JOURNAL OF ECONOMIC THEORY **50**, 381–413 (1990)

Strategic Stability and Uniqueness in Signaling Games*

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Received June 9, 1987; accepted May 2, 1989

A class of signaling games is studied in which a unique Universally Divine equilibrium outcome exists. We identify a monotonicity property under which a variation of Universal Divinity is generically equivalent to strategic stability. Further assumptions guarantee the existence of a unique Universally Divine outcome. *Journal of Economic Literature* Classification Numbers: 021, 022, 026. © 1990 Academic Press, Inc.

1. INTRODUCTION

This paper identifies a class of signaling games in which a unique Universally Divine equilibrium outcome (Banks and Sobel [4]) exists and characterizes this outcome. The unique Universally Divine outcome involves separation of all but the highest types (those with the most attractive information) of the informed signaler. Although Universal Divinity forces the informed signaler to reveal as much information as possible, a separating equilibrium need not always exist in the class of models that we analyze. However, if one exists, then the unique Universally Divine equilibrium outcome coincides with the Pareto dominating equilibrium outcome often studied in applications.

This paper continues the approach, pioneered by Kreps [13] and developed by Cho and Kreps [6], of studying the refinement of equilibria

* We thank Jeff Banks, Mark Machina, Georg Noldeke, Garey Ramey, Lones Smith, Iñigo Zapater, and seminar participants for helpful comments. Detailed remarks of two referees and an associate editor greatly improved the substance and presentation of this paper. We are grateful to the University of Chicago, the National Science Foundation, and the Sloan Foundation for financial support.

0022-0531/90 $3.00

in signaling games by restriction of beliefs off the equilibrium path. The restriction that we use is related to the concept of Universal Divinity. We also use a theorem of Kohlberg and Mertens [12], which proves that the set of stable equilibrium outcomes is not empty. This result guarantees the existence of sequential equilibria satisfying our condition since any stable outcome in a generic signaling game must be Universally Divine. We find the restrictions imposed on the beliefs off the equilibrium path easier to understand and work with than the full implications of stability. Hence we base our uniqueness arguments on refinements of the sequential equilibrium concept rather than a direct appeal to strategic stability. Our paper identifies a class of models in which a simple restriction on beliefs off the equilibrium path is equivalent to strategic stability.

Section 3 presents a condition under which the outcomes that satisfy our restriction coincide with the (generally smaller) set of strategically stable outcomes. The sufficient condition states that the informed player's preferences over his opponent's actions do not depend on his private information. In the Spence [22] signaling game, this property holds because, regardless of their productivity, all workers prefer higher wages to lower wages.

The assumption needed to characterize stable outcomes in terms of our refinement concept does not guarantee uniqueness. In Section 4, we present additional conditions that, combined with our refinement ideas, lead to unique predictions in signaling games. We need to make two basic assumptions. The first assumption ranks private information: *Better* information makes the Receiver take *better* actions. In labor market signaling models, this condition holds: The higher the productivity of a worker, the higher the wage an employer is willing to pay. The second assumption guarantees that the better the private information, the lower the costs associated with sending certain kinds of signals. Once again, this condition is satisfied in the Spence model: More productive workers have lower marginal disutilities of acquiring education than do less productive workers. Independently, Kohlberg [23] identified virtually same conditions for the existence of a unique stable equilibrium outcome. He focused on a subset of the signaling games analyzed in this paper where the sender's payoff is additively separable with respect to the sender's signal and the receiver's response. We strongly recommend this readable paper to readers. Section 5 analyzes an example that illustrates the construction of Section 4 and allows us to compare our results to those of Cho and Kreps [6]. Section 6 reviews some signaling games that have been analyzed by others and discusses the limitations of our methods and results. Section 7 is a brief conclusion.

