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Nash, John Forbes (born 1928)

Nash originated general non-cooperative game theory in seminal articles in the early 1950s by formally distinguishing between non-cooperative and cooperative models and by developing the concept of equilibrium for non-cooperative games. Nash developed the first bargaining solution characterized by axioms, pioneered methods and criteria for relating cooperative-theory solution concepts and non-cooperative games, and also made fundamental contributions in mathematics. Nash was the 1994 recipient of the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel, jointly with John C. Harsanyi and Reinhard Selten.

The context of for Nash's work: von Neumann and Morgenstern

Nash's contributions to the theory of games were fundamental to the development of the discipline and its interface with applied fields of study. This section provides is a short account of the state of affairs before Nash's work. For a more detailed account, see the suggestions for further reading at the end of this article.

The first significant step in mathematical modelling of strategic situations was Augustin Cournot's (1838) book on oligopoly, where Cournot presented models of firm interaction that were analysed using what we now call Nash equilibrium. But Cournot did not attempt, or perhaps even recognize, how the analysis might generalize. Further, in the ensuing years confusion persisted regarding whether it would be appropriate for a firm to incorporate a response by its rivals when considering whether to change its own action. The concept of *strategic independence* – that the players' strategies can be considered to be chosen simultaneously and independently – began to be clarified by Emile Borel's (1921) description of a *method of play*.

Game theory became a discipline with the work of John von Neumann (1928), which was incorporated into the path-breaking book by von Neumann and Oscar Morgenstern (1944; 1947). In the book, von Neumann and Morgenstern formally defined both the extensive form (tree-based) and normal form (strategy-based) representations of games, related by the notion of a strategy; they studied for the first time a general class of games, defining solutions and proving existence using fixed-point methods; they introduced the idea of analysing how coalitions of players can take advantage of binding agreements; and they provided a theory of utility and decision-making under risk (the expected utility criterion). With one book, game theory was created and put on solid footing.

Von Neumann and Morgenstern were interested in developing a positive theory of behaviour in games – for any given game, a 'solution'. In a nutshell, their analysis progresses as follows:

1. Formulate a solution concept for two-player *zero–sum games*, which have the defining property that, for each *strategy profile* (one strategy for each player), the players' payoffs sum to zero. Such a game is special because the only economic concern is distributional; in other words, the game models a situation of pure conflict between the players, where one player's winnings come at the other's expense.
2. Analyse *n*-player zero–sum games by assuming that coalitions of players could bind together and play as a team against the other players. This requires assuming that coalitions can communicate before the game and make binding agreements on how to play. The value of forming a co-

alition is calculated in reference to the implied zero–sum game that the coalitions play against one another, which ultimately is a two-player game to which the solution from Part 1 above is applied.

3. To evaluate a non-zero–sum, n-player game, imagine the existence of a fictitious player $n+1$ whose payoff is defined as negative of the sum of the other players' payoffs. This creates a zero–sum game to which the preceding applies.

For an illustration of von Neumann and Morgenstern's analysis of two-player zero–sum games (Part 1 above), consider a simple example. Suppose that players 1 and 2 interact in the normal form game depicted in the following table.

1\2	X	Y	Z
A	4, – 4	0, 0	– 2, 2
B	3, – 3	1, – 1	1, – 1
C	2, – 2	1, – 1	1, – 1

Player 1 selects between strategies A, B, and C. Simultaneously, player 2 chooses between X, Y, and Z. The players' payoffs, which might as well be in monetary terms, are shown in the cells of the table, with player 1's payoff written first. Note that this is a zero–sum game in that, in each cell of the table, the players' payoffs sum to zero.

Von Neumann and Morgenstern motivated their solution concept by considering sequential variations of games in which one player would move first and then the other player, having seen what the first selected, would respond. Their key concept is what is generally known as a 'maximin strategy', also called a 'security strategy'. A security strategy for a given player is a strategy that gives the highest guaranteed payoff level; that is, it maximizes the minimum that the player could get, where the minimum is calculated over all of the strategies of the other player.

