A Note on the Relation Between Models in Discrete and Continuous Time

Joel Watson*

November 2020

Abstract

This note provides a uniform convergence result that relates (i) a sequence in discrete time that is defined inductively with respect to a transition function and (ii) the solution of a differential initial-value problem in continuous time that is derived from the same transition function.

1 Introduction

Consider a discrete-time modeling exercise with a solution that can be expressed inductively. The model specifies a state space $X \subset \mathbb{R}^n$, an initial state $x^0 \in X$, and a period length $\Delta > 0$. The solution (the model’s prediction) is a sequence $\{x^k(\Delta)\}_{k=0}^{\infty}$, where $x^k$ denotes the state in period $k$. Suppose that the solution is characterized by a function $f: X \times \mathbb{R}_+ \rightarrow X$, such that

$$x^0(\Delta) = x^0 \quad \text{and} \quad x^{k+1}(\Delta) = f(x^k(\Delta), \Delta) \quad \text{for each } k \in P,$$

where $P = \{0, 1, 2, \ldots\}$ is the set of natural numbers.

In some settings, whereas the transition function $f$ can be calculated at least implicitly, it would be difficult to solve for $\{x^k(\Delta)\}$. Nevertheless, it may be possible to calculate a continuous-time limit as $\Delta \rightarrow 0$ using the transition function. Specifically, suppose that

$$h(x) \equiv \lim_{\Delta \rightarrow 0^+} \frac{f(x, \Delta) - x}{\Delta},$$

is well defined (that is, the limit exists) for every $x \in X$, and suppose there is a unique solution $y: \mathbb{R}_+ \rightarrow X$ to the system of differential equations $Dy = h(y)$ with initial value $y(0) = x^0$. Here $Dy$ denotes the first derivative (Jacobian) matrix of $y$.

A key issue is whether $y$ is a good approximation of $\{x^k(\Delta)\}$ for small values of $\Delta$. To make this precise, let us convert the sequence into a function that gives the state

x as a function of the time t on the continuum, so that it is comparable to y. For simplicity, let it be the piece-wise linear function \( \hat{x}: \mathbb{R}_+ \times \mathbb{R}_+ \times X \rightarrow X \) defined by

\[
\hat{x}(t, x^0; \Delta) \equiv x^{[t/\Delta]}(\Delta) + \left( t - \frac{[t/\Delta]}{\Delta} \right) \left( x^{[t/\Delta]+1}(\Delta) - x^{[t/\Delta]}(\Delta) \right),
\]

where for any number \( a \geq 0 \), \([a]\) denotes the largest integer \( k \) satisfying \( k \leq a \). Thus, for every time \( t \) satisfying \( t = k\Delta \) for some integer \( k \), we have \( \hat{x}(t, \Delta) = x^k(\Delta) \), and the function is continuous with linear segments.

Suppose we are interested in the comparison for an interval \([0, T]\) in continuous time. Then we should ask whether \( \hat{x}(\cdot, x^0; \Delta) \) converges to \( y(t) \) uniformly in \( t \in [0, T] \) as \( \Delta \) approaches zero. If so, we can say that the function \( y \) well approximates the sequence \( \{x^k(\Delta)\}_{k=0}^{\lfloor T/\Delta \rfloor} \) on \([0, T]\). This note shows that convergence is assured under appropriate conditions on the primitives that one would hope to be able to check in applications. It also gives a similar result for more challenging settings in which \( x^{k+1} \) is a function of both \( x^k \) and \( x^{k-1} \). One component of the analysis utilizes a standard result on the convergence of Euler’s one-step method for approximating the solution of an initial-value problem for ordinary differential equations.

### 2 Results

Note that if \( h \) is well defined and \( f \) is differentiable, then it must be that \( f(x, 0) = x \) and \( h(x) = D_\Delta f(x, 0) \) for all \( x \in X \), the latter conclusion following from L’Hôpital’s rule. Some assumptions described below are made directly on \( h \) for convenience but could also be put in terms of \( f \). Let \( D \) denote the first-derivative, as already mentioned, and \( D^2 \) the second derivative. Subscripts will indicate differentiation with respect to a subset of the variables (either \( x \) or \( \Delta \)). Depending on the functions being dealt with, the derivatives may be matrices or scalars. Let \( P_+ \) denote the positive integers \( \{1, 2, \ldots\} \).

Let us start by comparing \( \{x^k(\Delta)\}_{k=0}^{\lfloor T/\Delta \rfloor} \) with another sequence \( \{z^k(\Delta)\}_{k=0}^{\lfloor T/\Delta \rfloor} \) that is defined by the natural discrete-time approximation of \( y \) using the known derivative \( h \):

\[
z^0(\Delta) = x^0 \quad \text{and} \quad z^{k+1}(\Delta) = z^k(\Delta) + h(z^k(\Delta))\Delta \quad \text{for each} \ k \in P.
\]

We translate this sequence into a function of continuous time, \( \hat{z}: \mathbb{R}_+ \times \mathbb{R}_+ \times X \rightarrow X \), by defining

\[
\hat{z}(t, x^0; \Delta) \equiv z^{[t/\Delta]}(\Delta) + \left( t - \frac{[t/\Delta]}{\Delta} \right) \left( z^{[t/\Delta]+1}(\Delta) - z^{[t/\Delta]}(\Delta) \right).
\]