2. The Model and Refinement Concepts

We deal with a subset of two-player signaling games. One player, the Sender, obtains private information. This information is the Sender's type t, an element of a finite set. We take this set to be the first T positive integers; we also denote the set of types by T. The Sender's type is drawn according to some probability distribution π over T; π is common knowledge. After the Sender learns his type, he sends a signal m to the other player, the Receiver. The set of possible signals is M. The Receiver responds to the Sender's signal by taking an action a from a set A. We consider both finite signaling games, where both M and A are finite sets, and those signaling games with A a compact real interval and M a product of compact intervals in $\mathbb{R}^n$. The players have von Neumann–Morgenstern utility functions defined over type, signal, and action. The Sender's payoff function is $u(t, m, a)$ and the Receiver's payoff function is $v(t, m, a)$.

We must introduce more notation to discuss the equilibria of these games. We represent a behavior strategy for the Sender by $q(\cdot)$; for each t, $q(\cdot \mid t)$ is a probability distribution over M. We represent a behavior strategy for the Receiver by $r(\cdot)$; for each m, $r(\cdot \mid m)$, which we abbreviate $r(m)$, is a probability distribution over A. We restrict attention to equilibria in which these distributions have finite support.[1] Hence $q(m \mid t)$ is the probability that a type t Sender sends the signal m, and $r(a \mid m)$ is the probability that the Receiver plays the pure strategy a in response to the signal m. We extend $u(\cdot)$ to behavior strategies of R by taking expected values; that is, if r is a probability distribution over A with finite support, then $u(t, m, r) = \sum_{a \in A} u(t, m, a)\, r(a)$. We denote the Receiver's assessment of or beliefs about the type of the Sender after a signal m by a probability distribution $\mu = \mu(\cdot \mid m)$ over T. We let $\mathrm{BR}(\mu, m)$ be the set of actions that are best responses to m given the assessment μ. That is,

$$\mathrm{BR}(\mu, m) \equiv \arg\max_{a \in A} \sum_{t=1}^{T} v(t, m, a)\, \mu(t).$$

Define

$$\mathrm{BR}(I, m) \equiv \bigcup_{\{\mu \,:\, \mu(I \mid m) = 1\}} \mathrm{BR}(\mu, m);$$

$\mathrm{BR}(I, m)$ is the set of best responses by the Receiver to assessments concentrated on the subset I of T. Let $\mathrm{MBR}(\mu, m)$ and $\mathrm{MBR}(I, m)$ represent the corresponding sets of mixed-strategy best responses. We assume that $\mathrm{MBR}(\mu, m)$ is an upper hemi-continuous correspondence in μ. This property follows if A is finite or if $v(t, m, a)$ is a continuous function of a.

[1] In Section 4 we make assumptions which guarantee that the Receiver never randomizes in equilibrium.

A sequential equilibrium for a signaling game is a triple of strategies and assessments $((q(m \mid t))_{t \in T}, ((r(a \mid m)))_{m \in M}, (\mu(t \mid m))_{m \in M})$ that satisfy sequential rationality and consistency.

Sequential Rationality: For all t, if $q(m' \mid t) > 0$, then $m' \in \arg\max_{m \in M} u(t, m, r(m))$ and $r(m) \in \mathrm{MBR}(\mu(\cdot \mid m), m)$ for all $m \in M$.

Consistency: If $\sum_{s=1}^{T} \pi(s)\, q(m \mid s) > 0$, then $\mu(t \mid m) = \pi(t)\, q(m \mid t)/[\sum_{s=1}^{T} \pi(s)\, q(m \mid s)]$.

Consistency imposes no restrictions on the beliefs $\mu(\cdot \mid m)$ when $\sum_{s=1}^{T} \pi(s)\, q(m \mid s) = 0$. Consequently, many qualitatively different sequential equilibria exist. We refine the set of sequential equilibria by imposing conditions on $\mu(\cdot \mid m)$ for those signals sent with probability zero in equilibrium. We require the Receiver to assign zero weight to the types of the Sender that are "unlikely" to send the signal m. We propose three related criteria. First, we introduce more terminology.

