot G(\bar{p}_1)+\upsilon(x_2)\cdot[1-G(\bar{p}_1)] & \text{for } x_1 \le x_2 \\ \upsilon(x_1)\cdot[1-G(\bar{p}_2)]+\upsilon(x_2)\cdot G(\bar{p}_2) & \text{for } x_1 \ge x_2 \end{cases} \tag{101}$$

Since $V_{RD}(x_1,\bar{p}_1;x_2,\bar{p}_2)$ is a special case of the form $W_{RS}(x_1,x_2)$, Figure 6b also serves to illustrate its indifference curves, which are seen to be exactly payoff-separable on the region below the 45° line in the figure, as well as on the region above it, and concentrate all their nonseparability on the 45° line, where they are locally nonseparable.[56]

Figure 7 illustrates the x_1,x_2 indifference curves of the *three-outcome* rank-dependent formula $V_{RD}(x_1,\frac{1}{3};x_2,\frac{1}{3};\bar{x}_3,\frac{1}{3})$, for fixed probabilities of $\frac{1}{3}$ and payoff x_3 fixed at $\bar{x}_3$. The regions I, II,. . . , VI correspond to the six strict payoff orderings $x_1<x_2<\bar{x}_3, x_2<x_1<\bar{x}_3, \ldots, \bar{x}_3<x_2<x_1$, and hence the six distinct regions over which rank-dependent preferences will be exactly separable in the payoffs. The three lines of kink points in the diagram again correspond to the boundary points of these regions, along which preferences are generally locally nonseparable. Similarly, the x_1,x_2 indifference curves of the *four-outcome* formula $V_{RD}(x_1,\frac{1}{4};x_2,\frac{1}{4};\bar{x}_3,\frac{1}{4};\bar{x}_4,\frac{1}{4})$ for fixed $\bar{x}_3<\bar{x}_4$ would have

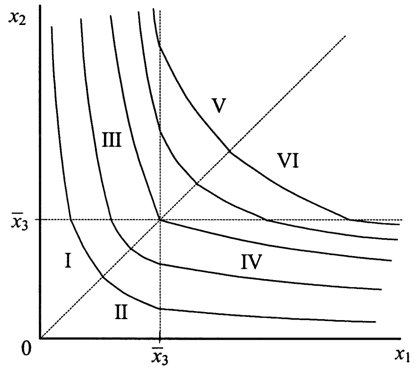

Figure 7. Six regions of exact payoff-separability for $V_{RD}(x_1, \frac{1}{3}; x_2, \frac{1}{3}; \bar{x}_3, \frac{1}{3})$.

kinks along the 45° line, and along the two horizontal and two vertical lines where x_1 or x_2 equals $\bar{x}_3$ and/or $\bar{x}_4$, and thus 12 distinct regions of exact payoff separability, etc.

Since Figure 7's indifference curves are smooth over the *interior* of each region, they might seem to contradict property (F) (that $V_{RD}(\cdot)$ has locally non-separable kinks at every lottery). Instead, they illustrate property (A): Since any small change from an interior lottery $\mathbf{P} = (x_1, \frac{1}{3}; x_2, \frac{1}{3}; \bar{x}_3, \frac{1}{3})$ *that stays within the plane of the figure* represents a *whole-probability* payoff change in x_1 and/or $x_2, V_{RD}(\cdot)$ indeed *does* respond smoothly, with smooth left = right derivatives (90), which (as mentioned in Note 49) satisfy the local additivity property (10). But as seen in Section 4.3, $V_{RD}(\cdot)$ will be locally non-separably kinked in the payoff variables y_a, y_b as they depart from this same lottery $\mathbf{P} = (x_1, \frac{1}{3}; x_2, \frac{1}{3}; \bar{x}_3, \frac{1}{3}) = (\ldots; x_i, \frac{1}{3} - \rho_a - \rho_b; y_a, \rho_a; y_b, \rho_b; \ldots)|_{y_a = y_b = x_i}$, for some choices of ρ_a, ρ_b and choices of $x_1 = x_1, x_2$ or $\bar{x}_3$. (Similar remarks apply to Figure 6b.)

Finally, observe that the properties illustrated in Figures 6b and 7 extend to rank-dependent preferences over general n-outcome lotteries. That is, for any probabilities $(\bar{p}_1, \ldots, \bar{p}_n)$, rank-dependent preferences over the lotteries $(x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n)$, are *regionwise separable* in the payoffs, but generally locally nonseparably kinked on regional boundaries. For each permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ of the integers $\{1, \ldots, n\}$, define the payoff region $\mathscr{X}_\sigma = \{(x_1, \ldots, x_n) \in [0, M]^n \mid x_{\sigma_1} < \ldots < x_{\sigma_n}\}$. Since the ordering $(\hat{x}_1, \ldots, \hat{x}_n) = (x_{\sigma_1}, \ldots, x_{\sigma_n})$ of the variables $(x_1, \ldots, x_n)$ remains fixed over any such region, their respective probability values $(\hat{p}_1, \ldots, \hat{p}_n) = (\bar{p}_{\sigma_1}, \ldots, \bar{p}_{\sigma_n})$ are also fixed, so $V_{RD}(x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n)$ takes the following payoff-separable form over the region $\mathscr{X}_\sigma$:

