Econ Theory (2011) 48:425–436
DOI 10.1007/s00199-011-0652-4

SYMPOSIUM

Event-Separability in the Ellsberg urn

Mark J. Machina

Received: 5 April 2011 / Accepted: 9 June 2011 / Published online: 16 July 2011

Abstract Ellsberg's three-color urn involves two distinct sources of uncertainty—the color composition of the urn (which is subjective) and the identity of the drawn ball (which is objective)—and bets on it can involve mixed objective/subjective uncertainty. It is known that typical betting preferences on this urn violate both the Sure-Thing Principle and probabilistic sophistication over its mixed uncertainty but are consistent with both of them over its purely subjective uncertainty. In this paper, we show that the standard Ellsberg-type preference reversal is actually implied by the Independence Axiom over its purely objective uncertainty.

Keywords Ellsberg paradox · Ambiguity · Ambiguity aversion · Event-separability · Ellsberg urn

JEL Classification D81 · 200 · 315 · 1345

Preface

Loosely speaking, the *Sure-Thing Principle* and *Independence Axiom* of classical expected utility consist of the following principles:

I would like to thank Chris Chambers, Larry Epstein, Haluk Ergin, Simon Grant, Peter Klibanoff, Duncan Luce, Anthony Marley, David Schmeidler, Uzi Segal, Joel Sobel, and especially Robert Nau and Peter Wakker for helpful comments on this material.

M. J. Machina (✉)
Department of Economics, University of California, San Diego,
La Jolla, CA 92093, USA
e-mail: mmachina@ucsd.edu

- If two subjective acts offer the same prize over some event, then replacing it with any other prize will not change the ranking of the acts.
- If two objective lotteries offer the same prize with some probability, then replacing it with any other prize will not change the ranking of the lotteries.

Besides being the central tenets of the subjective and objective expected utility hypotheses, these principles are usually taken to be the hallmarks of rational behavior under uncertainty.

A classic counterexample to the first principle was offered by Ellsberg (1961) in his famous three-color urn example, as depicted below. In addition to Ellsberg's four bets, the table also includes $f_2'(\cdot) =$ [\$0 if red; \$0 if black; \$100 if yellow] and $f_3'(\cdot) =$ [\$100 if red; \$100 if black; \$0 if yellow]. Because of their informational symmetry, we will assume that agents are always indifferent between $f_2(\cdot)$ and $f_2'(\cdot)$ and between $f_3(\cdot)$ and $f_3'(\cdot)$.

Three-color Ellsberg paradox (with additional bets $f_2'(\cdot)$ and $f_3'(\cdot)$)

	30 Balls	60 Balls		
	Red	Black	Yellow	
$f_1(\cdot)$	\$100	\$0	\$0	$\succ$
$f_2(\cdot)$	\$0	\$100	\$0	$\sim$
$f_2'(\cdot)$	\$0	\$0	\$100	
$f_3'(\cdot)$	\$100	\$100	\$0	$\sim$
$f_3(\cdot)$	\$100	\$0	\$100	$\prec$
$f_4(\cdot)$	\$0	\$100	\$100	

Agents' typical rankings of $f_1(\cdot)$ over $f_2(\cdot)$ and $f_4(\cdot)$ over $f_3(\cdot)$ are seen to violate the first of the above principles, and hence the Sure-Thing Principle, since replacing the prize of \$0 on the event {yellow} with a prize of \$100 transforms the acts $f_1(\cdot)$ and $f_2(\cdot)$ into $f_3(\cdot)$ and $f_4(\cdot)$, respectively, so that anyone who prefers $f_1(\cdot)$ over $f_2(\cdot)$ should prefer $f_3(\cdot)$ over $f_4(\cdot)$. For this reason, the three-color Ellsberg urn is considered the classic paradox in the theory of choice under subjective uncertainty.

But applying the second of the above principles to the Ellsberg urn yields a very different result. Replacing the 1/3 probability of \$0 in the (purely objective) bet $f_1(\cdot)$ with a 1/3 probability of \$100 yields the purely objective bet $f_4(\cdot)$, and replacing a 1/3 probability of \$0 in bet $f_2(\cdot)$ with a 1/3 probability of \$100 yields bet $f_3'(\cdot)$. By the second of the above principles, anyone who prefers $f_1(\cdot)$ over $f_2(\cdot)$ should prefer $f_4(\cdot)$ over $f_3'(\cdot)$ and hence prefer $f_4(\cdot)$ over $f_3(\cdot)$. Do the Sure-Thing Principle and Independence Axiom generate precisely opposite predictions?

Perhaps, but the above argument is not quite rigorous enough to establish this. The reason is that while it may be just as normatively appealing, the second principle cannot really be taken to be a statement of the Independence Axiom. The Independence Axiom works with objective probability mixtures of the form [p chance of z; $(1-p)$ chance of $\hat{z}$], where z and $\hat{z}$ are well-defined prizes from some set $\mathcal{Z}$ of such prizes. In the above example, none of the acts $f_2(\cdot)$, $f_2'(\cdot)$, $f_3'(\cdot)$, or $f_3(\cdot)$ can be represented in this form, since an object such as "[\$100 if black; \$0 if yellow]"—involving an incomplete event space—does not constitute a well-defined prize.