Any behavior strategies q and r induce a probability distribution over the endpoints of the game. This probability distribution over $T \times M \times A$ is the *outcome* of the game associated with the strategies q and r. If q and r are sequential equilibrium strategies, then we call the corresponding outcome a *sequential equilibrium outcome.* We say that an assessment μ supports this equilibrium if (q, r, μ) is a sequential equilibrium. Associated with equilibrium strategies is the *equilibrium path,* which we identify with the set of information sets arrived at with positive probability $\{m \in M : \sum_{t=1}^{T} \pi(t)\, q(m \mid t) > 0\}$. We impose restrictions on $\mu(\cdot \mid m)$ for off-the-equilibrium-path signals, $\{m \in M : \sum_{t=1}^{T} \pi(t)\, q(m \mid t) = 0\}$, and refer to the corresponding beliefs $\mu(\cdot \mid m)$ as off-the-equilibrium-path beliefs.

Fix a sequential equilibrium (q, r, μ). Let $u^*(t)$ be the equilibrium expected utility of the type t Sender: $u^*(t) = \sum_{m \in M} \sum_{a \in A} u(t, m, a)\, q(m \mid t)\, r(a \mid m)$. Choose an off-the-equilibrium-path signal m and define

$$P(t \mid m) = \{r \in \mathrm{MBR}(T, m) : u^*(t) < u(t, m, r)\}$$

and

$$P^0(t \mid m) = \{r \in \mathrm{MBR}(T, m) : u^*(t) = u(t, m, r)\}.$$

$P(t \mid m)$ is the set of best responses of the Receiver to m that induce the type t Sender deviate from his equilibrium strategy. If R responds to the signal m with an action $r \in P^0(t \mid m)$, then sending the signal m yields the same utility as following the equilibrium. Therefore, if the Receiver takes an action $r \in P(t \mid m) \cup P^0(t \mid m)$ in response to m, then the type t Sender has a weak incentive to deviate from his equilibrium strategy.

We require that if the Receiver considers every type in a subset J of T unlikely to send the signal m, then following the signal m, he should have beliefs that place probability zero on types in J. In this case, we require that $\mu(t \mid m) = 0$ for all $t \in J$ (provided that $J \neq T$). We now discuss three ways to define the subset J of types unlikely to send m. Banks and Sobel [4] and Cho and Kreps [6] introduced these procedures.

The first criterion for reasonable beliefs is based on the idea of Divinity (Banks and Sobel [4]). If the type t' Sender has an incentive to deviate whenever the type t Sender has a weak incentive to deviate, then the beliefs of the Receiver should not assign positive weight to type t at the information set m. The criterion, which we call criterion D1 as in Cho and Kreps [6], eliminates strategy m of the type t Sender if there exists t' such that

$$P(t \mid m) \cup P^0(t \mid m) \subset P(t' \mid m). \tag{1}$$

We say that a sequential equilibrium survives criterion D1 if and only if, for all off-the-equilibrium-path signals m, $\mu(t \mid m) = 0$ whenever (1) holds for some t' such that $P(t' \mid m) \neq \varnothing$.[2] We call a sequential equilibrium (sequential equilibrium outcome) that survives this criterion a D1 equilibrium (D1 equilibrium outcome). Types t for which (1) does not hold are said to survive criterion D1.

It is more difficult for a type to survive the second criterion, Universal Divinity (Banks and Sobel [4]), than D1. Universal Divinity eliminates strategy m of the type t Sender if

$$P(t \mid m) \cup P^0(t \mid m) \subset \bigcup_{t \neq t'} P(t' \mid m). \tag{2}$$

Hence, the signal m for the type t is deleted if, whenever t has a weak incentive to deviate to m, there exists some other type of Sender that has a strong incentive to deviate to m. D1 further requires that the other type to be independent of the Receiver's response to m.

