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CARDINAL PROPERTIES OF "LOCAL UTILITY FUNCTIONS"

This paper outlines the cardinal properties of "local utility functions" of the type used by Allen [1985], Chew [1983], Chew and MacCrimmon [1979], Dekel [1985], Epstein [1985], Fishburn [1984], Karni and Safra [1985], Machina [1982,1983, 1984], Yaari [1985] and others.

1. SMOOTH PREFERENCES AND LOCAL UTILITY FUNCTIONS

Consider the set D[a,b] of all cumulative distribution functions F(·) over some interval [a,b], and an individual whose preference ranking over this set can be represented by a real-valued *preference functional* V(F), in the sense that the distribution F*(·) is weakly preferred to F(·) if and only if V(F*) ≥ V(F). Note that V(·) is *ordinal* in the sense that some other functional V*(·) will represent the same preference ranking if and only if V*(·) = ϕ(V(·)) for some increasing function $\phi(\cdot)$.

If the individual is an expected utility maximizer, we know that V(·), or some increasing transformation of it, will take the linear form $V(F) \equiv \int U(x)dF(x)$, where U(·) is known as the individual's *von Neumann-Morgenstern utility function*. In such a case we know that the individual will rank the distributions F*(·) and $F_0(\cdot)$ according to the sign of

$$V(F^*) - V(F_0) = \int U(x)\cdot[dF^*(x)-dF_0(x)] \qquad (1)$$

Expected utility theory has provided us with many results linking the von Neumann-Morgenstern utility function U(·) with properties of risk preferences. For example, the individual will prefer all first order stochastically dominating shifts if and only if U(x) is increasing in x, and will be *risk averse*, i.e. averse to all mean preserving increases in risk, if and only if U(x) is concave in x.

We also know that the von Neumann-Morgenstern utility

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function $U(\cdot)$ is *cardinal* in the sense that the linear preference functional $V^*(F) \equiv \int U^*(x)dF(x)$ will represent the same preferences as $V(\cdot)$ if and only if $U^*(\cdot) = \beta \cdot U(\cdot) + \gamma$ for some constants $\beta > 0$ and γ.

Consider now some $V(\cdot)$ which is *not* of the linear (i.e. expected utility) form. In such a case there exists no von Neumann-Morgenstern utility function $U(\cdot)$. However, if $V(\cdot)$ is at least *differentiable* (i.e. "smooth"), we know that it will be *locally linear* in the sense that we may take a first order Taylor expansion about any distribution $F_o(\cdot)$:

$$V(F^*) - V(F_o) = \int U(x;F_o)\cdot[dF^*(x) - dF_o(x)] + o(\|F^* - F_o\|) \qquad (2)$$

where $\|\cdot\|$ is the L^1 metric $\|F^* - F_o\| \equiv \int|F^*(x) - F_o(x)|dx$, and the term $o(\cdot)$ denotes a function which is zero at zero and of higher order than its argument. Comparing the first order term in this expansion with equation (1), it is clear that the individual will evaluate alternative *differential* shifts of probability mass from the distribution $F(\cdot)$ precisely as would an expected utility maximizer with von Neumann-Morgenstern utility function $U(\cdot;F_o)$. We refer to the function $U(\cdot;F_o)$ as the individual's *local utility function* at the distribution $F_o(\cdot)$.[2]

The technique of "generalized expected utility analysis" (e.g. Machina [1982,1983]) essentially consists of exploiting the linear approximation (2) to generalize the fundamental tools, concepts and results of expected utility analysis to the case of nonlinear but smooth preferences over probability distributions. Thus, equation (2) implies that $V(\cdot)$ will prefer all differential first order stochastically dominating shifts from $F_o(\cdot)$ if and only if the local utility function $U(\cdot;F_o)$ is increasing, and $V(\cdot)$ will be averse to all differential mean preserving increases in risk about $F_o(\cdot)$ if and only if $U(\cdot;F_o)$ is concave. To extend these results to *non-differential* changes, consider any *path* $\{F(\cdot;\alpha) \mid \alpha \in [0,1]\}$ from the distribution $F_o(\cdot) = F(\cdot;0)$ to the distribution $F^*(\cdot) = F(\cdot;1)$ which is "smooth" enough so that the derivative $d[\|F(\cdot;\alpha) - F(\cdot;\bar{\alpha})\|]/d\alpha$ exists at $\alpha = \bar{\alpha}$ for all $\bar{\alpha} \in [0,1]$, as for example with the "straight line" path $F(\cdot;\alpha) \equiv \alpha\cdot F^*(\cdot) + (1-\alpha)\cdot F_o(\cdot)$. By (2) we therefore have

$$\left.\frac{dV(F(\cdot;\alpha))}{d\alpha}\right|_{\bar{\alpha}} = \left.\frac{d}{d\alpha}\left[\int U(x;F(\cdot;\bar{\alpha}))dF(x;\alpha)\right]\right|_{\bar{\alpha}} \qquad (3)$$

and the Fundamental Theorem of Integral Calculus yields

$$V(F^*) - V(F_0) = \int_0^1 \left[\frac{d}{d\alpha} \left[\int U(x;F(\cdot;\overline{\alpha}))dF(x;\alpha) \right] \Big|_{\overline{\alpha}} \right] d\overline{\alpha} \qquad (4)$$

which shows how the individual's ranking of these two distributions depends upon the local utility functions along the path between them. In Machina [1982] such path integrals were used to demonstrate that $V(\cdot)$ will exhibit *global* first order stochastic dominance preference and *global* risk aversion if and only if its local utility functions $U(x;F)$ are respectively nondecreasing and concave at each distribution $F(\cdot) \in D[a,b]$.