$$
\begin{aligned}
V_{RD}&(x_1, \bar{p}_1; \ldots; x_n, \bar{p}_n) \\
&\equiv v(x_{\sigma_1}) \cdot G(\bar{p}_{\sigma_1}) \\
&+ v(x_{\sigma_2}) \cdot [G(\bar{p}_{\sigma_1} + \bar{p}_{\sigma_2}) - G(\bar{p}_{\sigma_1})] \\
&\quad \vdots \\
&+ v(x_{\sigma_i}) \cdot [G(\bar{p}_{\sigma_1} + \ldots + \bar{p}_{\sigma_i}) - G(\bar{p}_{\sigma_1} + \ldots + \bar{p}_{\sigma_{i-1}})] \qquad (x_1, \ldots, x_n) \in \mathscr{X}_\sigma \\
&\quad \vdots
\end{aligned}
$$

$$+v(x_{\sigma_n})\cdot[G(1)-G(\bar{p}_{\sigma_1}+\ldots+\bar{p}_{\sigma_{n-1}})]$$
$$=\sum_{i=1}^{n}v(x_{\sigma_i})\cdot\left[G\left(\sum_{j\le i}\bar{p}_{\sigma_j}\right)-G\left(\sum_{j<i}\bar{p}_{\sigma_j}\right)\right] \tag{102}$$

Thus, as in the two- and three -outcome examples of the figures, $V_{RD}(x_1,\bar{p}_1;\ldots;x_n,\bar{p}_n)$ is exactly payoff-separable over each of the $n!$ distinct payoff regions $\mathscr{X}_\sigma$, and concentrates its nonseparability on the boundaries of these regions—that is, at lotteries $\mathbf{P}=(x_1,\bar{p}_1;\ldots;x_n,\bar{p}_n)$ where some payoff x_i equals one or more of the other payoffs x_j—at which points $V_{RD}(\cdot)$ is generally locally nonseparable in these variables. Axiomatically, this feature of the rank-dependent model derives from its foundational *Comonotonic Independence Axiom*,[57] which imposes exact separability in the variables $(x_1,\ldots,x_n)$ *within* each region $\mathscr{X}_\sigma$, though not *across* these regions. For any interior lottery $\mathbf{P}=(x_1,\bar{p}_1;\ldots;x_n,\bar{p}_n)$, the remarks of the previous paragraph still hold, namely that $V_{RD}(\cdot)$ will be smooth for small changes in the variables $(x_1,\ldots,x_n)$, but will be locally nonseparably kinked in the payoff variables y_a,y_b as they depart from $\mathbf{P}=(x_1,\bar{p}_1;\ldots;x_n,\bar{p}_n)=(\ldots;x_i,p_i-\rho_a-\rho_b;y_a,\rho_a;y_b,\rho_b;\ldots)|_{y_a=y_b=x_i}$, for some choices of ρ_a,ρ_b and choice of $i\in\{1,\ldots,n\}$.

5.2. *Are observed risk preferences kinked in the payoffs?*

Payoff kinds were not among the empirical phenomena reported or modeled in the classic expected utility analysis of Friedman and Savage (1948), or its modification by Markowitz (1952), who defined the function $U(\cdot)$ over changes from current wealth and also observed that individuals are generally averse to small symmetric gambles. In Kahneman and Tversky's (1979) and Tversky and Kahneman's (1986) analysis of the non-expected utility form $\sum_{i=1}^{n}v(x_i)\cdot\pi(p_i)$, the function $v(\cdot)$ was asserted to be: (i) defined over changes from current wealth; (ii) generally concave over gains and convex over losses; and (iii) steeper for losses than for gains, which is defined in (1979, p. 279) as the property $v(-y)-v(-x)>v(x)-v(y)$ for $x>y\ge 0$. Although these conditions are consistent with a kink at $x=0$, they do not imply one,[58] and whereas the graph of $v(\cdot)$ in (1979, Fig. 3) has such a kink, the graph of $v(\cdot)$ in (1986, Fig. 1) appears smooth at $x=0$.

It is impossible to infer the existence of a payoff kink from any finite set of pairwise rankings or pairwise choices over lotteries. However in some cases kinks can be inferred from the *infinite* number of pairwise choices implicit in the selection of an optimal lottery from one or more budget lines. As seen in Section 4, a natural setting of uncertain payoff choices along budget lines are insurance decisions. We briefly examine the implications of the following three apparently general phenomena, involving the demand for and/or nature of insurance contracts:

- individuals frequently purchase *complete* coverage of certain forms of insurance offered at actuarially unfair prices, and do so at general wealth levels[59]

- individuals frequently purchase *zero* coverage of certain forms of insurance offered at better-than-fair (e.g., government subsidized) prizes, again at general wealth levels
- as noted above, many insurance policies provide no indemnity payment under certain types of events, such as acts of war, insurer bankruptcy, etc.

Concerning the first of these phenomena, we saw in Figure 3a how a von Neumann-Morgenstern utility function $U(\cdot)$ with a kink at a given wealth level (say x^*) can lead to first order risk aversion about x^* and hence the possibility of complete purchase of actuarially unfair insurance. However, it was also seen that this phenomenon occurs only in the knife-edge case when the budget line from the original uninsured position C hits the 45° line exactly at the point (x^*, x^*)—any steeper or flatter unfair budget line out of C will lead to a partial-insurance optimum. Though one can posit many kink points in $U(\cdot)$, it is fair to say that payoff-kinked expected utility cannot be used to model the phenomena of full purchase of unfair insurance at *general* wealth levels. Indeed, as noted in the discussion of Figure 3b, a kinked utility function $U(\cdot)$ implies that the Engel curve for insurance can (in whole or part) take an unusual $\wedge$-shaped form.