The criterion of Never Weak Best Response (NWBR) (Kohlberg and Mertens [12] and Cho and Kreps [6]) goes beyond (2) by relaxing the conditions under which a particular signal m can be eliminated. This test eliminates strategy m of the type t Sender if

$$P^0(t \mid m) \subset \bigcup_{t \neq t'} P(t' \mid m). \tag{3}$$

In words, NWBR requires that the Receiver place positive probability on the type t Sender if there exists a response to the signal m that supports the equilibrium outcome and yields expected utility $u^*(t)$. That is, if (3) holds for the type t and the signal m, then in every sequential equilibrium that induces the same equilibrium path as the given equilibrium, it is never a weak best response for t to send m. We say that type t survives NWBR for

[2] By requiring that $\mu(t \mid m) = 0$ only when (1) holds for some t' such that $P(t' \mid m) \neq \varnothing$, the upper hemi-continuity of $\mathrm{BR}(\cdot, m)$ guarantees that criterion D1 never requires $\mu(t \mid m) = 0$ for all t. If $P(t \mid m) = \varnothing$ for all $t = 1, 2, ..., T$, then no Sender type would prefer sending m to obtaining $u^*(t)$. Therefore, any specification of $\mu(\cdot \mid m)$ supports a sequential equilibrium.

the signal m if (3) does not hold. An equilibrium is said to survive NWBR if it can be supported by beliefs concentrated on types that survive NWBR or if no type survives NWBR.

Kohlberg and Mertens [12] demonstrate that equilibria satisfying these criteria exist in finite signaling games. For the remainder of the section we review those results of Kohlberg and Mertens needed for our analysis.

Fix a finite l-player game G in normal form. Let $\sigma^* = (\sigma_1^*, ..., \sigma_l^*)$, where σ_i^* is a completely mixed strategy for player i in G. For $\delta > 0$, consider the set of all games G' having the same strategy space as G and for which there exist $\delta_i \in (0, \delta)$, $i = 1, ..., l$, such that if the strategy vector $(\sigma_1, ..., \sigma_l)$ is played in G', then the payoffs are the same as when each player i plays $(1 - \delta_i)\sigma_i + \delta_i \sigma_i^*$ in G. Call any game in this set a (σ^*, δ) perturbation of G. A set of Nash equilibria of G is said to be *stable* if it is minimal in the set of all N with the property: N is a closed set of Nash equilibria of G such that for each $\varepsilon > 0$ there exists $\delta > 0$ for which any (σ^*, δ) perturbation of G has a Nash equilibrium less than ε in distance to N.[3] If every strategy combination in a stable set gives rise to the same outcome, then the common outcome is a *stable outcome*. We refer the reader to Kohlberg and Mertens [12] for further discussion of these definitions.

The theory of Kohlberg and Mertens is most useful to us in games that have a stable outcome. In a precise sense, this result holds for "most" games. Let $|K|$ denote the cardinality of the set K. We can identify any finite signaling game with a point in $\mathbb{R}^{2|T \times A \times M|}$ that determines payoffs. A property of a signaling game is generic if there exists $D \subset \mathbb{R}^{2|T \times A \times M|}$ such that the property holds for all signaling games in D and the closure of $\mathbb{R}^{2|T \times A \times M|} \backslash D$ is a set of Lebesgue measure zero. If a property of a signaling game is generic, we say that it holds for generic signaling games.

With this background, we state the results of Kohlberg and Mertens [12] that we use in this paper.

PROPOSITION 2.1. (1) *A generic set of signaling games has a stable outcome* (*Kohlberg and Mertens* [12, *p.* 1027]).

(2) *A stable set contains a stable set of any game obtained by deleting a strategy that is Never a Weak Best Response to all the equilibria of the set* (*Kohlberg and Mertens* [12, *p.* 1029]).

Proposition 2.1 implies that there exist outcomes that survive our criteria in generic finite signaling games. In Section 3 we identify an assumption under which conditions (1), (2), and (3) are equivalent to each other and, for generic finite signaling games, equivalent to stability in the sense that any D1 outcome is also a stable outcome.

[3] The distance is standard Euclidean distance; since the game is finite, a vector of strategies is a point in some Euclidean space.

In Section 4 we deal only with infinite signaling games. When we do so, we are no longer able to use the theorem of Kohlberg and Mertens to guarantee existence. However, for the class of games discussed in Section 4, we construct a D1 equilibrium.