The purpose of this paper is to show that the difficulty raised by this example can in fact be formalized—in other words, that even though the Sure-Thing Principle and the Independence Axiom are both normatively similar event-separability conditions, they yield different predictions in the Ellsberg urn. Accordingly, the reader is asked to put aside this informal Preface and treat the paper as starting with the following section.

1 Introduction

Separability across mutually exclusive events—the *Independence Axiom* under objective uncertainty and the *Sure-Thing Principle* under subjective uncertainty—has long been considered the hallmark of rational and consistent choice over uncertain prospects and forms the foundation of the canonical expected utility model of choice under uncertainty.[1] For this reason, empirically or experimentally observed violations of separability—especially systematic violations of this property—are of more concern than they would be in most other branches of consumer theory.[2]

Experimenters have uncovered two systematic and widespread violations of event-separability in decision-makers' preferences. Under objective uncertainty, the Independence Axiom is systematically violated by the *Allais Paradox* and related phenomena. The Sure-Thing Principle is taken to be violated by the subject of this paper, the three-color *Ellsberg Paradox*, and similar examples.[3]

In contrast to Allais' purely objective prospects, Ellsberg urns involve both objective and subjective uncertainties and accordingly allow for three types of uncertain events, partitions, and bets: purely subjective (involving the color composition of the urn), purely objective (involving the identity of the drawn ball), and mixed objective/subjective (involving the color of the drawn ball). In this paper, we show that, rather than being an overarching "metaproperty" of preferences, event-separability can have distinct—and even conflicting—implications when applied to these three different forms of uncertainty.

The following section reviews the Independence Axiom, the Sure-Thing Principle, and a third event-separability property known as the *Strong Comparative Probability Axiom.* Section 3 illustrates their implications for betting preferences on two simplified versions of the original Ellsberg urn, containing only three balls each. Section 4 extends the argument to the standard 90-ball version, and Sect. 5 concludes.

2 Three principles and a paradox

The Independence Axiom, the Sure-Thing Principle, and the Strong Comparative Probability Axiom represent three distinct forms of event-separability of preferences over

[1] Even attempts to model more general 'non-expected utility' preferences have primarily relied upon some notion of event-separability—either applied to smaller groups of prospects (e.g., Schmeidler 1989) or else locally in the sense of differentiability (e.g., Machina 2005).

[2] A possible exception might be intertemporal choice, which relies extensively on separability across time periods.

[3] Allais (1953), Ellsberg (1961). See MacCrimmon and Larsson (1979), Segal (1987), Schmeidler (1989), Starmer (2000), Halevy (2007), Ergin and Gul (2009), and Seo (2009) for surveys, experimental findings, and theoretical responses to these two phenomena.

objectively and/or subjectively uncertain prospects.[4] Denoting an objective lottery yielding prize z_i with probability p_i as $\mathbf{P} = (z_1, p_1; \ldots; z_m, p_m)$, the *Independence Axiom* consists of the property:

$$\begin{aligned}&\left(z_1^*, p_1^*; \ldots; z_i^*, p_i^*\right) \succcurlyeq (z_1, p_1; \ldots; z_i, p_i)\\ &\quad \Leftrightarrow \left(z_1^*, \alpha p_1^*; \ldots; z_i^*, \alpha p_i^*; \hat{z}_{i+1}, (1-\alpha)\hat{p}_{i+1}; \ldots; \hat{z}_m, (1-\alpha)\hat{p}_m\right)\\ &\quad \succcurlyeq (z_1, \alpha p_1; \ldots; z_i, \alpha p_i; \hat{z}_{i+1}, (1-\alpha)\hat{p}_{i+1}; \ldots; \hat{z}_m, (1-\alpha)\hat{p}_m)\end{aligned} \tag{1}$$

for all lotteries $(z_1^*, p_1^*; \ldots; z_i^*, p_i^*)$, $(z_1, p_1; \ldots; z_i, p_i)$, $(\hat{z}_{i+1}, \hat{p}_{i+1}; \ldots; \hat{z}_m, \hat{p}_m)$ and all $\alpha \in (0,1]$. This can be described as separability over objective probability mixtures, in that the individual's ranking of two lotteries is unchanged with they are each mixed with some common third lottery, using some common mixture probability.

The Independence Axiom also implies the following event-separability property:

$$\begin{aligned}&\left(z_1^*, p_1; \ldots; z_i^*, p_i; z_{i+1}, p_{i+1}; \ldots; z_m, p_m\right)\\ &\quad \succcurlyeq (z_1, p_1; \ldots; z_i, p_i; z_{i+1}, p_{i+1}; \ldots; z_m, p_m)\\ &\quad \Leftrightarrow \left(z_1^*, p_1; \ldots; z_i^*, p_i; \hat{z}_{i+1}, p_{i+1}; \ldots; \hat{z}_m, p_m\right)\\ &\quad \succcurlyeq (z_1, p_1; \ldots; z_i, p_i; \hat{z}_{i+1}, p_{i+1}; \ldots; \hat{z}_m, p_m)\end{aligned} \tag{1$'$}$$

for all prizes and probabilities. Property $(1)'$ can be described as separability across objectively uncertain events, in that the individual's ranking of a pair of lotteries is unchanged when one or more of their common prizes $\{z_{i+1}, \ldots, z_m\}$ are replaced by the common prizes $\{\hat{z}_{i+1}, \ldots, \hat{z}_m\}$, with all other prizes, and all probabilities, unchanged.