However, from Figures 5b and 6b it is clear that individuals with rank-dependent preferences *can* exhibit first order risk aversion, and hence full purchase of actuarially unfair insurance, at general wealth levels. This feature of rank-dependent preferences, noted by Karni (1992, 1995) and others, constitutes an argument for modeling risk preferences via the rank-dependent form.

The second phenomenon, zero purchase of actuarially more-than-fair insurance, does not provide any discriminatory power between the expected utility, Fréchet differentiable, or rank-dependent models. None of the models can generate *risk averse* preferences that would exhibit this behavior, and all three can generate *risk-loving* preferences that exhibit it, via indifference curves that lie subtangent to the fair odds lines along the 45° line in the Hirshleifer-Yaari diagram, leading to zero purchase of actuarially subsidized insurance for low enough subsidies.

The third phenomenon (uninsured events) is probably more a feature of insurance *supply* (nondiversifiable risks) than of risk preferences or insurance *demand*. Nevertheless, its existence does bring out another implication of the rank-dependent model. Figure 7 can be used to illustrate the implications of uninsured states of nature on rank-dependent insurance demand. Say state 3 is uninsured, has probability $\frac{1}{3}$ and yields payoff $\bar{x}_3$. Then, by an argument similar to that of Figure 3b, the Engel curves for insurance arising from the rank-dependent preferences in Figure 7 can take a similar $\wedge$-shaped form. An important difference between the two cases is that while the expected utility kinks in Figure 3b derive from an exogenously hypothesized kink in $U(\cdot)$, the rank-dependent kinks in Figure 7 arise generically whenever there is any positive-probability uninsured event, even when $\upsilon(\cdot)$ and the nonlinear function $G(\cdot)$ are arbitrarily smooth. As noted above, if there is *more than one* uninsured state—i.e., if the uninsured event itself involves any uncertainty—there will be multiple vertical and horizontal dashed lines in the figure, and even though $\upsilon(\cdot)$ and $G(\cdot)$ may be smooth, rank-dependent Engel curves for insurance could actually "zigzag" (be $\wedge\!\!\wedge\!\!\vee$-shaped). This potential implication of rank-dependent preferences might provide an argument against their use in insurance

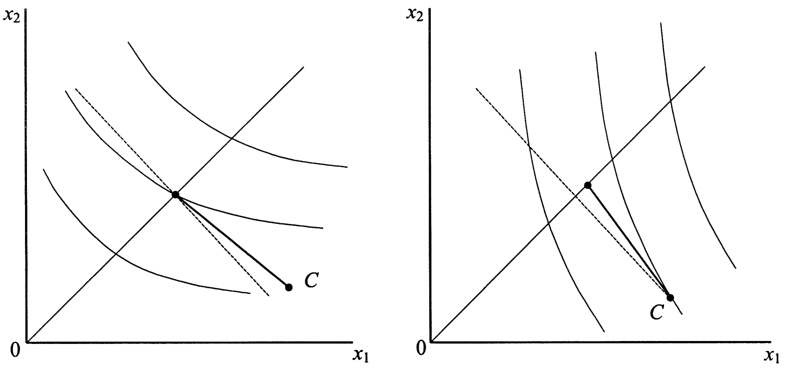

Figure 8. (a and b) Full purchase of actuarially unfair insurance due to pessimistic beliefs; zero purchase of actuarially better-than-fair insurance due to optimistic beliefs.

analysis and similar settings of choice over state-contingent payoff levels $(x_1, \ldots, x_n)$, such as state-contingent asset or commodity markets.

On the other hand, since almost all insurance decisions involve subjective rather than objective uncertainty, it is not clear that the above three phenomena are necessarily reflecting features of individuals' *risk preferences*. An alternative and perhaps more parsimonious way to reconcile the three phenomena—both with each other and also with the hypothesis of risk aversion—might be to attribute them to heterogeneity of individuals' *subjective beliefs*. Consider some event such as a devastating local earthquake. Figure 8a and 8b illustrate the insurance decisions of two risk averters who have the same risk preferences—that is, the same $V(x_1, p_1; x_2, p_2)$ function—but have distinct *subjective probabilities* $(\bar{p}_1, \bar{p}_2)$ and $(\bar{\bar{p}}_1, \bar{\bar{p}}_2)$, with $\bar{p}_1/\bar{p}_2 < \bar{\bar{p}}_1/\bar{\bar{p}}_2$. Since state 2 is the loss state, the left individual's beliefs are relatively *pessimistic* and the right individual's beliefs are relatively *optimistic*.

These figures illustrate how disparate beliefs can lead some individuals to buy full insurance at actuarially unfair rates, and others to buy no insurance at actuarially more-than-fair rates. The "actuarially neutral" odds for the states 1 and 2 (however arrived at) are represented by the common dashed line in each figure, and the distinct insurance rates face by the individuals are represented by their different budget lines. Figure 8a shows how pessimistic beliefs can lead to full purchase of insurance that is actuarially unfair. Figure 8b shows how optimistic beliefs can lead a risk averter to purchase zero insurance, even when it is better than actuarially fair.