3. Monotonic Signaling Games

Banks and Sobel [4], Cho and Kreps [6], and Grossman and Perry [10] present examples to show that equilibria satisfying criterion D1 need not be Universally Divine, that Universally Divine outcomes need not survive the NWBR criterion, and that outcomes which satisfy the NWBR criterion need not be stable. However, we show in this section that D1 is equivalent to NWBR for a class of signaling models that satisfy a natural monotonicity property. Further, for generic finite games in this class, we show that D1 is equivalent to strategic stability. A1 states the monotonicity property that we use in this section.

A1. For all $m \in M$ and r and $r' \in \mathrm{MBR}(T, m)$, if

$$u(s, m, r) > u(s, m, r') \qquad \text{for some} \quad s \in T,$$

then

$$u(t, m, r) > u(t, m, r') \qquad \text{for all} \quad t \in T.$$

We call a signaling game in which A1 holds a *monotonic signaling game.*

Note that A1 may not hold even if all Sender types have identical preferences over the Receiver's pure strategy best responses; they must have the same preferences over mixed best responses as well. Thus, even if a Receiver's action represents a monetary payment to the Sender and all Sender types prefer more money to less, A1 need not hold without making further assumptions on the risk preferences of the Sender.

There are special cases where agreement of all the Sender types on the ranking of the Receiver's pure strategies implies A1. If A is an interval and the Receiver has a strictly quasi-concave utility function, then it is never optimal for the Receiver to randomize. In this case, A1 holds without any reference to the Sender's risk preferences. If A is finite, and A1 holds for all pure-strategy best responses to m, it is possible to order these responses so that $u(t, m, a_i) \leqslant u(t, m, a_{i+1})$ for all t and $i = 1, ..., |A| - 1$. If the Receiver's set of mixed best responses contains only randomizations between "adjacent" pure strategies (that is, the randomizations are supported on two-point sets of the form $\{a_i, a_{i+1}\}$), then A1 holds. To see this, let r and r' be behavior strategies supported on $\{a_i, a_{i+1}\}$ and $\{a_j, a_{j+1}\}$, respectively, and observe that $u(t, m, r) > u(t, m, r')$ if and only if $i > j$ or

$i=j$ and $r'(a_i)>r(a_i)$, independent of t. When the Receiver has only two pure-strategy best responses, the Receiver's set of mixed best responses necessarily contains only randomizations between adjacent pure strategies. Therefore, A1 holds in this case whenever all the Sender types agree on their ranking of the pure-strategy best responses of the Receiver.

We now can state the first result of this section.

PROPOSITION 3.1. *In monotonic signaling games criterion* D1 *is equivalent to Universal Divinity and the* NWBR *criterion.*

Proposition 3.1 is a consequence of Lemma 3.1, which we state and prove later in the section. The proposition demonstrates that D1 and NWBR are equivalent for monotonic signaling games. Hence, whenever A1 holds, D1 is as powerful as the generally stronger NWBR criterion. This result is reassuring in light of examples constructed to show that NWBR may eliminate some equilibria for counterintuitive reasons. We find one feature of NWBR particularly unattractive. Suppose that a refinement criterion (either D1, Universal Divinity, or NWBR) requires that the Receiver's beliefs be supported on a subset K of types following the signal m. One could ask: In order to pass the refinement test, is it sufficient that the original sequential equilibrium outcome is still a sequential equilibrium outcome in a game in which only types in K are able to send the signal m? The answer to this equation is "yes" for the criteria of D1 and Universal Divinity. However, Cho and Kreps [6, p. 207] present an example in which an equilibrium outcome that fails the test of NWBR does so because a previously eliminated type (and no other type) wishes to defect when the Receiver's beliefs are restricted to exclude that type. Proposition 3.1 implies that this problem does not arise in monotonic signaling games.