**Theorem 1:** Let \( X \) be an open subset of \( \mathbb{R}^n \), where \( n \) is finite, and let \( \Gamma \) be an open interval of \( \mathbb{R} \) containing 0. Let \( f: X \times \Gamma \rightarrow X \) be a thrice continuously differentiable function satisfying \( f(x, 0) = x \), and define \( h: X \rightarrow \mathbb{R}^n \) by \( h(x) = D_\Delta f(x, 0) \) for all
\( x \in X \). Assume \( D^2_\Delta f(\cdot, 0) \) and \( D^2 h \) are bounded. Let functions \( \hat{x} \) and \( \hat{z} \) be defined by Expressions 1, 3, 4, and 5. Fix a positive number \( T \). Then \( \hat{x}(\cdot; \cdot; \Delta) \) and \( \hat{z}(\cdot; \cdot; \Delta) \) converge uniformly on \([0, T] \times X\) as \( \Delta \to 0 \).

The next result establishes the uniform convergence of \( \hat{z} \) to \( y \) which, together with the first theorem, establishes that \( \hat{x} \) converges uniformly to \( y \). This theorem is a slight extension of a standard convergence result for Euler’s one-step approximation of the solution of an initial-value problem (see, for instance, Suli and Mayers 2003, pages 317–323).

**Theorem 2:** Let \( X \) be an open subset of \( \mathbb{R}^n \), where \( n \) is finite, and consider any twice-differentiable function \( h : X \to \mathbb{R}^n \). Assume that the second derivatives of \( h \) are bounded. For a given \( x^0 \in X \), suppose there is a unique solution to the system of differential equations \( Dy = h(y) \) with initial value \( y(0) = x^0 \). Let function \( \hat{z} \) be defined by Expressions 4 and 5. Consider any function \( \tilde{x}^0 : \mathbb{R}_+ \to X \) such that \( \tilde{x}^0(\Delta) \) converges to \( x^0 \) as \( \Delta \to 0 \), and let \( T \) be any positive number. Then \( \hat{z}(\cdot; \tilde{x}^0(\Delta); \Delta) \) converges to \( y \) uniformly on \([0, T] \) as \( \Delta \to 0 \).

The third result is a version of Theorem 1 for a setting in which the first two elements of the sequence \( \{x^k(\Delta)\} \) are given, and the transition function has arguments \( x^k \) and \( x^{k-1} \). That is, the state in the next period is a function of the current-period state and its value in the previous period:

\[
x^0(\Delta) = x^0, \quad x^1(\Delta) = x^1, \quad \text{and} \quad x^{k+1}(\Delta) = f(x^k(\Delta), x^{k-1}(\Delta), \Delta) \quad \text{for each} \quad k \in \mathbb{P}. \quad (6)
\]

Let us limit attention to the case of \( n = 1 \), so all arguments are scalars. Let \( f_1 \) denote the derivative of \( f \) with respect to its first argument, \( x^k \); let \( f_2 \) denote the derivative of \( f \) with respect to its second argument, \( x^{k-1} \), and let \( f_\Delta \) denote the derivative of \( f \) with respect to its third argument, \( \Delta \). The definition of function \( \hat{x} : \mathbb{R}_+ \times \mathbb{R}_+ \times X \to X \) is the same as in the first setting except now we write \( \hat{x}(t, x^0, x^1; \Delta) \) to show the dependence on the first two elements of the sequence.

Before stating the theorem, it is useful to present some intuition. Consider a point in continuous time \( t \in \mathbb{R}_+ \). For any \( \Delta \), the period closest to \( t \) is \( k = [T/\Delta] \). Let us describe the slope

\[
\frac{\hat{x}(t + \Delta, x^0, x^1; \Delta) - \hat{x}(t, x^0, x^1; \Delta)}{\Delta} = \frac{x^{k+1}(\Delta) - x^k(\Delta)}{\Delta}
\]

and give a heuristic characterization of the limit as \( \Delta \) converges to 0. Using Equation 6, this slope is

\[
\frac{f(x^k(\Delta), x^{k-1}(\Delta), \Delta) - x^k(\Delta)}{\Delta}.
\]

For convenience, imagine that \( x^k(\Delta) \) is a constant \( x \). The key thing to notice is that \( x^{k-1}(\Delta) \) changes as \( \Delta \) approaches 0, and for the discrete-time model to converge, it
must be that \( x^{k-1}(\Delta) \to x \). Then as \( \Delta \) approaches 0, its direct effect on the slope is roughly \( \lim_{\Delta \to 0^+} \frac{f(x, x^k(\Delta), 0) - x}{\Delta} = -f_2(x, x, 0) \cdot \lim_{\Delta \to 0^+} \frac{x - x^{k-1}(\Delta)}{\Delta} \).

We thus conclude that the slope of \( y \) at time \( t \) is

\[
h(x) = f_\Delta(x, x, 0) - f_2(x, x, 0) \cdot \lim_{\Delta \to 0^+} \frac{x - x^{k-1}(\Delta)}{\Delta}.
\]

Further, \( \lim_{\Delta \to 0^+} \frac{x - x^{k-1}(\Delta)}{\Delta} \) must be the same slope \( h(x) \). So we have \( h(x) = f_\Delta(x, x, 0) - f_2(x, x, 0)h(x) \); rearranging yields \( h(x) = f_\Delta(x, x, 0)/(1 + f_2(x, x, 0)) \). As before, bounds on the derivatives of \( f \), and additional conditions on \( f_2 \) and the initial values, will be sufficient for a convergence result.