Denoting a subjective act yielding prize z_j in event E_j as $f(\cdot) = [z_1 \text{ if } E_1; \ldots; z_n \text{ if } E_n]$, the *Sure-Thing Principle* consists of the property:

$$\begin{aligned}&\left[z_1^* \text{ if } E_1; \ldots; z_j^* \text{ if } E_j; z_{j+1} \text{ if } E_{j+1}; \ldots; z_n \text{ if } E_n\right]\\ &\quad \succcurlyeq \left[z_1 \text{ if } E_1; \ldots; z_j \text{ if } E_j; z_{j+1} \text{ if } E_{j+1}; \ldots; z_n \text{ if } E_n\right]\\ &\quad \Leftrightarrow \left[z_1^* \text{ if } E_1; \ldots; z_j^* \text{ if } E_j; \hat{z}_{j+1} \text{ if } E_{j+1}; \ldots; \hat{z}_n \text{ if } E_n\right]\\ &\quad \succcurlyeq \left[z_1 \text{ if } E_1; \ldots; z_j \text{if } E_j; \hat{z}_{j+1} \text{ if } E_{j+1}; \ldots; \hat{z}_n \text{ if } E_n\right]\end{aligned} \tag{2}$$

for all prizes and events. This can be described as separability across subjectively uncertain events, in that the individual's ranking of a pair of acts is unchanged when one or more of their common prizes $\{z_{j+1}, \ldots, z_n\}$ are replaced by the common prizes $\{\hat{z}_{j+1}, \ldots, \hat{z}_n\}$, with all other prizes, and all events, unchanged.

[4] See Savage (1954, p. 23, Axiom P2), Samuelson (1952, p. 672, Axiom II), and Machina and Schmeidler (1992, p. 761, Axiom P4*) for formal statements of these principles.

A second, distinct form of separability across subjectively uncertain events is the *Strong Comparative Probability Axiom*, which consists of the property:

$$\begin{aligned}&\left[z_1^* \text{ if } E_1; z_2^* \text{ if } E_2; z_3 \text{ if } E_3; \ldots; z_n \text{ if } E_n\right]\\ &\quad \succcurlyeq \left[z_2^* \text{ if } E_1; z_1^* \text{ if } E_2; z_3 \text{ if } E_3; \ldots; z_n \text{ if } E_n\right]\\ &\quad \Leftrightarrow \left[\hat{z}_1^* \text{ if } E_1; \hat{z}_2^* \text{ if } E_2; \hat{z}_3 \text{ if } \hat{E}_3; \ldots; \hat{z}_n \text{ if } \hat{E}_n\right]\\ &\quad \succcurlyeq \left[\hat{z}_2^* \text{ if } E_1; \hat{z}_1^* \text{ if } E_2; \hat{z}_3 \text{ if } \hat{E}_3; \ldots; \hat{z}_n \text{ if } \hat{E}_n\right] \qquad (3)\end{aligned}$$

for all prizes and events for which $z_1^* \succ z_2^*$ and $\hat{z}_1^* \succ \hat{z}_2^*$. This can be described as the existence of well-defined likelihood rankings, in that the individual's preference for which way to stake the more preferred versus less preferred of two prizes on some pair of events will not depend upon the two prizes themselves nor upon the other prizes and events in the act. It has been shown to imply the property of *probabilistic sophistication*—the existence of a subjective probability measure $\mu(\cdot)$ over events such that an act $[z_1 \text{ if } E_1; \ldots; z_n \text{ if } E_n]$ is evaluated solely on the basis of its implied lottery $(z_1, \mu(E_1); \ldots; z_n, \mu(E_n))$.

Although they are both defined over subjective partitions $\{E_1, \ldots, E_n\}$, the Sure-Thing Principle and the Strong Comparative Probability Axiom are logically independent properties: the state-dependent expected utility preference function $\Sigma U(z_j|E_j)\mu(E_j)$ (e.g., Karni 1985) satisfies the Sure-Thing Principle but not necessarily the Strong Comparative Probability Axiom, whereas the probabilistically sophisticated non-expected utility preference function $V(z_1, \mu(E_1); \ldots; z_n, \mu(E_n))$ (e.g. Machina and Schmeidler 1992) satisfies Strong Comparative Probability but not necessarily the Sure-Thing Principle.

It is important to note that the above separability properties are taken to hold "for all prizes." Thus, both the Sure-Thing Principle and the Strong Comparative Probability Axiom allow the prizes in a subjective act $[\ldots; (\ldots; z_{ij}, p_{ij}; \ldots) \text{ if } E_j; \ldots]$ to be objective lotteries, and the Independence Axiom allows the prizes in an objective lottery $[\ldots; (\ldots; z_{ij} \text{ if } E_{ij}; \ldots), p_i; \ldots]$ to be subjective acts. The only requirement, of course, is that each prize $(\ldots; z_{ij}, p_{ij}; \ldots)$ be a fully specified objective lottery satisfying $\sum_i p_{ij} = 1$ and each prize $[\ldots; z_{ij} \text{ if } E_{ij}; \ldots]$ be a fully specified subjective act satisfying $\cup_j E_{ij} = \Omega$.