We must introduce more notation in order to prove the proposition. For each off-the-equilibrium-path signal m define

$$J_m = \{t \in T : \text{there does not exist } t' \text{ such that}$$
$$P(t \mid m) \cup P^0(t \mid m) \subset P(t' \mid m)\}.$$

J_m is precisely the set of types that survive criterion D1. Also, let ψ_m be given by

$$\psi_m = \{(\mu, S) : \exists r \in \mathrm{MBR}(\mu, m) \text{ such that } u^*(t) = u(t, m, r) \text{ for } t \in S,$$
$$\text{and } u^*(t) > u(t, m, r) \text{ for } t \notin S\}.$$

It is possible to support a sequential equilibrium outcome with beliefs μ given m if and only if $(\mu, S) \in \psi_m$ for some S. We can describe NWBR in terms of the set ψ_m. Recall that m is never a weak best response for type

t exactly when there is no optimal response to m that supports the equilibrium and yields utility $u^*(t)$ to type t. Consequently, the signal m is NWBR for the type t if and only if there is no $S \subset T$ with $t \in S$ such that $(\mu, S) \in \psi_m$ for some μ.

In order to demonstrate that D1 and NWBR are equivalent, it suffices to show that any type that survives D1 also survives NWBR. Lemma 3.1 contains this result.

LEMMA 3.1. *Assume that* A1 *holds. For all* $(\mu, S) \in \psi_m$, *if* $S \neq \varnothing$, *then* $J_m \subset S$.

Proof of Lemma 3.1. If $S = T$, then there is nothing to prove. If there exists $t \notin S$, then there exists $r \in \mathrm{MBR}(\mu, m)$ such that

$$u(t, m, r) < u^*(t) \qquad \text{and} \qquad u(s, m, r) = u^*(s) \text{ for all } s \in S. \tag{4}$$

If $r' \in P(t \mid m) \cup P^0(t \mid m)$, then $u(t, m, r) < u^*(t) \leqslant u(t, m, r')$.

Therefore, by A1 and (4), $r' \in P(s \mid m)$ for all $s \in S$. Consequently, $P(t \mid m) \cup P^0(t \mid m) \subset P(s \mid m)$ for all $s \in S$. Since $S \neq \varnothing$, $t \notin J_m$.

Proof of Proposition 3.1. Since any equilibrium that survives NWBR is Universally Divine and Universal Divinity is more restrictive than D1, we need only show that any D1 equilibrium satisfies NWBR. Fix a D1 equilibrium and a signal m that is sent with probability zero in the equilibrium. If no type survives NWBR, then any beliefs given m support the equilibrium. Hence the equilibrium satisfies NWBR. If type t survives NWBR, then there exists $(\mu, S) \in \psi_m$ with $t \in S$. Lemma 3.1 implies that S contains all of J_m. Therefore, every element of J_m survives NWBR. Since J_m is the set of types not eliminated by criterion D1 and we started with a D1 equilibrium, it follows that the equilibrium survives NWBR.

We now limit discussion to monotonic signaling games with a finite number of pure strategies. We do so in order to compare D1 with strategic stability, which Kohlberg and Mertens [12] define just for finite games. Also, the characterization theorem that we use (Lemma 3.2) only applies to finite games.

We show that for generic, finite, monotonic signaling games, D1 is equivalent to strategic stability. This result is (generically) stronger than Proposition 3.1 because when A1 does not hold, there exist generic finite signaling games in which there are unstable equilibria that survive NWBR.

Our proposition follows from Lemma 3.1 and the next result, which appears in Banks and Sobel [4] and Cho and Kreps [6]. For $S \subset T$, let $\Delta(S)$ be the set of all probability distributions over S, and let $\mathrm{co}(y, Z)$ denote the convex hull of a point y and a set Z. (We let $\Delta(\varnothing) = \varnothing$, and let $\mathrm{co}(y, \varnothing) = y$.)

LEMMA 3.2. *For generic finite signaling games, an equilibrium outcome is strategically stable if and only if for all unsent signals m, and all $\theta \in \Delta(T)$, there exists $(\mu, S) \in \psi_m$ such that $\mu \in \mathrm{co}(\theta, \Delta(S))$.*

The appendix of Banks and Sobel [4] sketches a proof of Lemma 3.2. Here we will only provide intuition for the result. The condition $\mu \in \mathrm{co}(\theta, \Delta(S))$ states that one can move from the perturbation θ to μ by "adding" probabilities from $t \in S$. Given the perturbation θ, we construct a "nearby" equilibrium in which only types in S send m with positive probability. In the equilibrium of the perturbed game, the Receiver believes that μ is the probability distribution over types given m and takes an action $r \in \mathrm{MBR}(\mu, m)$ such that for all t, $u^*(t) \geqslant u(t, m, r)$ with equality if and only if $t \in S$. Since $(\mu, S) \in \psi_m$, such an action exists.