**Theorem 3:** Let \( X \) be an open and bounded subset of \( \mathbb{R} \) and let \( \Gamma \) be an open interval of \( \mathbb{R} \) containing 0. Let \( f : X \times X \times X \to X \) be a twice continuously differentiable function satisfying \( f(x, x, 0) = x \). Define \( h : X \to \mathbb{R}^n \) by \( h(x) \equiv f_\Delta(x, x, 0)/(1 + f_2(x, x, 0)) \), for all \( x \in X \). Assume that \( f_2 \) is bounded below 1, the second derivatives of \( f \) are bounded, and \( D^2 h \) is bounded. Let functions \( \hat{x} \) and \( \tilde{z} \) be defined by Expressions 6, 3 (but including \( x^1 \) as an argument of \( \hat{x} \)), 4, and 5. Consider any functions \( \tilde{x}^0 : \mathbb{R}_+ \to X \) and \( \tilde{x}^1 : \mathbb{R}_+ \to X \) such that \( \tilde{x}^0(\Delta) \) and \( \tilde{x}^1(\Delta) \) both converge to a given \( x^0 \in X \) as \( \Delta \to 0 \), and suppose \( (\tilde{x}^1(\Delta) - \tilde{x}^0(\Delta)) / \Delta \) is bounded. Let \( T \) be any positive number. Then \( \hat{x}(\cdot, \tilde{x}^0(\Delta), \tilde{x}^1(\Delta); \Delta) \) and \( \hat{z}(\cdot, \tilde{x}^0(\Delta), \tilde{x}^1(\Delta); \Delta) \) converge uniformly on \([0, T]\) as \( \Delta \to 0 \).

### 3 Proofs

**Proof of Theorem 1:** Let \( B \) be a number such that, for every \( x \), the components of \( D^3 f, D h, \) and \( D^2 h \) are all between \(-B\) and \( B \). (That \( D^2 h \) is bounded implies \( D h \) is bounded.) The Euclidean norm is denoted by \(| \cdot | \) below.

The main steps of the proof involve examining the difference between \( x^{k+1}(\Delta) \) and \( z^{k+1}(\Delta) \); implications for the functions \( \hat{x} \) and \( \hat{z} \) are worked out at the end. Using Equations 1 and 4,

\[
x^{k+1}(\Delta) - z^{k+1}(\Delta) = f(x^k(\Delta), \Delta) - z^k(\Delta) - h(z^k(\Delta))\Delta.
\]

Substituting for the first term using the first-degree Taylor polynomial centered at \((x^k(\Delta), 0)\), and using the identity \( h(x) = D_\Delta f(x, 0) \) and that \( f(x, 0) = x \), we obtain

\[
x^{k+1}(\Delta) - z^{k+1}(\Delta) = f(x^k(\Delta), 0) + D_\Delta f(x^k(\Delta), 0)\Delta + E^2(k, \Delta) - z^k(\Delta) - h(z^k(\Delta))\Delta
\]

\[
= x^k(\Delta) - z^k(\Delta) + D_\Delta f(x^k(\Delta), 0)\Delta - h(z^k(\Delta))\Delta + E^2(k, \Delta)
\]

\[
= x^k(\Delta) - z^k(\Delta) + [h(x^k(\Delta)) - h(z^k(\Delta))]\Delta + E^2(k, \Delta),
\]
where $E^2(k, \Delta) = \frac{1}{2}D^2_{\Delta} f(x^k(\Delta), \hat{\Delta}) \Delta^2$ for some $\hat{\Delta} \in [0, \Delta]$, and therefore

$$|E^2(k, \Delta)| \leq B \Delta^2.$$ 

Because $h$ maps one $n$-dimensional vector to another, let us write

$$h(x) = \begin{pmatrix} h^1(x) \\ h^2(x) \\ \vdots \\ h^n(x) \end{pmatrix},$$

where $h^i$ denotes the $i$th component. We can use the first-degree Taylor approximation for $h$, centered at $z^k(\Delta)$, to write

$$h(x^k(\Delta)) = h(z^k(\Delta)) + Dh(z^k(\Delta))(x^k(\Delta) - z^k(\Delta)) + \tilde{E}^2(k, \Delta),$$

where $\tilde{E}^2(k, \Delta)$ is a vector whose $i$th component is equal to

$$\frac{1}{2} (x^k(\Delta) - z^k(\Delta))^T D^2 h^i(\hat{x}^i)(x^k(\Delta) - z^k(\Delta))$$

for some $\hat{x}^i$ between $x^k(\Delta)$ and $z^k(\Delta)$. Here $D^2 h^i$ is the Hessian matrix of $h^i$, which is a real-valued function. The bound $B$ implies that

$$|\tilde{E}^2(k, \Delta)| \leq n^2 B |x^k(\Delta) - z^k(\Delta)|^2.$$

Using Equation 7 to substitute for $h(x^k(\Delta))$ in the expression for $x^{k+1}(\Delta) - z^{k+1}(\Delta)$, we obtain

$$x^{k+1}(\Delta) - z^{k+1}(\Delta) = x^k(\Delta) - z^k(\Delta) + [h(x^k(\Delta)) - h(z^k(\Delta))] \Delta + E^2(k, \Delta)$$

$$= x^k(\Delta) - z^k(\Delta) + Dh(z^k(\Delta)) (x^k(\Delta) - z^k(\Delta)) \Delta$$