It is also important to note that, regardless of what prizes may or may not be assigned to them, the events in a "subjective" partition $\{E_1, \ldots, E_n\}$ may have objective attributes. Thus, for the three-color urn of Table 1, the event red is purely objective with probability 1/3, and the events black and yellow have the attribute that their union has an objective probability of 2/3. We term a partition $\{E_1, \ldots, E_n\}$ *purely subjective* if it contains no objective information whatsoever.

As noted elsewhere, the Sure-Thing Principle is violated by a choice problem known as the *three-color Ellsberg paradox* and by similar examples due to or inspired by Ellsberg (1961). As illustrated in Table 1, this example consists of an urn containing 90 balls, 30 of which are known to be red, with each of the remaining 60 either black or yellow, although the number of black versus yellow balls is unknown. A ball is drawn from this urn, and the bet $f_1(\cdot) = [\$100 \text{ if red}; \$0 \text{ if black}; \$0 \text{ if yellow}]$ yields \$100

Table 1 Three-color Ellsberg paradox

	30 Balls	60 Balls		
	Red	Black	Yellow	
$f_1(\cdot)$	\$100	\$0	\$0	$\succ$
$f_2(\cdot)$	\$0	\$100	\$0	
$f_3(\cdot)$	\$100	\$0	\$100	$\prec$
$f_4(\cdot)$	\$0	\$100	\$100	

if it is red, \$0 if it is black and \$0 if it is yellow, similarly for the bets $f_2(\cdot)$, $f_3(\cdot)$, and $f_4(\cdot)$. When faced with these prospects, most individuals express a strict preference for $f_1(\cdot)$ over $f_2(\cdot)$ and for $f_4(\cdot)$ over $f_3(\cdot)$, as indicated to the right of the table. But since bets $f_3(\cdot)$ and $f_4(\cdot)$ can be respectively obtained from $f_1(\cdot)$ and $f_2(\cdot)$ by replacing their common outcome [\$0 if yellow] by the common outcome [\$100 if yellow], these preferences violate the Sure-Thing Principle over the mixed objective/subjective partition {red,black,yellow}. Since $f_1(\cdot) \succ f_2(\cdot)$ would reveal the likelihood ranking μ(red) > μ(black) but $f_3(\cdot) \prec f_4(\cdot)$ would reveal μ(red) < $\mu(black)$, this pair of rankings also violates the Strong Comparative Probability Axiom, and hence probabilistic sophistication, over the partition {red, black, yellow}.[5]

The intuition behind these preferences is clear: $f_1(\cdot)$ offers the \$100 prize on an objective 1/3-probability event {red}, whereas $f_2(\cdot)$ offers it on one element of an informationally symmetric but subjective partition {black, yellow} of a 2/3-probability event. Similarly, $f_4(\cdot)$ offers the prize on an objective 2/3-probability event, whereas $f_3(\cdot)$ offers it on the union of a 1/3-probability event and the other element of that subjective partition. Such preferences are termed *ambiguity averse*.

3 Analysis of two simplified examples

For simplicity, we start with a simpler version of the three-color urn, involving only three balls and including two additional bets, as illustrated in Table 2. For reasons similar to those in the standard 90-ball version, we define *Ellsberg preferences* (or an *Ellsberg reversal*) as the pair of rankings $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ and $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$. Thus, in addition to the typical rankings $f_1(\cdot) \succ f_2(\cdot)$ and $f_3(\cdot) \prec f_4(\cdot)$, we posit indifference between the informationally symmetric acts $f_2(\cdot)$ and $f_2'(\cdot)$, and between the informationally symmetric acts $f_3(\cdot)$ and $f_3'(\cdot)$.

Bets on an Ellsberg urn are most clearly represented when its mixed objective/subjective partition {red, black, yellow} is replaced by an "orthogonal" representation of the urn's underlying objective and subjective uncertainties. The orthogonal

5 If we require that subjective beliefs satisfy μ(red) = 1/3 and define $f_2'(\cdot)$ = [\$0 if red; \$0 if black; \$100 if yellow] as in the following table, then the ranking $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ would itself violate probabilistic sophistication over the mixed partition {red, black, yellow}, since it would imply 1/3 = μ(red) > μ(black) = μ(yellow) even though μ(red), μ(black), and μ(yellow) must sum to unity.

Table 2 Three-color Ellsberg paradox (simplified version)

	1 Ball	2 Balls		
	Red	Black	Yellow	
$f_1(\cdot)$	\$100	\$0	\$0	$\succ$
$f_2(\cdot)$	\$0	\$100	\$0	$\sim$
$f_2'(\cdot)$	\$0	\$0	\$100	
$f_3'(\cdot)$	\$100	\$100	\$0	$\sim$
$f_3(\cdot)$	\$100	\$0	\$100	
$f_4(\cdot)$	\$0	\$100	\$100	$\prec$

Table 3 Orthogonal representation of the three-color partition

serial number of drawn ball				
3	black	yellow	black	yellow
2	black	black	yellow	yellow
1	red	red	red	red
	BB	*BY*	*YB*	*YY*

composition of the urn

representation of mixed prospects—which dates back to the equations of Anscombe and Aumann (1963) and the diagrams of Pratt et al. (1964)—has been applied to Ellsberg urns by Segal (1987), Klibanoff et al. (2005), Nau (2006), Chew and Sagi (2008), Ergin and Gul (2009), and Seo (2009), and we adopt this approach here.