PROPOSITION 3.2. *In a generic monotonic finite signaling game, an equilibrium outcome is strategically stable if and only if the outcome survives criterion* D1.

Notice that A1 holds for a subset of finite signaling games that has nonempty interior in the set of all finite signaling games. Therefore, it makes sense to talk about generic properties of monotonic, finite signaling games.

Proof of Proposition 3.2. Fix an equilibrium outcome that survives D1 and a signal m that is sent with probability zero in equilibrium. Fix $\theta \in \Delta(T)$ and let $\mathrm{proj}_1 \psi_m$ denote the projection map onto the first component of ψ_m. We show that there exists $(\mu, S) \in \psi_m$ such that $\mu \in \mathrm{co}(\theta, \Delta(S))$. Therefore, Lemma 3.2 implies that in generic finite signaling games the equilibrium outcome is strategically stable. Since any strategically stable outcome survives D1, this result establishes that D1 is equivalent to strategic stability for generic monotonic signaling games.

First, assume that $\theta \in \mathrm{proj}_1 \psi_m$. In this case, we know that there exists an $S \subset T$ such that $(\theta, S) \in \psi_m$. Since $\theta \in \mathrm{co}(\theta, \Delta(S))$, Lemma 3.2 implies that the D1 equilibrium is strategically stable.

Now, assume that $\theta \notin \mathrm{proj}_1 \psi_m$. In this case, for all $r \in \mathrm{MBR}(\theta, m)$, there exists t' such that

$$u^*(t') < u(t', m, r). \tag{5}$$

Consequently $P(t' \mid m) \neq \varnothing$. Since the equilibrium outcome survives criterion D1, there exist $\tilde{\mu} \in \Delta(J_m)$ and $\tilde{r} \in \mathrm{MBR}(\tilde{\mu}, m)$ such that

$$u^*(t) \geqslant u(t, m, \tilde{r}) \qquad \text{for all} \quad t. \tag{6}$$

We claim that there exists $\bar{\alpha} \in (0, 1]$ such that

$$\bar{\alpha}\tilde{\mu} + (1 - \bar{\alpha})\theta \in \text{proj}_1 \psi_m. \tag{7}$$

Define

$$\mathscr{A} = \{\alpha \in [0, 1] : \exists r \in \text{MBR}(\alpha\tilde{\mu} + (1 - \alpha)\theta, m) \text{ such that } u^*(t) \geq u(t, m, r) \text{ for all } t\},$$

and let $\bar{\alpha} = \inf \mathscr{A}$. We know that $0 \notin \mathscr{A}$ from (5) and $1 \in \mathscr{A}$ from (6). Moreover, $\mathscr{A}$ is closed since $\text{MBR}(\mu, m)$ is upper hemi-continuous in μ. Therefore $\bar{\alpha} > 0$ and $\bar{\alpha} \in \mathscr{A}$. Condition (7) holds because $\text{MBR}(\mu, m)$ is upper hemi-continuous in μ.

We have shown that if $\mu^* = \bar{\alpha}\tilde{\mu} + (1 - \bar{\alpha})\theta$, then there exists $S \neq \varnothing$, $S \subset T$ such that

$$(\mu^*, S) \in \psi_m. \tag{8}$$

Lemma 3.1 implies that $\Delta(J_m) \subset \Delta(S)$. Thus, $\tilde{\mu} \in \Delta(J_m)$ implies that

$$\mu^* \in \text{co}(\theta, \Delta(J_m)) \subset \text{co}(\theta, \Delta(S)). \tag{9}$$

By Lemma 3.2, (8) and (9) imply that the outcome is stable.