$$+ E^2(k, \Delta) \Delta + E^2(k, \Delta).$$

Using the error bounds for $\tilde{E}^2$ and $E^2$, and noting that

$$|Dh(z^k(\Delta)) (x^k(\Delta) - z^k(\Delta)) \Delta| \leq nB \Delta |(x^k(\Delta) - z^k(\Delta))|,$$

we find that

$$|x^{k+1}(\Delta) - z^{k+1}(\Delta)| \leq |x^k(\Delta) - z^k(\Delta)| + nB \Delta |(x^k(\Delta) - z^k(\Delta))|$$

$$+ n^2 B \Delta |x^k(\Delta) - z^k(\Delta)|^2 + B \Delta^2 \quad (8)$$

For any fixed $\Delta$ and $T$, we next construct an upper bound on $|x^k(\Delta) - z^k(\Delta)|$, denoted $b^k(\Delta)$, for each $k = 1, 2, \ldots, \lfloor T/\Delta \rfloor$. Set $b^1(\Delta) = B \Delta$, which is clearly an upper bound on $|x^1(\Delta) - z^1(\Delta)|$, from Inequality 8, the fact that $\Delta \leq 1$, and because $x^0(\Delta) =
By letting \( \Delta \) be small enough, the upper bounds \( b^2(\Delta), b^3(\Delta), \ldots, b^{[T/\Delta]}(\Delta) \) will be constructed in a way that ensures they are all weakly greater than \( B\Delta \) and weakly less than 1. With reference to the right side of Inequality 8, this means that
\[
b^k(\Delta) + nB\Delta b^k(\Delta) + n^2B\Delta b^k(\Delta)^2 + B\Delta^2 \leq b^k(\Delta) (1 + \Delta (1 + nB + n^2B)).
\]
Inductively define \( b^2(\Delta), b^3(\Delta), \ldots, b^{[T/\Delta]}(\Delta) \) by setting
\[
b^{k+1}(\Delta) = b^k(\Delta) (1 + \Delta (1 + nB + n^2B)) \tag{9}
\]
for all \( k = 1, 2, \ldots, [T/\Delta] - 1 \). Then these are valid upper bounds if they are all in the interval \([B\Delta, 1]\). Clearly the sequence is increasing, so \( b^k(\Delta) \geq B\Delta \) for each \( k \). Recall that we have set \( b^1(\Delta) = B\Delta \), and so we can use Equation 9 to explicitly solve for \( b^{[T/\Delta]}(\Delta) \):
\[
b^{[T/\Delta]}(\Delta) = B\Delta (1 + \Delta (1 + nB + n^2B)) ^{[T/\Delta]}.
\]
Note that, from the definition of the natural number \( e \), the right side is less than \( B\Delta e^{1+nB+n^2B} \). For \( \Delta \) small enough, this number is below 1 and our presumptions hold.

Further, for any desired \( \varepsilon > 0 \) we can find a number \( \overline{\Delta} > 0 \) such that \( \Delta < \overline{\Delta} \) implies that \( B\Delta e^{1+nB+n^2B} < \varepsilon \), and so \( b^k(\Delta) < \varepsilon \) for every \( k = 1, 2, \ldots, [T/\Delta] - 1 \). This means \( |\hat{x}(k\Delta, x^0; \Delta) - \hat{x}(k\Delta, x^0; \Delta)| < \varepsilon \) for every \( k \in P \) and every \( x^0 \in X \). From the definitions of these functions as piece-wise linear, the same inequality holds at all other values of \( t \in [0, T] \), proving the theorem. \( \Box \)

**Proof of Theorem 2:** This proof is similar to the previous proof. For simplicity, let us focus on the case in which \( \hat{x}^0(\Delta) = x^0 \) for every \( \Delta \). Extending the proof to the case of a nontrivial limit of initial values is straightforward because of the uniform nature of convergence and that \( y \) is uniformly continuous in its initial value.