Just how much diversity is needed to credibly attribute full insurance to diversity of beliefs rather than to payoff kinks? Consider earthquake insurance priced on the basis of an actuarial probability of .005 and a load factor of 20%. Every risk averter with smooth preferences but whose own subjective probability happens to exceed .006 will buy full insurance.

5.3. Induced payoff and probability kinks

Most economically important situations of choice under uncertainty (e.g., agriculture, insurance, real investment) involve *delayed-resolution* risk or uncertainty. In such cases, there are invariably "auxiliary" decisions that must also be made prior to learning the outcome of the uncertain choice—if nothing else, consumption/savings decisions—in which case we refer to the delayed-resolution risk as *temporal risk*. Researchers such as Markowitz (1959, Ch. 11), Mossin (1969), Spence and Zeckhauser (1972), Kreps and Porteus (1979), Machina (1984) and Kelsey and Milne (1999) have examined how agents' *induced preferences* over such temporal lotteries—that is, the preferences obtained by maximizing out the auxiliary decision(s)—can systematically differ from their underlying risk preferences. In this section we illustrate the types of kinks that can arise in such induced preferences.

Consider an agent with an (expected utility or non-expected utility) preference function $V(\mathbf{P}, \alpha)$ that is jointly smooth over lotteries $\mathbf{P}$ and an auxiliary choice variable α selected from a set A. Maximizing out α yields the agent's induced preference function over temporal lotteries

$$W(\mathbf{P}) \stackrel{\text{def}}{\equiv} V(\mathbf{P}; \alpha^*(\mathbf{P})) \qquad \text{where} \qquad \alpha^*(\mathbf{P}) \stackrel{\text{def}}{\equiv} \underset{\alpha \in A}{\arg\max}\, V(\mathbf{P}; \alpha) \tag{103}$$

or equivalently, $W(x_1, p_1; \ldots; x_n, p_n) \stackrel{\text{def}}{\equiv} V(x_1, p_1; \ldots; x_n, p_n; \alpha^*(x_1, p_1; \ldots; x_n, p_n))$ where $\alpha^*(x_1, p_1; \ldots; x_n, p_n) \stackrel{\text{def}}{\equiv} \arg\max_{\alpha \in A} V(x_1, p_1; \ldots; x_n, p_n; \alpha)$. Depending upon the decision, the auxiliary variable α could either be continuous or discrete. If it is continuous, and $V(\cdot;\cdot)$ is such that the optimal choice $\alpha^*(\mathbf{P})$ varies smoothly in $\mathbf{P}$, then the induced preference function $W(\mathbf{P}) \equiv V(\mathbf{P}; \alpha^*(\mathbf{P}))$ will also be smooth $\mathbf{P}$, and subject to generalized expected utility analysis (Machina (1984)).

However, when the auxiliary variable α can only take discrete values, the induced preference function $W(\cdot)$ is in general only *regionwise smooth*, which as seen in Section 6.1, implies it will generally exhibit locally nonseparable payoff kinks on the boundaries of these regions. For example, when $A = \{\alpha', \alpha''\}$, the induced preference function $W(\cdot)$ consists of the upper envelope of the two smooth functions $\{V(\cdot;\alpha'), V(\cdot;\alpha'')\}$, and as such, will have kinks along those *ridges* of lotteries $\mathbf{P}$ where the two functions cross. Figure 9 illustrates a portion of the (x_1, x_2) indifference curves of $W(x_1, \bar{p}_1; x_2, \bar{p}_2)$ for fixed $(\bar{p}_1, \bar{p}_2)$, with locally quasiconvex kinks along an upward sloping ridge of kink points.[60] When such a ridge is not parallel to either the x_1 or x_2 axis, $W(\cdot)$'s payoff kinks at such ridge lotteries will be locally nonseparable.[61]

Although the induced preference function $W(\cdot)$ and the rank-dependent form $V_{RD}(\cdot)$ both exhibit locally nonseparable payoff kinks, their kink structure is otherwise quite different. For smooth underlying $V(\mathbf{P}, \alpha)$ and finite auxiliary choice set, say $A = \{\alpha', \alpha''\}$, the induced preference form (103) differs from the rank-dependent form (67) in that it

- can exhibit kinks in *whole probability* payoff shifts[62]
- *does not* exhibit first order risk aversion (or risk loving) about certainty[63]

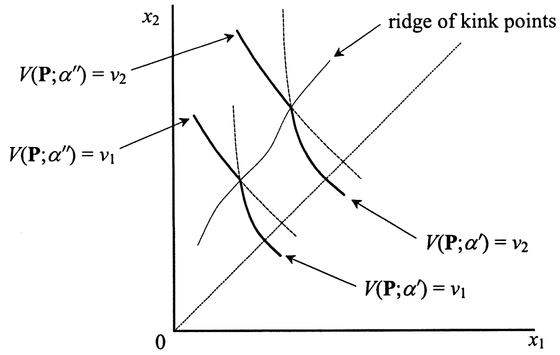

Figure 9. (Solid) Indifference curves of the induced preference function $W(\mathbf{P}) \equiv \max\{V(\mathbf{P};\alpha'), V(\mathbf{P};\alpha'')\}$.