To separately represent the underlying objective and subjective uncertainties in this urn, we follow Ergin and Gul (2009) and label objective events by assigning serial numbers to the individual balls and subjective events by the color composition of the urn. Ball number 1 is red, and balls 2 and 3 can take the respective colors black/black, black/yellow, yellow/black, or yellow/yellow. The objective uncertainty thus consists of the serial number i of the drawn ball, whereas the subjective uncertainty consists of which of the states *BB*, *BY*, *YB*, or *YY* obtains. For example, the event that ball number 2 is actually drawn is purely objective, with probability 1/3, and the event that (drawn or not) its color is black is purely subjective, depending on whether or not the one of the states *BB* or *BY* obtains. The event that a black ball—any black ball—is drawn is thus mixed objective/subjective, depending upon whether or not the serial number and state are one of the pairs (2, *BB*), (2, *BY*), (3, *BB*), or (3, *YB*).

Table 3 displays this orthogonal representation of the urn's purely objective, purely subjective, and mixed uncertainty, with the serial number of the chosen ball represented vertically, the black/yellow composition of the urn represented horizontally, and the resulting drawn ball color in each cell. This twelve-element orthogonal partition is seen to be a refinement of the three-element mixed partition {red, black, yellow} of Table 2. Table 4 depicts bets $f_1(\cdot)$, $f_2(\cdot)$, $f_2'(\cdot)$, $f_3'(\cdot)$, $f_3(\cdot)$, and $f_4(\cdot)$ as bets over

Table 4 Orthogonal representation of the three-color bets

f_1: [\$100 if R; \$0 if B; \$0 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$0	\$0	\$0	\$0
2	\$0	\$0	\$0	\$0
1	\$100	\$100	\$100	\$100

$\succ$

f_2: [\$0 if R; \$100 if B; \$0 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$100	\$0	\$100	\$0
2	\$100	\$100	\$0	\$0
1	\$0	\$0	\$0	\$0

$\sim$

f_2': [\$0 if R; \$0 if B; \$100 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$0	\$100	\$0	\$100
2	\$0	\$0	\$100	\$100
1	\$0	\$0	\$0	\$0

f_4: [\$0 if R; \$100 if B; \$100 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$100	\$100	\$100	\$100
2	\$100	\$100	\$100	\$100
1	\$0	\$0	\$0	\$0

composition of the urn

$\succ$

f_3': [\$100 if R; \$100 if B; \$0 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$100	\$0	\$100	\$0
2	\$100	\$100	\$0	\$0
1	\$100	\$100	\$100	\$100

composition of the urn

$\sim$

f_3: [\$100 if R; \$0 if B; \$100 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$0	\$100	\$0	\$100
2	\$0	\$0	\$100	\$100
1	\$100	\$100	\$100	\$100

composition of the urn

this orthogonal partition. Reading cell by cell, each bet can be viewed as a bet over the joint objective/subjective uncertainty, with sure consequences as prizes. Reading column by column, each can be viewed as a subjective act over the space {*BB*, *BY*, *YB*, *YY*}, with objective lotteries as prizes. Reading row by row, each can be viewed as an objective lottery, with subjective acts over {*BB*, *BY*, *YB*, *YY*} as prizes.[6]

Tables 3 and 4 illustrate how event-separability over these three different forms of uncertainty has different implications for how preferences over $f_1(\cdot)$, $f_2(\cdot)$, and $f_2'(\cdot)$ are linked to preferences over $f_3'(\cdot)$, $f_3(\cdot)$, and $f_4(\cdot)$. By providing an alternative representation of the three-color partition {red, black, yellow} of Table 2, they again illustrate how the pair of rankings $f_1(\cdot) \succ f_2(\cdot)$ and $f_3(\cdot) \prec f_4(\cdot)$ violate both the Sure-Thing Principle and the Strong Comparative Probability Axiom (and thus probabilistic sophistication) over the urn's *mixed objective/subjective uncertainty*.

Table 4 also illustrates why the Ellsberg rankings of $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ *do not* violate event-separability—neither the Sure-Thing Principle nor the Strong Comparative Probability Axiom—over the urn's *purely subjective uncertainty* (its unknown color composition). When applied to its purely subjective partition {*BB, BY, YB, YY*}, the Sure-Thing Principle *has no implications* for preferences across the two triples of bets. Bets $f_1(\cdot)$, $f_2(\cdot)$ and $f_2'(\cdot)$ do share a common prize over the partition {*BB, BY, YB, YY*}, namely the objective lottery (\$100, 1/3; \$0, 2/3) in the events *BY* and *YB*, and this is indeed replaced by the common prize (\$100, 2/3; \$0, 1/3) in bets $f_3(\cdot)$, $f_3'(\cdot)$ and $f_4(\cdot)$. However, for the Sure-Thing Principle to have any implication, two of the bets $f_3(\cdot)$, $f_3'(\cdot)$, or $f_4(\cdot)$ would have to differ from two of the bets $f_1(\cdot)$, $f_2(\cdot)$, or $f_2'(\cdot)$ *only* in this replacement, which is not the case. In contrast, the Strong Comparative Probability Axiom does have an implication for