Let \( B \) be a number such that, for every \( x \), the components of \( h, Dh \), and \( D^2h \) are all between \(-B\) and \( B \). (That \( D^2h \) is bounded implies \( Dh \) and \( h \) are bounded.) Consider any \( \Delta \geq 0 \) and nonnegative integer \( k \). Using the definition of \( \{z^k(\Delta)\}_{k=0}^\infty \) and that \( y(t) = y(s) + \int_s^t h(y(r)) dr \) for \( t \geq s \), we have
\[
z^{k+1}(\Delta) - y(\Delta(k + 1)) = z^k(\Delta) + h(z^k(\Delta)) \Delta - y(\Delta k) - \int_{\Delta k}^{\Delta(k+1)} h(y(r)) dr \tag{10}
\]
Considering the upper limit of the integral as a variable, let us write the integral using the first-degree Taylor expansion centered at \( \Delta k \):
\[
\int_{\Delta k}^{\Delta(k+1)} h(y(r)) dr = 0 + h(y(\Delta k)) \Delta + E^2(k, \Delta),
\]
where \( E^2(k, \Delta) \) is a vector whose \( i \)th component is equal to
\[
(1/2) Dh^i(y(\hat{r}^i)) Dy(\hat{r}^i) \Delta^2 = (1/2) Dh^i(y(\hat{r}^i)) h(\hat{r}^i) \Delta^2
\]
for some \( r^i \in [\Delta k, \Delta (k + 1)] \). We thus know that
\[
|E^2(k, \Delta)| \leq nB^2\Delta^2.
\]
Substituting for the integral in Equation 10 yields
\[
z^{k+1}(\Delta) - y(\Delta(k + 1)) = z^k(\Delta) - y(\Delta k) + [h(z^k(\Delta)) - h(y(\Delta k))] \Delta - E^2(k, \Delta). \tag{11}
\]
We can write \( h(z^k(\Delta)) \) using the first-degree Taylor polynomial centered at \( y(\Delta k) \):
\[
h(z^k(\Delta)) = h(y(\Delta k)) + Dh(y(\Delta k)) (z^k(\Delta) - y(\Delta k)) + \tilde{E}^2(k, \Delta),
\]
where, similar to what we found in the proof of the first theorem,
\[
|\tilde{E}^2(k, \Delta)| \leq nB^2 |z^k(\Delta) - y(\Delta k)|^2.
\]
Substituting for \( h(z^k(\Delta)) \) in Equation 11 yields
\[
z^{k+1}(\Delta) - y(\Delta(k + 1)) = z^k(\Delta) - y(\Delta k) + Dh(y(\Delta k)) (z^k(\Delta) - y(\Delta k)) + \tilde{E}^2(k, \Delta + E^2(k, \Delta).
\]
Using the error bounds for \( \tilde{E}^2 \) and \( E^2 \), and noting that
\[
|Dh(y(\Delta k)) (z^k(\Delta) - y(\Delta k)) \Delta| \leq nB\Delta |z^k(\Delta) - y(\Delta k)|,
\]
we find that
\[
|z^{k+1}(\Delta) - y(\Delta(k + 1))| \leq |z^k(\Delta) - y(\Delta k)| + nB\Delta |z^k(\Delta) - y(\Delta k)|
+ n^2B\Delta |z^k(\Delta) - y(\Delta k)|^2 + nB^2\Delta^2. \tag{12}
\]
Similar to the final step in the proof of the first theorem, for any fixed \( \Delta \) and \( T \) we construct an upper bound on \( |z^k(\Delta) - y(\Delta k)| \), denoted \( b^k(\Delta) \), for each \( k = 1, 2, \ldots, \lfloor T/\Delta \rfloor \). Set \( b^1(\Delta) = nB^2\Delta \), which is clearly an upper bound on \( |z^k(\Delta) - y(\Delta k)| \), from Inequality 12 and because \( z^0(\Delta) = y(0) = x^0 \). By letting \( \Delta \) be small enough, the upper bounds \( b^2(\Delta), b^3(\Delta), \ldots, b^{\lfloor T/\Delta \rfloor}(\Delta) \) will be constructed in a way that ensures they are all weakly greater than \( nB^2\Delta \) and weakly less than 1. With reference to the right side of Inequality 12, this means that
\[
b^k(\Delta) + nB\Delta b^k(\Delta) + n^2B^2\Delta b^k(\Delta)^2 + nB^2\Delta^2 \leq b^k(\Delta)(1 + \Delta(1 + nB + n^2B)).
\]
Inductively define \( b^2(\Delta), b^3(\Delta), \ldots, b^{\lfloor T/\Delta \rfloor}(\Delta) \) by setting
\[
b^{k+1}(\Delta) = b^k(\Delta)(1 + \Delta(1 + nB + n^2B)) \tag{13}
\]
for all \( k = 1, 2, \ldots, \lfloor T/\Delta \rfloor - 1 \). Then these are valid upper bounds if they are all in the interval \([nB^2\Delta, 1]\). Clearly the sequence is increasing, so \( b^k(\Delta) \geq nB^2\Delta \) for each
k. Recall that we have set \( b^1(\Delta) = nB^2\Delta \), and so we can use Equation 13 to explicitly solve for \( b^{T/\Delta}(\Delta) \):

\[
b^{T/\Delta}(\Delta) = nB^2\Delta (1 + \Delta (1 + nB + n^2B))^{T/\Delta}.
\]

Note that, from the definition of the natural number \( e \), the right side is less than \( nB^2\Delta e^{1+nB+n^2B} \). For \( \Delta \) small enough, this number is below 1 and our presumptions hold.

Further, for any desired \( \varepsilon > 0 \) we can find a number \( \Delta > 0 \) such that \( \Delta < \Delta \) implies that \( nB^2\Delta e^{1+nB+n^2B} < \varepsilon \), and so \( b^k(\Delta) < \varepsilon \) for every \( k = 1, 2, \ldots, \lfloor T/\Delta \rfloor - 1 \).

The implication for \( |\hat{z}(\cdot, x^0; \Delta) - y| \) is derived with one more step. Note that \( \hat{z}(\cdot, x^0; \Delta) \) is a piecewise linear spline with mesh points \( t = 0, \Delta, 2\Delta, \ldots \), where the \( k \)th line segment has the slope \( h(k\Delta) \). Writing the first-degree Taylor polynomial of \( y \) centered at \( k\Delta \) to approximate \( y(k|de+\Delta) \), we find that \( Dy \) is equal to \( (y(k\Delta + \Delta) - y(k\Delta))/\Delta \) plus an error term of order \( \Delta^2 \). Doing the same to approximate \( y(t) \) for any \( t \in [k\Delta, k\Delta + \Delta] \) gives \( y(t) \) equals \( y(k\Delta) + Dy(k\Delta)(t - k\Delta) \) plus an error term of order \( \Delta^2 \). Using the first calculation to substitute for \( Dy(k\Delta) \) in the second, we get

\[
y(t) = y(k\Delta) + \left( \frac{t - k\Delta}{\Delta} \right) (y(k\Delta + \Delta) - y(k\Delta)) + E^2(k, \Delta),
\]

where \( E^2(k, \Delta) \) is of order \( \Delta^2 \). Likewise, from Equation 5 and the bound on \( b^k(\Delta) \) calculated above, we have

\[
\hat{z}(t, x^0; \Delta) = y(k\Delta) + \left( \frac{t - k\Delta}{\Delta} \right) (y(k\Delta + \Delta) - y(k\Delta)) + \tilde{E}^2(k, \Delta),
\]

where \( |\tilde{E}^2(k, \Delta)| < 3\varepsilon \), proving that \( |\hat{z}(t, x^0; \Delta) - y(t)| < 3\varepsilon \) for all \( t \in [0, T] \).