- can only exhibit *locally quasiconvex* payoff kinks[64]
- is only kinked over *lower dimensional manifolds* (ridges) of lotteries in $\mathscr{L}$[65]
- also exhibits locally nonseparable *probability kinks* at such ridge lotteries[66]

Notes

1. Axiomatic characterizations of smooth preferences under certainty and uncertainty have been provided by Debreu (1972), Allen (1987) and others.
2. Such as income and substitution effect analysis.
3. E.g., in the standard case we find that all Cobb-Douglas utility functions have zero cross-price elasticities.
4. Although many of our results can be extended to general probability measures on R^1 (or R^n), we do not do so here.
5. See Savage (1954), as well as the economic applications in Arrow (1953, 1964), Debreu (1959, Ch. 7), Hirshleifer (1965, 1966, 1989) and Chambers and Quiggin (1999).
6. Important analyses of self insurance vs. self protection include Boyer and Dionne (1983, 1989), Dionne and Eeckhoudt (1985), Chang and Ehrlich (1985), Briys and Schlesinger (1990), Briys, Schlesinger and Schulenburg (1991), Sweeney and Beard (1992), and in a non-expected utility framework, Konrad and Skaperdas (1993).
7. Defining $\delta_x(\cdot)$ as the measure that assigns unit mass to x, the three paths are $\{\sum_{j\neq i} p_j \cdot \delta_{x_j}(\cdot) + p_i \cdot \delta_{x_i+t}(\cdot)|t\}$, $\{\sum_{j\neq i} p_j \cdot \delta_{x_j}(\cdot) + (p_i - \rho)\cdot \delta_{x_i}(\cdot) + \rho \cdot \delta_{x_i+t}(\cdot)|t\}$ and $\{\sum_{xj\neq x} p_j \cdot \delta_{x_j}(\cdot) + (\sum_{xj=x} p_j)\cdot \delta_{x+t}(\cdot)|t\}$, which typically have no common elements beyond their starting point at $t=0$.
8. A final condition for $V(x_1, p_1 + t\cdot\Delta p_1; \ldots; x_n, p_n + t\cdot\Delta p_n)$ to remain within the set of lotteries for all small enough t is that if any p_i is initially 0, we require $\Delta p_i \geq 0$ and also evaluate the limit in (12) as t approaches zero from above ("$t \downarrow 0$"). This condition will be understood to hold for all probability derivatives considered in this paper.
9. Namely the path $\{\sum_{k\neq i,j,i*} p_k \cdot \delta_{x_k}(\cdot) + (p_{i*} - t)\cdot \delta_{x_{i*}}(\cdot) + (p_i + p_j + t)\cdot \delta_x(\cdot)|t\}$ starting at $t=0$.
10. Thus at each distribution $\mathbf{P}$, the local utility function $U(\cdot;\mathbf{P})$ for the expected utility formula $\sum_{i=1}^n U(x_i)\cdot p_i$ is simply its von Neumann-Morgenstern utility function $U(\cdot)$.
11. This equivalence must be expressed in terms of *weakly increasing* (i.e., nondecreasing) functions and *weak* FSD preference, for the usual calculus reason that globally positive derivatives are *not* equivalent to strict monotonicity, as exemplified by the univariate function $g(z) = z^3$ which is globally strictly increasing even though $g'(0) = 0$.
12. See Allen (1987), Bardsley (1993), Chew, Epstein and Zilcha (1988), Chew and Nishimura (1992), Karni (1987, 1989), Machina (1982, 1989, 1995), Röell (1987) and Wang (1993) for additional results and extensions.