In the example, B and C are both security strategies for player 1 because, regardless of what player 2 does, player 1 gets a payoff of at least 1 when using either of these strategies, whereas it is feasible for player 1 to obtain a lower payoff (0 or – 2, in particular) by selecting strategy A. For player 2, Y and Z are security strategies and they guarantee a payoff of at least – 1.

Von Neumann and Morgenstern's general analysis focuses on mixed strategies (probability distributions over pure strategies) in finite two-player games, to which the maximin definition extends. They prove that the players' security levels (the amounts that the security strategies guarantee) sum to zero. Thus, when each player selects his security strategy, each player obtains exactly his security level payoff. Further, when one player selects his security strategy, the other player can do no better than select her own security strategy; that is, the two players' security strategies are optimal responses to each other. Security strategies also describe optimal play in zero–sum games that are played sequentially. For example, if player 1 had the privilege of selecting among A, B, and C *after* observing player 2's choice, both players would still select security strategies. Finally, security strategies are interchangeable in that the preceding conclusions hold equally well for any combination of security strategies, for instance (B, Y) as well as (B, Z).

Although von Neumann and Morgenstern had developed a theory that applied to all finite games, their theory is essentially empty for non-zero–sum games. For example, in converting a two-player game into a three-player game by adding the fictitious player 3, von Neumann and Morgenstern basically change the rules of the game for the original two players, who now can make binding agreements. The resulting prediction is that the two players will bind themselves to a strategy profile that maximizes the sum of their payoffs, with each player getting at least his security level. Von Neumann and Morgenstern's theory is therefore incomplete and unsatisfying on two fronts. First, for non-zero–sum games, it offers no treatment of rationality in the absence of binding commitments. Second, it offers no way of predicting the outcome of a two-player bargaining problem beyond Francis Ysidro Edgeworth's (1881) contract curve and it relies on transferable utility. Nearly all interesting economic examples involve efficiency concerns and hence are not zero–sum in nature, so economics had little to benefit from game theory until another significant step could be made in the modelling of rational behaviour.

Nash's contributions

Nash's contributions to the emerging discipline of game theory were equally as bold as were von Neumann and Morgenstern's and, in terms of applicability, even more significant. Nash's main contributions were made in a series of four papers published between 1950 and 1953 and summarized in this section.

In his articles in the *Proceedings of the National Academy of Sciences* in 1950 and the *Annals of Mathematics* in 1951, which reported his dissertation research, Nash (*a*) introduced and made clear the distinction between cooperative and non-cooperative games – the latter being games in which players act independently (that is, without the assumption about coalitions that von Neumann and Morgenstern adopted) – and (*b*) defined a solution concept for non-cooperative games. The first four paragraphs from Nash's *Annals of Mathematics* article describe the context and the contribution succinctly:

> Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior*. This book also contains a theory of *n*-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.
>
> Our Theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.
>
> The notion of an *equilibrium point* is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is the set of all pairs of opposing 'good strategies.'
>
> In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game. (1951, p. 286)

Nash's equilibrium concept became known as 'Nash equilibrium'. It and the cooperative/non-cooperative distinction were cited by the Royal Swedish Academy of Sciences in awarding Nash the Nobel Prize.

In more mathematical and modern language, here are the definitions of *best response* (in Nash's words, a 'good strategy') and Nash equilibrium. Consider any game defined by a number n of players; a strategy set S_i for each player $i = 1, 2, \ldots, n$; and, for each player i, a payoff function $u_i : S \to \mathbf{R}$, where S is the set of strategy profiles. The strategy sets may be defined as mixed strategies for some underlying set of pure strategies, in which case the payoff functions, as expectations, are linear in the mixed strategies. For a player i, we write '$-i$' to refer to the other players. Given a strategy vector s_{-i} for the other players, player i's strategy s_i is called a best response if player i can do no better than to select s_i; that is, we have $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for every strategy s_i' of player i. Then strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is called a Nash equilibrium if every player is best responding to the others—that is, if for each player i, it is the case that s_i^* is a best response to s_{-i}^*.