For a more formal and complete development of these concepts as well as applications of this approach, the reader is referred to Machina [1982,1983] as well as the references cited below. We turn now to the cardinal properties of local utility functions.

2. ADDITIVE INVARIANCE

Equations (2) and (4) show how the set of local utility functions $\{U(\cdot;F) | F(\cdot) \in D[a,b]\}$ derived from a preference functional can be used to determine its ranking of both differential and global shifts from any distribution $F_0(\cdot)$. Our first result is that any other set of functions $\{U^*(\cdot;F) | F(\cdot) \in D[a,b]\}$ satisfying $U^*(x;F) \equiv U(x;F) + \gamma(F)$ will *also* generate the same differential and global rankings via these equations. In other words, we can add a different constant $\gamma(F)$ to each local utility function $U(\cdot;F)$ without changing the original ranking.

To see that they generate the same rankings over differential shifts, recall that the individual will prefer (not prefer) such a shift from $F_0(\cdot)$ if and only if the first order term in (2) is positive (negative). However, since the term $[dF^*(x) - dF_0(x)]$ integrates to zero, we have that

$$\begin{aligned} & \int U^*(x;F_0)\cdot[dF^*(x) - dF_0(x)] \\ = & \int [U(x;F_0) + \gamma(F_0)]\cdot[dF^*(x) - dF_0(x)] \qquad (5) \\ = & \int U(x;F_0)\cdot[dF^*(x) - dF_0(x)], \end{aligned}$$

so that the first order terms generated by the local utility functions $U^*(\cdot;F_0)$ and $U(\cdot;F_0)$ will be identical for all differential shifts about all $F_0(\cdot)\in D[a,b]$.

To extend this equivalence to global rankings, note that

$$\frac{d}{d\alpha}\Big[\int U^*(x;F(\cdot;\bar{\alpha}))dF(x;\alpha)\Big]\Big|_{\bar{\alpha}}$$

$$= \frac{d}{d\alpha}\Big[\int [U(x;F(\cdot;\bar{\alpha}))+\gamma(F(\cdot;\bar{\alpha}))]\cdot dF(x;\alpha)\Big]\Big|_{\bar{\alpha}} \qquad (6)$$

$$= \frac{d}{d\alpha}\Big[\int U(x;F(\cdot;\bar{\alpha}))dF(x;\alpha)\Big]\Big|_{\bar{\alpha}}$$

so that the right side of equation (4) will be identical for any path $\{F(\cdot;\alpha)|\alpha\in[0,1]\}$.

3. MULTIPLICATIVE INVARIANCE

Given the set $\{U(\cdot;F)|F(\cdot)\in D[a,b]\}$ of local utility functions generated from a preference functional $V(\cdot)$, our second result is that any other set of functions $\{U^*(\cdot;F)|F(\cdot)\in D[a,b]\}$ obtained by means of the transformation $U^*(x;F) \equiv \beta(V(F))\cdot U(x;F)$ will also generate the same ranking, for any positive continuous function $\beta(\cdot)$. Note that this implies we can multiply different local utility functions by different positive constants, provided that the local utility functions corresponding to indifferent distributions are multiplied by the same constant.

To see this, define the function $V(\cdot)$ and functional $V^*(\cdot)$ by

$$\phi(\upsilon)\equiv\int_{-\infty}^{\upsilon}\beta(s)\cdot ds \qquad \text{and} \qquad V^*(F)\equiv\phi(V(F)) \qquad (7)$$

By the chain rule, we have that the local utility function of $V^*(\cdot)$ is given by $\phi'(V(F))\cdot U(\cdot;F) = \beta(V(F))\cdot U(\cdot;F) = U^*(\cdot,F)$, which establishes that the family of local utility functions $\{U^*(\cdot;F)|F(\cdot)\in D[a,b]\}$ from the previous paragraph comes from an increasing transformation $\phi(V(F))$ of the original preference function $V(\cdot)$, and hence represents the same global preference ranking.

Finally, we note that while the local utility functions used in some of the following references are derived from notions of differentiability which make weaker assumptions on the higher order term (1), these local utility functions can also be shown to exhibit the cardinal properties discussed above.

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NOTES

1. I am grateful to Chew Soo Hong and Joel Sobel for helpful comments.

2. Applying the integration by parts formula from Machina [1982,Lemma 2] (see also Feller [1971,p.150]) yields that the first order term in the Taylor expansion is equal to $-\int[F^*(x)-F(x)]\cdot U'(x;F)\cdot dx$, which is seen to be precisely the classical variational derivative of the functional $V(\cdot)$ with respect to the function $F(\cdot)$.

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