13. E.g., Huber (1981, Sect. 2.5), Wang (1993).
14. Machina (1982, Lemma 1) shows that the norm (22) in fact induces the *topology of weak convergence* on probability distributions over $[0, M]$ (e.g., Billingsley (1971; 1986, Ch. 25)).
15. E.g., Rudin (1987, pp. 144–146), Kolmogorov and Fomin (1970, pp. 336–340).
16. With reference to the discussion following (10), Fréchet differentiability serves to link $V(\cdot)$'s responses to differential movements along the three distinct paths of Note 7, which differ in the amounts p_i, ρ, and $\sum_{xj=x} p_j$ of probability mass shifted from the payoff level x_i or x.
17. Since (22) implies $\|(\ldots; x_i+\alpha\cdot t, p_i; x_j+\beta\cdot t, p_j; \ldots) - (\ldots; x_i, p_i; x_j, p_j; \ldots)\| \equiv |\alpha\cdot p_i\cdot t| + |\beta\cdot p_j\cdot t|$ for all sufficiently small t, whether or not x_i and x_j are distinct, a similar derivation establishes (31).
18. Again, (22) implies $\|(x_1+k, p_1; \ldots; x_n+k, p_n) - (x_1, p_1; \ldots; x_n, p_n)\| \equiv |k|$, whether or not the values $x_1, \ldots, x_n$ are distinct.
19. See for example Kolmogorov and Fomin (1970, pp. 334–336) or Feller (1971, pp. 35–36).
20. Though see Dekel's (1986) ingenious use of the Cantor function in a counterexample to a theoretical proposition.
21. S stands for "separable" and L for "Leontief." An additive term z_1+z_2 can be appended to these functions to make each of them strictly increasing, with no relevant change in their respective kink properties.
22. A convex set $E_j \subseteq R^n$ is a *convex cone* if $(z_1, \ldots, z_n) \in E_j \Rightarrow (\lambda\cdot z_1, \ldots, \lambda\cdot z_n) \in E_j$ for all $\lambda > 0$. Note that while only one cone in the partition $\{E_1, \ldots, E_J\}$ will actually contain the origin, it will be in closure of each cone E_j.
23. At $(1,1)$, both functions' left derivatives with respect to z_1 and z_2 equal 1, and their right derivatives equal 0.
24. Although additivity does hold when k_1, k_2 have opposite signs, it fails for negative k_1, k_2. For general piecewise-linear $\widehat{H}(\cdot, \ldots, \cdot)$ as in (36) or (37), additivity only holds when the individual variables' directions $(k_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, k_n)$, and hence their joint direction $(k_1, \ldots, k_n)$, all lie in the *same* convex cone E_j, on which $\widehat{H}(\cdot, \ldots, \cdot)$ is linear.
25. Do the properties of multivariate functions at kink points, or along one-dimensional loci of kink points, really matter for economic analysis? From standard consumer theory (and Figures 3a, b below), we know that a maximizing agent is at least as likely to be at a kink point, and move along a locus of kink points, as be anywhere else.
26. In each case, "all sufficiently small $t > 0$" denotes "for all positive t less than some $t_{z_0, \Delta z^A, \Delta z^B, \Delta z} > 0$."
27. Or rather, its natural extension from locally linear functions to locally piecewise-linear functions.
28. As before, if $V(\cdot)$ does not already satisfy $dV(\mathbf{P}_\alpha)/d\alpha > 0$ at every point on a given path $\{\mathbf{P}_\alpha | \alpha \in [\underline{\alpha}, \bar{\alpha}]\}$, it has an increasing transformation $V^*(\cdot)$ that does.
29. Although a multivariate expected utility function such as $\sum_{i=1}^n U(x_i, y_i)\cdot p_i \equiv \sum_{i=1}^n \min[x_i, y_i]\cdot p_i$ can exhibit nonseparable kinks *across commodities* x_i, y_i, it remains globally separable *across mutually exclusive bundles* $(x_i, y_i), (x_j, y_j)$.
30. Hirshleifer (1965, 1966), Yaari (1965, 1969). Each point (x_1, x_2) in the diagram represents the lottery $(x_1, \bar{p}_1; x_2, \bar{p}_2)$.
31. As mentioned, we restrict attention to locally piecewise-linear kinks (e.g., $U(\cdot)$ cannot be the Cantor function.)
32. Actuarially unfair insurance induces a budget line from C to a point on the certainty line with lower expected value.
33. Any actuarially unfair budget line that intersects the certainty line at a point (x, x) other than (100, 100) will cut the smooth indifference curve through that point from above, which implies an optimum strictly southeast of (x, x).
34. Equation (57) differs from (26) in that it is treated as a function of the payoffs $x_1, \ldots, x_n$ alone for fixed probabilities $\bar{p}_1, \ldots, \bar{p}_n$, and hence has higher order term $o(\|(x_1, \ldots, x_n) - (\bar{x}_1, \ldots, \bar{x}_n)\|)$ rather than $o(\|\mathbf{P} - \mathbf{P}_0\|)$.
35. More generally, the appropriate left/right directional derivatives on the right side of an equation such as (64) will be determined by the direction of change of t and the respective signs of α and β in the same manner as in (40).