**Proof of Theorem 3:** Because \( f \) is continuously differentiable, it must be that \( f(x, x, 0) = x \). Let \( B \) be a number such that, for every \( x \), the components of \( Df \) and \( D^2f \), and values \( h \), \( Dh \), and \( (x^1(\Delta) - x^0(\Delta))/\Delta \) are all between \(-B\) and \( B \), and additionally assume that \( B \) is larger than \( \sup\{x - x' \mid x, x' \in X\} \). (That \( D^2h \) and the second derivatives of \( f \) are bounded implies bounds on \( Dh \) and \( Df \).) Define

\[
m^k(\Delta) \equiv \frac{x^{k+1}(\Delta) - x^k(\Delta)}{\Delta} - h(x^{k+1}(\Delta)).
\]

**Lemma:** There exist a number \( \beta \), a decreasing function \( \kappa : \mathbb{R}_+ \to \mathbb{R}_+ \), and a number \( \Delta > 0 \) with the following properties for all \( \Delta < \Delta \). First, \( |m^k(\Delta)| < \beta \Delta \) for every integer \( k > \kappa(\Delta) \). Second, \( |m^k(\Delta)| \leq 2B \) for every integer \( k \leq \kappa(\Delta) \). Third, \( \Delta \kappa(\Delta) \) converges to 0 as \( \Delta \to 0 \).
// Proof of the Lemma: By the definition of $h$ and the transition function $f$, we have

$$m^k(\Delta) = \frac{f(x^k(\Delta), x^{k-1}(\Delta), \Delta) - x^k}{\Delta} = \frac{f_\Delta(x^k(\Delta), x^k(\Delta), 0) - f_2(x^k(\Delta), x^k(\Delta), 0)}{1 + f_2(x^k(\Delta), x^k(\Delta), 0)}.$$  

Note that, using the first-degree Taylor polynomial centered at $(x^k(\Delta), x^k(\Delta), 0)$, we can write

$$f(x^k(\Delta), x^{k-1}(\Delta), \Delta) = f(x^k(\Delta), x^k(\Delta), 0) + f_2(x^k(\Delta), x^k(\Delta), 0)(x^{k-1} - x^k) + f_\Delta(x^k(\Delta), x^k(\Delta), 0)\Delta + E^2(k, \Delta),$$

where $E^2(k, \Delta)$ is $1/2$ times the Hessian matrix of $f$ as a function of its second and third arguments, pre- and post-multiplied by the vector $(x^{k-1} - x^k, \Delta)$. Because the second derivatives of $f$ are bounded by $B$, the error term $E^2(k, \Delta)$ satisfies

$$|E^2(k, \Delta)| \leq B|x^k(\Delta) - x^{k-1}(\Delta)|^2 + 2B|x^k(\Delta) - x^{k-1}(\Delta)|\Delta + B\Delta^2$$

$$= \frac{|x^k(\Delta) - x^{k-1}(\Delta)|^2}{\Delta^2} + 2B \left| \frac{x^k(\Delta) - x^{k-1}(\Delta)}{\Delta} \right|^2 + B\Delta^2.$$  

Using $f(x^k(\Delta), x^k(\Delta), 0) = x^k(\Delta)$, combining the fractions, and simplifying terms yields:

$$m^k(\Delta) = -f_2(x^k(\Delta), x^k(\Delta), 0) \left( x^k(\Delta) - x^{k-1}(\Delta) - h(x^k) \right) + \frac{E^2(k, \Delta)}{\Delta},$$

and so

$$m^k(\Delta) = -f_2(x^k(\Delta), x^k(\Delta), 0) \cdot m^{k-1}(\Delta) + \frac{E^2(k, \Delta)}{\Delta}. \quad (14)$$

Recall that $|h|$ and $|x^k(\cdot) - x^0(\cdot)|/\Delta$ are bounded by $B$, implying that $|m_0(\Delta)| \leq 2B$. If we find that $|m^k(\Delta)| < 2B$ for all $k$, then $|x^k(\Delta) - x^{k-1}(\Delta)|/\Delta < 4B$, which implies that $|E^2(k, \Delta)| < (16B^3 + 8B^2 + B)\Delta^2$. This will indeed be the case for $\Delta$ less than $\overline{\Delta}$ defined below. Limiting attention to $\Delta < 1$, we thus have that the sum of these terms, in absolute value, is bounded above by $c\Delta$, where $c = 16B^3 + 8B^2 + B$. Let $a < 1$ be an upper bound on $f_2$, the existence of which has been assumed. Then for all $\Delta < 1$ and every $k \in P_+$, Equation 14 implies:

$$|m^k(\Delta)| < a|m^{k-1}(\Delta)| + c\Delta.$$  

Define the indexed sequence $\{b^k(\Delta)\}_{k=0}^\infty$ recursively by setting $b^0(\Delta) = 2B$ and, for every $k \in P_+$, let $b^k(\Delta) = ab^{k-1}(\Delta) + c\Delta$. Then for every $k \in P$, we have that $|m^k(\Delta)| \leq b^k$.