6 That such tables allow for these three perspectives was noted by Pratt et al. (1964, p. 364).

preferences across the two triples of bets, since the ranking $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ exhibits equal revealed likelihoods for the events *BB* and *YY*. However, this is the only revealed likelihood information in this ranking, and it is shared by the ranking $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$.[7] The compatibility of Ellsberg preferences with the Sure-Thing Principle and the Strong Comparative Probability Axiom over the urn's subjective uncertainty has been known since the work of Klibanoff et al. (2005) and Nau (2006), who have shown how Ellsberg-type ambiguity aversion, is consistent with a subjectively separable preference function of the form $\sum_{j=1}^{n} g(\sum_{i=1}^{m} U(z_{ij}) \cdot p_{ij}) \cdot \mu(E_j)$.

That Ellsberg preferences could violate event-separability over mixed partitions but not purely subjective partitions should come as no surprise, since the latter is a strictly weaker property. What does come as a surprise, however, is that given the typical Ellsberg ranking $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ over the first triple of acts, the reversed ranking $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ over the second triple is actually *implied* by event-separability—that is, by the Independence Axiom—over the urn's *purely objective uncertainty.*

As seen in Table 4, bets $f_4(\cdot)$, $f_3'(\cdot)$ and $f_3(\cdot)$, respectively, differ from $f_1(\cdot)$, $f_2(\cdot)$, and $f_2'(\cdot)$ by the replacement of a common 1/3 probability of the prize (subjective act) [\$0 if *BB*; \$0 if *BY*; \$0 if *YB*; \$0 if *YY*] with a 1/3 probability of the subjective act [\$100 if *BB*; \$100 if *BY*; \$100 if *YB*; \$100 if *YY*], with all remaining prizes (acts) and their probabilities unchanged. By the Independence Axiom (separability across objective events), preferences should satisfy the relationship $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot) \Leftrightarrow f_4(\cdot) \succ f_3'(\cdot) \sim f_3(\cdot)$. Anyone who satisfies the Independence Axiom over the urn's purely objective uncertainty and who strictly prefers $f_1(\cdot)$ over $f_2(\cdot)$ *must* exhibit the Ellsberg reversal and strictly prefer $f_4(\cdot)$ over $f_3(\cdot)$.

The urn of Table 2 is offered as a 1:30 "scale replica" of the original 90-ball urn to allow for the simple representation of Tables 3 and 4, and as such it retains the information structure of the original urn, namely the individual color uncertainty of each non-red ball. In correspondence, Robert Nau has observed that by adopting a slightly different information structure, we can construct an urn with an even simpler orthogonal representation, which exhibits the same event-separability properties as the urn of Table 2 and Ellsberg's original 90-ball urn.

Modify the urn of Table 2 so that the second and third balls are either both black or both yellow, but keep Table 2's descriptions of the six bets the same, so that the Ellsberg rankings of $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ continue to violate both the Sure-Thing Principle and the Strong Comparative Probability Axiom over the mixed objective/subjective partition {red, black, yellow}. Table 5 presents the resulting orthogonal representation of the bets, which is essentially Table 4 with the states *BY* and *YB* omitted. Since $f_1(\cdot)$ shares no common column with either $f_2(\cdot)$ or $f_2'(\cdot)$, the Sure-Thing Principle again has no implications over the urn's purely subjective uncertainty. Since the rankings $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ and $f_4(\cdot) \succ f_3'(\cdot) \sim f_3(\cdot)$ both

[7] It may seem that the ranking $f_2(\cdot) \sim f_2'(\cdot)$ also exhibits equal revealed likelihoods for the events *BY* and *YB*, since it would appear that the prizes in these two events get switched. But as noted elsewhere, these prizes are both the *same* objective lottery (\$100, 1/3; \$0, 2/3), so no comparative likelihood of *BY* versus *YB* is revealed. As reflected in the statement of the Strong Comparative Probability Axiom, comparative likelihood information is only revealed by the swapping of two distinct, *non-indifferent* prizes.

Table 5 Orthogonal representation of the three-color bets (Nau's example)

f_1:[\$100 if R;\$0 if B;\$0 if Y]

serial number of drawn ball	BB	YY
3	\$0	\$0
2	\$0	\$0
1	\$100	\$100

$\succ$

f_2:[\$0 if R;\$100 if B;\$0 if Y]

serial number of drawn ball	BB	YY
3	\$100	\$0
2	\$100	\$0
1	\$0	\$0

$\sim$

f_2':[\$0 if R;\$0 if B;\$100 if Y]

serial number of drawn ball	BB	YY
3	\$0	\$100
2	\$0	\$100
1	\$0	\$0

f_4:[\$0 if R;\$100 if B;\$100 if Y]

serial number of drawn ball	BB	YY
3	\$100	\$100
2	\$100	\$100
1	\$0	\$0

composition of the urn

$\succ$

f_3':[\$100 if R;\$100 if B;\$0 if Y]