36. Chew, Epstein and Segal (1991) have shown that when the function $\kappa(\cdot,\cdot)$ in the quadratic form is *not* smooth, but instead takes the Leontief form $\kappa(x_i,x_j)=\min\{x_i,x_j\}$, then the quadratic formula $\sum\sum\kappa(x_i,x_j)\cdot p_i\cdot p_j$ reduces to the rank-dependent form (67) with $\upsilon(x)\equiv x$ and $G(p)\equiv\sqrt{p}$, which is not Fréchet differentiable.
37. $U(\cdot)$ exhibits *second order conditional risk aversion* about x^* if $\partial\pi/\partial t|_{t=0}$ from (66) equals 0 and $\partial^2\pi/\partial t^2|_{t\downarrow 0}>0$.
38. Safra and Segal (2000) have further shown that both expected utility and Fréchet differentiable preferences are necessarily: (i) second order risk averse about x^* for almost all $x^*\in[0,M]$; (ii) for arbitrary p and $\tilde{x}$, second order *conditionally* risk averse about x^* for almost all $x^*\in[0,M]$.
39. See also Quiggin (1993), Weymark (1981) who proposed a similar form in the context of social welfare functions, Yaari (1987), who proposed a special case, and Allais (1988).
40. $\int_0^M \upsilon(x)\cdot d(G(F(x))$ is seen to equal to the *Choquet integral* $\int_0^M \upsilon(x)\cdot dG_F(x)$ of $\upsilon(\cdot)$ with respect to the *capacity* (monotonic but not necessarily additive measure) $G_F(\cdot)=G(\mu_F(\cdot))$, where $\mu_F(\cdot)$ is the measure induced by $F(\cdot)$ (e.g., Choquet (1953–54), Schmeidler (1989), Gilboa (1987), Gilboa and Schmeidler (1994), Denneberg (1994)).
41. As observed by Chew, Karni and Safra (1987) the result for infinite-outcome distributions is less straightforward.
42. As with all derivations of probability derivatives/local utility functions, we invoke the discussion preceding (15).
43. This uniqueness holds even when two or more of **P**'s outcomes are equal: Say $\hat{x}_{k-1}<\hat{x}_k=\hat{x}_{k+1}=\hat{x}_{k+2}<\hat{x}_{k+3}$ where $\hat{x}_k=\hat{x}_{k+1}=\hat{x}_{k+2}=x^*$. The value x^* only lies in the interval $[\hat{x}_{k+2},\hat{x}_{k+3})$, since $[\hat{x}_k,\hat{x}_{k+1})=[\hat{x}_{k+1},\hat{x}_{k+2})=\varnothing$.
44. For example, the equivalence of the "comparative probability equivalent," "comparative certainty equivalent," "comparative concavity" and "comparative arc concavity" conditions in the classic Arrow-Pratt characterization of comparative risk aversion (Pratt (1964, Thm. 1, conds. (b),(c),(d),(e))) does not require smoothness of $U(\cdot)$.
45. Although an even simpler example consists of assuming $G(1)-G(\frac{1}{2})\neq G(\frac{1}{2})-G(0)$ and looking at lotteries of the form $(x_1,\frac{1}{2};x_2,\frac{1}{2})$, we use the present setting to show that $V_{RD}(\cdot)$'s kinks do not just occur at degenerate lotteries.
46. As noted following (67), any payoff kinks resulting from exogenous kinks in $\upsilon(\cdot)$ *would* be locally separable.
47. This follows since both lines of (75) (and hence (76)/(76)$'$) reduce to $\ldots+\upsilon(x)\cdot\big[G(\frac{3}{4})-G\big(\frac{1}{4}\big)\big]+\ldots$ for $x_2=x_3=x$.
48. The formula for $\hat{x}_2(x_1,x_2,x_3)$ follows since two of the three terms in the sum $\sum\sum_{i<j}\max\{x_i,x_j\}=\max\{x_1,x_2\}+\max\{x_1,x_3\}+\max\{x_1,x_3\}$ must equal $\hat{x}_3$ and the other must equal $\hat{x}_2$. The formula for $\hat{x}_1(x_1,x_2,x_3)$ is derived by subtracting the formulas for $\hat{x}_3$ and $\hat{x}_2$ from $\sum_i x_i$. The formulas in (84), and also for $n>4$, can be derived similarly.
49. As illustrated in Section 4.2, $V_{RD}(\cdot)$ *will* satisfy the total derivative relationship (10) as long as x_i and x_j are not equal to each other, or to any other positive probability payoff x_k.
50. If a nonlinear $G(\cdot)$ were linear on each interval in (93), it would have to be kinked at one of their boundary points.
51. When $i^*=m$ and $\hat{x}_m=M,y_a,y_b$ *cannot* vary upward from $\hat{x}_{i^*}$. In this case pick positive ρ_a,ρ_b with $\rho_a+\rho_b\leq\hat{p}_m$ and $G(\sum_{j<m}\hat{p}_j+\rho_a+\rho_b)-G(\sum_{j<m}\hat{p}_j+\rho_a)-G(\sum_{j<m}\hat{p}_j+\rho_b)+G(\sum_{j<m}\hat{p}_j)\neq 0$, write $\mathbf{P}=(\ldots;y_a,\rho_a;y_b,\rho_b;\hat{x}_m,\hat{p}_m-\rho_a-\rho_b)|_{y_a=y_b=\hat{x}_m}$ and let y_a,y_b vary *downward* from $\hat{x}_m=M$ over $(\hat{x}_{m-1},\hat{x}_m]$. The term for $\hat{x}_m$ is $\upsilon(\hat{x}_m)\cdot[1-G(\sum_{j<m}\hat{p}_j+\rho_a+\rho_b)]$ and the terms for y_a and y_b can be written as $\upsilon(y_a)\cdot[G(\sum_{j<m}\hat{p}_j+\rho_a)-G(\sum_{j<m}\hat{p}_j)]+\upsilon(y_b)\cdot[G(\sum_{j<m}\hat{p}_j+\rho_b)-G(\sum_{j<m}\hat{p}_j)]+\upsilon(\max\{y_a,y_b\})\cdot[G(\sum_{j<m}\hat{p}_j+\rho_a+\rho_b)-G(\sum_{j<m}\hat{p}_j+\rho_a)-G(\sum_{j<m}\hat{p}_j+\rho_b)+G(\sum_{j<m}\hat{p}_j)]$. Since the Leontief term again receives nonzero weight, $V_{RD}(\cdot)$ is locally nonseparably kinked in y_a,y_b as they vary downward from $\hat{x}_m$ about **P**.
52. While this shows that $V_{RD}(\cdot)$ will be locally nonseparably kinked in certain *multivariate* payoff changes about every $\mathbf{P}\in\mathscr{L}$ and hence unamenable to additive marginal analysis, could $V_{RD}(\cdot)$ ever mimic the locally nonseparable example (46) and still be left/right smooth in all *univariate* or *directional* pay-

off changes from a particular $\mathbf{P}_0$? For what it's worth: Yes, whenever $G(\cdot)$ and $\mathbf{P}_0$ are such that the identity $G(\sum_{j\le i^*}\hat{p}_j)-G(\sum_{j\le i^*}\hat{p}_j-\rho)\equiv_\rho G(\sum_{j<i^*}\hat{p}_j+\rho)-G(\sum_{j<i^*}\hat{p}_j)$ holds on each of $\mathbf{P}_0$'s cumulative probability intervals (93).