For an illustration of Nash equilibrium and its relation to security strategies, consider the game depicted in the following table.

1\2	X	Y	Z
A	2,3	1,2	6,5
B	1,0	0,2	4,0
C	3,4	2,2	2,0

Observe that, in this game, C and Y are the players' security strategies, so a naive application of von Neumann and Morgenstern's maximin theory (absent binding agreements) would predict that strategy profile (C, Y) be played. However, this strategy profile is plainly inconsistent with the idea that players are rational in responding to each other. In particular, if player 1 is expected to select C then player 2 behaves quite irrationally by choosing Y. In fact, strategy Y is *not even rationalizable* for player 2; it does not survive iterated removal of dominated strategies (see below). Thus, the notion of a security strategy is not a good theory of behaviour for non-zero–sum games, demonstrating the limits of von Neumann and Morgenstern's analysis.

Next, observe that the game has two Nash equilibria in pure strategies, (C, X) and (A, Z). Both of these are reasonable predictions in the sense that, in both cases, the players are best responding to one another. For example, if player 1 is sure that player 2 will select X, then it is best for player 1 to select C; likewise, if player 2 is convinced that player 1 will select C, then it is optimal for player 2 to choose X. There is also a mixed-strategy Nash equilibrium in which player 1 randomizes between A and C, and player 2 randomizes between X and Z. That the game has multiple Nash equilibria demonstrates the general economic problem of coordination, in particular the possibility that the players will coordinate on the less efficient Nash equilibrium. Other games, such as the *Prisoner's Dilemma*, have only inefficient equilibria and thus reveal a fundamental tension between individual and joint incentives.

Nash's intuitive concept of equilibrium facilitated the analysis of *all* non-cooperative games, opening the door to widespread application of game theory. Indeed, Nash equilibrium has become the dominant solution concept for the analysis of games. Through an ingenious fixed-point argument, Nash also proved the existence of an equilibrium point in every finite game. Further, in his dissertation (1950) Nash offered two interpretations of the concept, one based on rational reasoning by individual players and the other describing stability of the distribution of strategies chosen by a population of individuals who interact over time. The latter is a precursor to the methodology of the literature on learning in games and to the modern theories of *evolutionary stability* in biology (John Maynard Smith, 1984). Nash's 1951 *Annals of Mathematics* article also contains a section that defines 'dominance' (meaning one strategy yields a strictly higher payoff than another, regardless of what the other players do) and explains how an iterated dominance procedure can be used to rule out strategies that are not equilibria. Thus, Nash also made observations that would resurface in the concept of 'rationalizable strategic behaviour' (B. Douglas Bernheim, 1984; David Pearce, 1984), the main non-equilibrium notion of rationality. Nash even was among the first to perform game experiments, as his co-authored article in the volume *Decision Processes* (1954) attests.

In his 1950 *Econometrica* article, Nash tackled the two-person bargaining problem with the objective of determining a unique solution (a precise 'value' that eluded von Neumann and Morgenstern) from the underlying set of alternatives and the players' preferences. Nash took a cooperate-theory approach by positing a system of four axioms that reasonably characterize properties one might expect the outcome of a bargaining process to exhibit: (*a*) a notion of equal bargaining power, (*b*) invariance to inessential utility transformations, (*c*) efficiency, and (*d*) independence of the solution to the removal of so-called irrelevant alternatives. Nash proved that a particular function of parameters (which maximizes the product of surpluses) is exactly characterized by the axioms. The analysis showed that it is possible to reasonably identify a precise outcome of a bargaining problem. It also initiated the axiomatic method for the analysis of bargaining (where theorists explore how different axioms characterize various functional solutions), starting a literature that thrived for several decades. The *Nash bargaining solution* is still the dominant solution in applied economic models.