serial number of drawn ball	BB	YY
3	\$100	\$0
2	\$100	\$0
1	\$100	\$100

composition of the urn

$\sim$

f_3:[\$100 if R;\$0 if B;\$100 if Y]

serial number of drawn ball	BB	YY
3	\$0	\$100
2	\$0	\$100
1	\$100	\$100

composition of the urn

exhibit equally revealed likelihoods for the events *BB* and *YY*, they are consistent with the Strong Comparative Probability Axiom over the urn's purely subjective uncertainty. But since bets $f_4(\cdot)$, $f_3'(\cdot)$, and $f_3(\cdot)$ can again be obtained from $f_1(\cdot)$, $f_2(\cdot)$, and $f_2'(\cdot)$ by replacing a 1/3 probability of the act [\$0 if *BB*; \$0 if *YY*] with a 1/3 probability of [\$100 if *BB*; \$100 if *YY*], the Independence Axiom continues to imply $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot) \Leftrightarrow f_4(\cdot) \succ f_3'(\cdot) \sim f_3(\cdot)$. Once again, anyone who strictly prefers $f_1(\cdot)$ over $f_2(\cdot)$ and who satisfies the Independence Axiom over urn's purely objective uncertainty must strictly prefer $f_4(\cdot)$ over $f_3(\cdot)$. Although this example does not retain the individual ball-by-ball color uncertainty of the Table 1 and 2 urns, it exhibits the same event-separability properties.

4 Analysis of the 90-ball urn

The arguments of the previous section can be extended to the 90-ball urn of Table 1. As before, we include the bets $f_2'(\cdot) =$ [\$0 if red; \$0 if black; \$100 if yellow] and $f_3'(\cdot)$ = [\$100 if red; \$100 if black; \$0 if yellow] and define Ellsberg preferences as the pair of rankings $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ and $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$. We have seen that these preferences violate both the Sure-Thing Principle and the Strong Comparative Probability Axiom over the urn's mixed objective/subjective uncertainty (the color of the drawn ball), but as the above-cited authors have shown, they are consistent with these two forms of event-separability over the urn's purely subjective uncertainty (its unknown color composition). Here, we generalize the argument of the previous section to demonstrate that the Ellsberg reversal is again implied by the Independence Axiom over the urn's purely objective uncertainty, namely the identity of the drawn ball.

Label the balls so that balls 1,…, 30 are red and each of balls 31,…, 90 is either yellow or black, and define …*BBYB*… as the state in which balls 31 through 90 take the respective colors …,B, B, Y, B,… etc., so that there are 2^{60} states. For $i = 31,\ldots,90$, define the event $\mathcal{B}_i$ as the union of all states in which ball i is black and event $\mathcal{Y}_i$ as its complement, that is, the set of all states in which ball i is yellow. The orthogonalized partition of the full objective/subjective uncertainty of the 90-ball urn thus consists of all pairs of the form (i,…$BBYB$…). The color of the drawn ball will be red if ball $i \in \{1,\ldots,30\}$ is drawn, it will be black if ball $i \in \{31,\ldots,90\}$ is drawn and the state is in $\mathcal{B}_i$, and it will be yellow if ball $i \in \{31,\ldots,90\}$ is drawn and the state is in $\mathcal{Y}_i$.

Each bet of the form $f(\cdot) = [z_{\text{red}}$ if red; z_{black} if black; z_{yellow} if yellow] can thus be represented as an objective lottery with subjective acts as prizes:

$$(z_{\text{red}}, 1/3; \ldots; [z_{\text{black}} \text{ if } \mathcal{B}_i; z_{\text{yellow}} \text{ if } \mathcal{Y}_i], 1/90; \ldots) \tag{4}$$

so that the bets of Table 1, along with $f_2'(\cdot)$ and $f_3'(\cdot)$, can be represented as:

$$\begin{aligned}
f_1(\cdot) &= (\$100, 1/3; \ldots; [\$0 \text{ if } \mathcal{B}_i; \$0 \text{ if } \mathcal{Y}_i], 1/90; \ldots) = (\$100, 1/3; \$0, 2/3)\\
f_2(\cdot) &= (\$0, 1/3; \ldots; [\$100 \text{ if } \mathcal{B}_i; \$0 \text{ if } \mathcal{Y}_i], 1/90; \ldots)\\
f_2'(\cdot) &= (\$0, 1/3; \ldots; [\$0 \text{ if } \mathcal{B}_i; \$100 \text{ if } \mathcal{Y}_i], 1/90; \ldots)\\
f_3'(\cdot) &= (\$100, 1/3; \ldots; [\$100 \text{ if } \mathcal{B}_i; \$0 \text{ if } \mathcal{Y}_i], 1/90; \ldots)\\
f_3(\cdot) &= (\$100, 1/3; \ldots; [\$0 \text{ if } \mathcal{B}_i; \$100 \text{ if } \mathcal{Y}_i], 1/90; \ldots)\\
f_4(\cdot) &= (\$0, 1/3; \ldots; [\$100 \text{ if } \mathcal{B}_i; \$100 \text{ if } \mathcal{Y}_i], 1/90; \ldots) = (\$100, 2/3; \$0, 1/3)
\end{aligned} \tag{5}$$