Recall that $a \in (0, 1)$. Let $\overline{\Delta}$ be any number in the interval $(0, 2(1-a)B/c)$. For $\Delta < \overline{\Delta}$, the equation $m = am + c\Delta$ is solved by the value $m = c\Delta/(1 - a)$ that is less
than $2B$, and the sequence $\{b^k(\Delta)\}$ is decreasing and converges to $c\Delta/(1-a)$. In fact, we have

$$b^k = \frac{c\Delta}{1-a} + \left(2B - \frac{c\Delta}{1-a}\right)a^k$$

for each $k \in P$.

Let us set $\beta = 2c/(1-a)$ and define function $\kappa$ so that, for every $\Delta$,

$$\frac{c\Delta}{1-a} + \left(2B - \frac{c\Delta}{1-a}\right)a^{\kappa(\Delta)} = \Delta \beta = \frac{2c\Delta}{1-a}.$$  

That is, letting $Q(\Delta) \equiv \frac{c\Delta}{1-a} / (2B - \frac{c\Delta}{1-a})$, we have $a^{\kappa(\Delta)} = Q(\Delta)$, and therefore $\kappa(\Delta) = \ln Q(\Delta)/\ln a$. By definition of $\kappa$, we have $b^k(\Delta) < \Delta \beta$, proving the first claim of the Lemma. The second claim follows from the fact that $b^0(\Delta) = 2B$ and $\{b^k(\Delta)\}$ is decreasing. Straightforward calculations show that $\lim_{\Delta \to 0^+} \Delta \ln Q(\Delta) = 0$ (using L'Hôpital's rule), which proves the third claim. \(/\)

With the Lemma in hand, we can put steps together to prove Theorem 3. As in the proof of Theorem 1, let us write $x^{k+1} - z^{k+1}$ as a function of $x^k - z^k$. Using Equations 4 and 6, we have

$$x^{k+1}(\Delta) - z^{k+1}(\Delta) = f(x^k(\Delta), x^{k-1}, \Delta) - z^k(\Delta) - h(z^k(\Delta))\Delta.$$  

In the expressions that follow, the dependence of the sequence elements on $\Delta$ is suppressed to save space. Substituting for the first term using the first-degree Taylor polynomial centered at $(x^k, x^k, 0)$, and using $f(x^k, x^k, 0) = x^k$ and the identity $h(x) \equiv f_\Delta(x^k, x^k, 0)/(1 + f_2(x^k, x^k, 0))$ to substitute terms, we obtain

$$x^{k+1} - z^{k+1} = f(x^k, x^k, 0) + f_2(x^k, x^k, 0)(x^{k-1} - x^k) + f_\Delta(x^k, x^k, 0)\Delta + E^2(k, \Delta)$$

$$- z^k - h(z^k)\Delta$$

$$= x^k - z^k + f_2(x^k, x^k, 0)(x^{k-1} - x^k) + h(x^k)\Delta$$

$$+ h(x^k)f_2(x^k, x^k, 0)\Delta + E^2(k, \Delta) - h(z^k)\Delta$$

$$= x^k - z^k + [h(x^k) - h(z^k)] \Delta + E^2(k, \Delta)$$

$$+ f_2(x^k, x^k, 0) [x^{k-1} - x^k + h(x^k)\Delta],$$

Because the second derivatives of $f$ are bounded by $B$, the error term satisfies

$$|E^2(k, \Delta)| \leq B|x^k(\Delta) - x^{k-1}(\Delta)|^2 + 2B|x^k(\Delta) - x^{k-1}(\Delta)|\Delta + B\Delta^2$$

$$= B \left|\frac{x^k(\Delta) - x^{k-1}(\Delta)}{\Delta}\right|^2 \Delta^2 + 2B \left|\frac{x^k(\Delta) - x^{k-1}(\Delta)}{\Delta}\right| \Delta^2 + B\Delta^2.$$  

Further, the Lemma gives us a way to construct a bound on $E^2(k, \Delta)$ in terms of only $\Delta$ and constants. Recall the definition of $m^k$. Limiting attention to $\Delta < \min\{\Delta, 1\}$
and letting $B' = \max\{\beta, 2B\}$, the Lemma establishes that $|m^k(\Delta)| \leq B'$ for all $k \in P$. That is

$$-B' \leq \frac{x^{k+1}(\Delta) - x^k(\Delta)}{\Delta} - h(x^{k+1}(\Delta)) \leq B'.$$

Subtracting $h(x^{k+1}(\Delta))$ from both sides of these inequalities and recalling that $|h| \leq B$, we find that

$$\left| \frac{x^{k+1}(\Delta) - x^k(\Delta)}{\Delta} \right| \leq B' + B.$$ 

So setting $C = B(B' + B)^2 + 2B(B' + B) + B$, we conclude that $|E^2(k, \Delta)| \leq C\Delta^2$ for every $k \in P$.