Nash's second paper on bargaining (the 1953 *Econometrica* article) took another major step by connecting the non-cooperative and cooperative approaches to strategic analysis. At the heart of this theoretical exercise is an underlying non-cooperative game, which gives a set of feasible payoffs, and a technology for the players to make binding commitments about the mixed strategies that they will play in the underlying game. In the model, players first simultaneously make threats, which are mixed strategies they are bound to play if they do not reach an agreement. Then the players interact in a non-cooperative bargaining game in which they simultaneously make payoff demands – this stage is now called the 'Nash demand game'. If their payoff demands are feasible in the underlying game, then the players obtain their demanded payoffs; otherwise, the players get what their threats imply.

Nash observed that the demand game has generally an infinite number of equilibria, revealing a coordination aspect to the bargaining problem. But Nash went further in developing a brilliant method to 'escape from this troublesome non-uniqueness' by looking at the limit of 'smooth' approximations of the demand game. Amazingly, Nash showed that the limit is unique and coincides with the prediction of his axiomatic model; that is, the limit is the Nash bargaining solution. Nash's limit argument was the forerunner to the enormous literature on *equilibrium refinements*, an area of research that thrived decades later and was the primary subject of Nash's Nobel co-recipients. More significantly, Nash argued that the relation between the cooperative solution concept and the equilibrium in the non-cooperative model justifies wide use of the cooperative solution as a reasonable shorthand for the actual non-cooperative setting. Nash's argument, and fascinating theoretical result, established the profession's understanding of the connection between cooperative and non-cooperative models and initiated the literature on what is now called the 'Nash program'.

After completing the work in game theory just described, Nash made fundamental contributions in pure mathematics – contributions that, in terms of mathematical depth and originality, were of an even higher order of sophistication and importance. According to leading mathematician John Milnor, Nash's

> subsequent mathematical work is far more rich and important [in this mathematical sense]. During the following years he proved that every smooth compact manifold can be realized as a sheet of a real algebraic variety, proved the highly anti-intuitive C1-isometric embedding theorem, introduced powerful and radically new tools to prove the far more difficult C1-isometric embedding theorem in high dimensions, and made a strong start on fundamental existence, uniqueness, and continuity theorems for partial differential equations. (Milnor, 1998, p. 1330)

It is not appropriate to provide here details on Nash's pure mathematics work (nor is it possible, due to the limitations of the author's fields of expertise).

Nash's personal life

Nash's character became legendary with the publication of a biography by Sylvia Nasar (1998) and a 2001 feature film produced by Brian Grazer and Ron Howard. Nash's remarkable personal journey began in Bluefield, West Virginia, where he was born and raised. He explored mathematics and conducted science experiments as a child, and attended Carnegie Institute of Technology, where the mathematics department discovered in him a budding genius. Nash's ideas on bargaining that were published as 'The Bargaining Problem' (1950) were developed while he was an undergraduate student at Carnegie, during the only economics course he took, on international trade.

Nash studied mathematics in the graduate program at Princeton University, where, as his biography describes, he was boorish, cocky, and a renowned adversary in strategic contests. At Princeton, Nash added to his prodigious achievements, finishing his dissertation – the work on non-cooperative games and equilibrium that would bring him the Nobel Prize – in his second year. (Nash also invented the board game *Hex*, a game independently created by Danish mathematician Piet Hein.) Nash taught at Princeton for one year and then took a position at Massachusetts Institute of Technology, where he was on the faculty until 1959. There he conducted the research that won him great acclaim in the mathematics community.

Nash's genius in advancing game theory and mathematics was paired with deep personal challenges. In 1959 Nash began experiencing the severe mental disturbances of paranoid schizophrenia. He resigned from MIT and began a phase of life marked by delusional thinking, an escape to Europe, repeated hospitalizations, unsuccessful medical treatments, and then a long, disengaged presence at Princeton. In the mid-1980s Nash miraculously began to emerge from the delusional haze in what he describes as a gradual rejection of psychotic thinking on intellectual grounds (Nash, 1995). After a quarter century of detachment, Nash's life regained a measure of normality.