As in the three-ball urn, bets $f_4(\cdot)$, $f_3'(\cdot)$, and $f_3(\cdot)$ are respectively obtained from $f_1(\cdot)$, $f_2(\cdot)$ and $f_2'(\cdot)$ solely by replacing a 1/3 probability of the prize \$0 with a 1/3 probability of the prize \$100. By the Independence Axiom, preferences should again satisfy the relationship $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot) \Leftrightarrow f_4(\cdot) \succ f_3'(\cdot) \sim f_3(\cdot)$. Anyone who satisfies the Independence Axiom over purely objective uncertainty and who strictly prefers $f_1(\cdot)$ over $f_2(\cdot)$ in the 90-ball urn must again exhibit the Ellsberg reversal and strictly prefer $f_4(\cdot)$ over $f_3(\cdot)$. Since preferences must also satisfy the relationship $f_1(\cdot) \prec f_2(\cdot) \sim f_2'(\cdot) \Leftrightarrow f_4(\cdot) \prec f_3'(\cdot) \sim f_3(\cdot)$, anyone who satisfies the Independence Axiom and exhibits the *ambiguity loving* ranking of $f_2(\cdot)$ over $f_1(\cdot)$ will exhibit a “reflected” Ellsberg reversal and strictly prefer $f_3(\cdot)$ over $f_4(\cdot)$.

5 Conclusion

The event-separability properties of the Ellsberg preferences of $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ in the three-color Ellsberg urn can be summarized as follows:

- Such preferences *do* violate both the Sure-Thing Principle and the Strong Comparative Probability Axiom (and therefore probabilistic sophistication) over the urn’s *mixed objective/subjective uncertainty*—that is, the color of the drawn ball.
- Such preferences *do not* violate either the Sure-Thing Principle or Strong Comparative Probability over its *purely subjective uncertainty*—that is, its unknown color composition.
- Such preferences (or their ambiguity-loving reflection) are in fact *implied* by the Independence Axiom over its *purely objective uncertainty*—that is, the identity of the drawn ball.

As mentioned, the first two of these points are well known—the purpose of this paper is to point out the third and thus the need for care when positing event-separability under uncertainty.

References

Allais, M.: Le comportement de l'Homme rationnel devant le risque, critique des postulats et axiomes de l'Ecole Américaine. Econometrica **21**, 503–546 (1953)

Allais, M., Hagen, O. (eds.): Expected Utility Hypotheses and the Allais Paradox. Dordrecht: D. Reidel Publishing Co (1979)

Anscombe, F., Aumann, R.: A definition of subjective probability. Ann Math Stat **34**, 199–205 (1963)

Chew, S.H., Sagi, J.: Small worlds: Modeling attitudes toward sources of uncertainty. J Econ Theory **139**, 1–24 (2008)

Ellsberg, D.: Risk, Ambiguity, and the Savage Axioms. Q J Econ **75**, 643–669 (1961)

Ellsberg, D.: Risk, Ambiguity and Decision. New York & London: Garland Publishing, Inc. (Published version of Ellsberg's 1962 Harvard doctoral thesis.) (2001)

Ergin, H., Gul, F.: A theory of subjective compound lotteries. J Econ Theory **144**, 899–929 (2009)

Halevy, Y.: Ellsberg Revisited: an experimental study. Econometrica **75**, 503–536 (2007)

Karni, E.: Decision Making Under Uncertainty: The Case of State Dependent Preferences. Cambridge, Mass: Harvard University Press (1985)

Klibanoff, P., Marinacci, M., Mukerji, S.: A smooth model of decision making under ambiguity. Econometrica **73**, 1849–1892 (2005)

MacCrimmon, K., Larsson, S.: Utility Theory: Axioms Versus 'Paradoxes'. In: Allais and Hagen (1979)

Machina, M.: 'Expected utility/subjective probability' analysis without the sure-thing principle or probabilistic sophistication. Econ Theory **26**, 1–62 (2005)

Machina, M., Schmeidler, D.: A more robust definition of subjective probability. Econometrica **60**, 745–780 (1992)

Nau, R.: Uncertainty aversion with second-order utilities and probabilities. Manag Sci **52**, 136–145 (2006)

Pratt, J., Raiffa, H., Schlaifer, R.: The foundations of decision under uncertainty: An elementary exposition. J Am Stat Assoc **59**, 353–375 (1964)

Samuelson, P.: Probability, utility, and the independence axiom. Econometrica **20**, 670–678 (1952)

Savage, L.: The Foundations of Statistics. New York: Wiley. Revised and Enlarged Edition, New York: Dover Publications, 1972 (1954)

Schmeidler, D.: Subjective probability and expected utility without additivity. Econometrica **57**, 571–587 (1989)

Segal, U.: The ellsberg paradox and risk aversion: An anticipated utility approach. Int Econ Rev **28**, 175–202 (1987)

Seo, K.: Ambiguity and second-order belief. Econometrica **77**, 1575–1605 (2009)

Starmer, C.: In non-expected utility theory: The hunt for a descriptive theory of choice under risk. J Econ Lit **38**, 332–382 (2000)