Next use the zero-degree Taylor approximation for $h$, centered at $z^k$, to write

$$h(x^k) = h(z^k) + Dh(\hat{x}^k)(x^k - z^k),$$

where $\hat{x}^k$ is a number between $x^k$ and $z^k$. Substituting for $h(x^k)$ in the expression for $x^{k+1} - z^{k+1}$, using the definition of $m'$, and rearranging terms, we obtain

$$x^{k+1}(\Delta) - z^{k+1}(\Delta) = x^k(\Delta) - z^k(\Delta) + Dh(\hat{x}^k(\Delta))(x^k(\Delta) - z^k(\Delta))\Delta + E^2(k, \Delta) - f_2(x^k(\Delta), x^k(\Delta), 0)m^{k-1}(\Delta)\Delta. \quad (15)$$

For any fixed $\Delta < 1$, we next construct an upper bound on $|x^k(\Delta) - z^k(\Delta)|$, denoted $b^k(\Delta)$, for each $k = 2, 3, \ldots$. The construction proceeds in two parts, first defining $b^k(\Delta)$ for each $k \leq [\kappa(\Delta)]$ and then defining the bound for $k > [\kappa(\Delta)]$. Continue to limit attention to $\Delta < \min\{\overline{\Delta}, 1\}$.

**Part one: $k \leq [\kappa(\Delta)]$**

With reference to Equation 15, recall that $X$ is bounded, so that $|x^k(\Delta) - z^k(\Delta)| \leq B$. Remember also that $|Dh| \leq B$, $|f_2| < 1$, and $|m^k(\Delta)| \leq 2B$ from the Lemma. Thus, we know that

$$|Dh(\hat{x}^k(\Delta))(x^k(\Delta) - z^k(\Delta))\Delta + E^2(k, \Delta) - f_2(x^k(\Delta), x^k(\Delta), 0)m^{k-1}(\Delta)\Delta| \leq B^2\Delta + C\Delta + 2B\Delta.$$

Let $G = \max\{B^2 + C + 2B, 1\}$. (Making sure that $G > 1$ comes in handy for Part two below.) Because $x^0(\Delta) = z^0(\Delta)$ and $x^1(\Delta) = z^1(\Delta)$, this implies that $|x^k(\Delta) - z^k(\Delta)| \leq kG\Delta$ for all $k \leq [\kappa(\Delta)]$. Therefore, for all $k \leq [\kappa(\Delta)]$ let us set $b^k(\Delta) = kG\Delta$.

**Part two: $k > [\kappa(\Delta)]$**

We next construct $b^{[\kappa(\Delta)]+1}(\Delta), b^{[\kappa(\Delta)]+2}(\Delta), \ldots$. Refer again to Equation 15 and recall that the Lemma gives a tighter bound on $m^k(\Delta)$ for $k > [\kappa(\Delta)]$ than was the case for smaller values of $k$. In particular, we now have $|m^{k-1}(\Delta)| < \beta\Delta$. Putting this together with other bounds that we already calculated, we have that

$$|x^{k+1}(\Delta) - z^{k+1}(\Delta)| \leq |x^k(\Delta) - z^k(\Delta)| + B|x^k(\Delta) - z^k(\Delta)|\Delta + C\Delta^2 + \beta\Delta^2. \quad (16)$$
By letting $\Delta$ be small enough, our constructed values $\{d^k(\Delta)\}$ will all weakly exceed $\Delta$, and so $d^k(\Delta)\Delta \geq \Delta^2$. With reference to the right side of Inequality 16, this means that

$$|x^k(\Delta) - z^k(\Delta)| + B |x^k(\Delta) - z^k(\Delta)| \Delta + C\Delta^2 + \beta\Delta^2 \
\leq b^k(\Delta) + b^k(\Delta)B\Delta + b^k(\Delta)C\Delta + b^k(\Delta)\beta\Delta.$$  

Inductively define $b^{[\kappa(\Delta)]+1}(\Delta), b^{[\kappa(\Delta)]+1}(\Delta), \ldots, b^{[T/\Delta]}(\Delta)$ by setting

$$b^{k+1}(\Delta) = b^k(\Delta)(1 + \Delta(B + C + \beta)) \quad (17)$$

for all $k = [\kappa(\Delta)], [\kappa(\Delta)] + 1, \ldots$. Then these are valid upper bounds if they are all weakly greater than $\Delta$, which is the case because the sequence is increasing and we have set $b^k(\Delta) = kG\Delta$ for $k \leq \kappa(\Delta)$, where $G$ is at least 1.

We can use Equation 17 to explicitly solve for $b^{[T/\Delta]}(\Delta)$:

$$b^{[T/\Delta]}(\Delta) = \kappa(\Delta)G\Delta(1 + \Delta(B + C + \beta))^{[T/\Delta] - [\kappa(\Delta)].}$$

From the definition of the natural number $e$, the right side is less than $\kappa(\Delta)G\Delta e^{B+C+\beta}$. Because $\lim_{\Delta \to 0^+} \kappa(\Delta)\Delta = 0$, for any $\varepsilon > 0$ we can find a number $\Delta > 0$ such that $\Delta < \Delta$ implies that $\kappa(\Delta)G\Delta e^{B+C+\beta} < \varepsilon$ and so we have $b^k(\Delta) < \varepsilon$ for every $k = 1, 2, \ldots, [T/\Delta]$. The last step of writing the implication for $|\hat{\varepsilon}(k\Delta, \hat{x}^0(\Delta), \hat{x}^1(\Delta); \Delta) - \hat{\varepsilon}(k\Delta, \hat{x}^0(\Delta); \Delta)|$ is the same as in the proof of Theorem 1. \hfill $\Box$

**Reference**