Nash's legacy in game theory and economics

There is no simple way of quantifying the enormous reach of Nash's ideas. The notions of Nash equilibrium, the Nash bargaining solution, the Nash demand game, and the Nash program have found such widespread acceptance and application that it has become customary, and perhaps even appropriate, for researchers to forgo formally citing Nash's articles when utilizing these concepts. Nash ideas helped to propel game theory from a mathematical sub-field into a full discipline, with major use and application in not only economics, where it is the main and worthy alternative to the competitive-market framework, but also in theoretical biology, political science, international relations, and law.

Beyond its theoretical content, Nash's work also made a stylistic departure from that of von Neumann and Morgenstern, whose book methodically records definitions, examples, and analysis for numerous special cases in the process of developing general theory. Nash, in contrast, used the terse style of the mathematician, presenting his ideas with minimal obscuring features. His 1950 *Proceedings of the National Academy of Sciences* entry, for instance, is generously allotted two pages and could have been typeset on one. The benefit of focusing on the basic mathematical concepts is that it allows for a broad range of interpretations and extensions. For example, there are several motivations for Nash equilibrium, including as a condition for self-enforcement of a contract (which is an important topic in the current literature). A hallmark of excellent theoretical modelling is precise and straightforward expression of assumptions and conclusions, with their relation shown in the most simple and elegant way possible.

Mathematician Milnor, after offering the assessment of Nash's work in pure mathematics that is quoted above, continues with by saying: 'However, when mathematics is applied to other branches of human knowledge, we must really ask a quite different question: To what extent does the new work increase our understanding of the real world? On this basis, Nash's thesis was nothing short of revolutionary' (1998, p. 1330). Two leading game theorists of today say 'Nash's theory of non-cooperative games should now be recognized as one of the outstanding intellectual advances of the twentieth century' (Myerson, 1999, p. 1067) and 'His work lay the foundation of non-cooperative game theory, now the predominant mode of analysis of strategic interactions in economics, political science, and biology' (Crawford, 2002, p. 380).

When viewed from the perspective of five short decades, game theory has caused a revolution in economics and other fields of study. It was with the work of John Nash that the flame so exquisitely ignited by von Neumann and Morgenstern became the torch that would eventually set the social sciences ablaze.

Joel Watson

See also

< xref = xyyyyyy > axiomatic theories;
< xref = B000073 > bargaining;

< xref = xyyyyyy > cooperative game theory (overview);
< xref = xyyyyyy > equilibrium;
< xref = xyyyyyy > game theory;
< xref = xyyyyyy > Morgenstern, Oskar;
< xref = N000137 > Nash program;
< xref = xyyyyyy > non-cooperative games;
< xref = xyyyyyy > von Neumann, John.

The author thanks Vincent Crawford, Joel Sobel, and Martin Dufwenberg for comments on a preliminary draft.

Bibliography

Items indicated with an asterisk provide good further background reading on John F. Nash, Jr. Also, the *Scandinavian Journal of Economics*, vol. 97, issue 1 (1995), contains articles on John Nash and his co-Nobel Prize recipients, John C. Harsanyi and Reinhard Selten. For a complete list of Nash's publications, including his papers in pure mathematics, see Milnor (1998).

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Index terms

coalitions
commitment
contract curve
cooperative games
Cournot, A. A.
dominance
equilibrium
equilibrium refinements
evolutionary stability
expected utility
fixed-point methods
game theory
maximin strategy
Morgenstern, O.
multiple equilibria
Nash bargaining solution
Nash demand game
Nash equilibrium
Nash program
Nash. J. R., Jr.
non-cooperative games
prisoner's dilemma
rational behaviour
strategic and extensive-form games
strategic independence
von Neumann, J.

Index terms not found:

multiple equilibria
strategic and extensive-form games