53. E.g., MacCrimmon and Larsson (1979). More recent experimental studies, such as Camerer (1989), Starmer (1992), Birnbaum and McIntosh (1996), Birnbaum and Chavez (1997) and Wu and Gonzalez (1998) have revealed a more varied pattern of departures from payoff separability.
54. Separability would imply that for any rectangle of points $(x_1',x_2'),(x_1'',x_2'),(x_1',x_2''),(x_1'',x_2'')$ with $x_1'<x_1''$ and $x_2'<x_2''$, their marginal rates of substitution satisfy $MRS(x_1',x_2')\cdot MRS(x_1'',x_2'')=MRS(x_1',x_2'')\cdot MRS(x_1'',x_2')$.
55. For example, $W_{RS}(\cdot,\cdot)$ generally violates the total derivative formula at $x_1=x_2$, since $dW_{RS}(x,x)/dx=f'(x)+g'(x)=f^{*\prime}(x)+g^{*\prime}(x)$, yet $\partial W_{RS}(x,x)/\partial x_1^L+\partial W_{RS}(x,x)/\partial x_2^L=f'(x)+g^{*\prime}(x)$ and $\partial W_{RS}(x,x)/\partial x_1^R+\partial W_{RS}(x,x)/\partial x_2^R=f^{*\prime}(x)+g'(x)$.
56. The rank-dependent indifference curves in this figure correspond to $\upsilon(\cdot)$ concave and $G(\bar{p}_1)+G(\bar{p}_2)<1$. Figure 5b illustrates rank-dependent indifference curves for a *linear* $\upsilon(\cdot)$, with $G(\bar{p}_1)+G(\bar{p}_2)<1$ (e.g., Yaari (1987)).
57. E.g., Schmeidler (1989), Wakker (1996), Wakker, Erev and Weber (1994).
58. For example, the function $\upsilon(\cdot)$ defined by $\upsilon(x)\equiv\{1-e^{-x}$ for $x\ge 0; -2+2\cdot e^{x/2}$ for $x<0\}$ satisfies the three conditions and is continuously differentiable at 0.
59. See for example, Kunreuther, et al. (1978), Kunreuther (1996), and the additional references cited there.
60. In many cases, such as when $\bar{p}_1=\bar{p}_2=\frac{1}{2}, V(\cdot;\alpha')$'s and $V(\cdot;\alpha'')$'s indifference curves could cross again on the other side of the 45° line, leading to at least one more ridge of kinks. For clarity, we omit this from the figure.
61. For $V(\cdot;\alpha')$ and $V(\cdot;\alpha'')$ indifference curves that cross along an upward sloping ridge as in Figure 9, the local piecewise linearization of $W(x_1,\bar{p}_1;x_2,\bar{p}_2)$ at a ridge lottery $\overline{\mathbf{P}}=(\bar{x}_1,\bar{p}_1;\bar{x}_2,\bar{p}_2)$ will have the locally Leontief form $\max\{k_1'\cdot x_1+k_2'\cdot x_2, k_1''\cdot x_1+k_2''\cdot x_2\}=k_1''\cdot x_1+k_2'\cdot x_2+\max\{(k_1'-k_1'')\cdot x_1,(k_2''-k_2')\cdot x_2\}$, where $k_1'>k_1''$ are the regular x_1 payoff derivatives of $V(\cdot;\alpha')$ and $V(\cdot;\alpha'')$ at $\overline{\mathbf{P}}$, and $k_2'<k_2''$ are their regular x_2 payoff derivatives there.
62. As illustrated in Figure 9, where $W(\cdot)$ is generally kinked in each payoff at any ridge lottery off the certainty line.
63. As follows from (32) and illustrated in Figure 9, all of $V(\cdot;\alpha')$'s and $V(\cdot;\alpha'')$'s indifference curves cross the certainty line at the common slope $\bar{p}_1/\bar{p}_2$, so $W(\cdot)$'s indifference curves, which are the lower envelopes of $V(\cdot;\alpha')$'s and $V(\cdot;\alpha'')$'s indifference curves, cannot be kinked at certainty points.
64. This follows from standard results of convex analysis as applied to the local linearizations of $V(\cdot;\alpha')$ and $V(\cdot;\alpha'')$.
65. Namely the manifold of lotteries solving $V(\mathbf{P};\alpha')=V(\mathbf{P};\alpha'')$. Whenever $V(\mathbf{P}_0;\alpha')\ne V(\mathbf{P}_0;\alpha'')$, we have either $W(\mathbf{P}_0)=V(\mathbf{P}_0;\alpha')>V(\mathbf{P}_0;\alpha'')$ or else $W(\mathbf{P}_0)=V(\mathbf{P}_0;\alpha'')>V(\mathbf{P}_0;\alpha')$. Whichever case holds will continue to hold for all for all lotteries sufficiently close to $\mathbf{P}_0$ in $\mathscr{L}$ (including ones that differ by partial-probability payoff shifts), which implies smoothness of $W(\cdot)$ about $\mathbf{P}_0$.
66. Probability kinks in induced preferences over lotteries have been illustrated in Markowitz (1959, Ch.11, Fig. 2), Kreps and Porteus (1979, Fig. 3), and Machina (1984, Fig. 1).

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