4. UNIQUENESS

This section gives conditions under which signaling games have unique outcomes that survive criterion D1. We only examine the case in which A is a compact interval and M is the product of compact intervals. Without further loss of generality, we take $A = [0, 1]$ and $M = [0, c]^N$ with $c > 0$. We denote an element of M by a vector $m = (m_1, ..., m_N)$ and we write $C = (c, ..., c)$. We maintain the assumption that the set of possible Sender types is finite. Ramey [19] obtains a similar uniqueness result when types are drawn from a real interval. We must assume that the strategy spaces are intervals in order to obtain our uniqueness results. This section concludes with examples illustrating that multiple equilibria could arise if either M or A (or both) are finite sets. The arguments below demonstrate that the compactness of the strategy spaces ensures the existence of outcomes surviving D1. We now state our main assumptions.

A0. The utility function $u(t, m, a)$ is a continuous function of (m, a) for each t.

A1′. If $a > b$, then $u(t, m, a) > u(t, m, b)$ for all t and m.

A2. The utility function $v(t, m, a)$ is a continuous function of (m, a) for each t and a strictly quasiconcave differentiable function of a.

A3. $\partial v/\partial a$ is a strictly increasing function of t.

A4. If $t<t'$ and $m<m'$,[4] then $u(t, m, a) \leqslant u(t, m', a')$ implies that $u(t', m\, a) < u(t', m', a')$.

After an appropriate relabeling of t an m, our analysis applies if "monotonic" replaces "increasing" in A3. If $\partial v/\partial a$ is strictly decreasing in t, but A0, A1′, A2, and A4 hold, then all five assumptions hold under the change of variables, say $s=-t$ and $l=-m$. Similarly, if either or both of the first two inequalities in A4 are reversed, then all assumptions hold after a reordering of types and/or signals.

The first condition is a standard regularity assumption. The next assumption, A1′, modifies A1 by labeling the Receiver's actions in a manner consistent with the (ordinal) preferences of the Sender. However, A1′ is less restrictive than A1 in the sense that A1′ requires each type of the Sender to have the same preferences only over the pure-strategy responses of the Receiver. A2 is the reason that we can relax A1. A2 guarantees that the best response correspondence of the Receiver, $\mathrm{BR}(\mu, m)$, is a continuous function of μ and m. Because we assume that the Receiver's best response is always a pure strategy, we need not impose monotonicity on arbitrary mixtures. A3 implies that $\mathrm{BR}(\mu, m)$ is increasing in μ (in the sense that if $\lambda \neq \mu$ and if λ first-order stochastically dominates μ, then $\mathrm{BR}(\lambda, m) > \mathrm{BR}(\mu, m)$). For our analysis, the only implications of A2 and A3 which we use are that $\mathrm{BR}(\mu, m)$ is a continuous function of μ and m and that $\mathrm{BR}(\mu, m)$ is increasing in μ. Taken together, A2 and A3 imply that high Sender types are stronger than low ones: The higher t is, the more R is willing to pay. Consequently, A1′ implies that any Sender type t would like R to believe that $t=T$.

A4 is crucial to our analysis. It states that if two signal–action pairs yield the same utility to some type of Sender, and one signal is greater (componentwise) than the other, then all higher types prefer to send the greater signal. Hence, the assumption guarantees that higher types are more willing to send higher signals than lower types.

When M is one dimensional, A4 is frequently derived from a condition which guarantees that the indifference curves of different Sender types through a fixed signal–action pair intersect only once. This single-crossing condition combines with A1′ to yield A4. Indeed, one can guarantee that indifference curves cross at most once (in the one dimensional case) by assuming that they are connected and that their slope $-(\partial u/\partial m)/(\partial u/\partial a)$ is strictly decreasing in t. In the multidimensional setting, we must ensure that if any two indifference surfaces cross at some signal m, then they intersect at no $m' > m$. It suffices to assume that $(\partial u/\partial m_i)/(\partial u/\partial a)$ is strictly increasing in t for all i (Engers [8] and Ramey [19] use a weaker condition), and that if $u(t, m, a) = u(t, m', a')$, then there is a path

[4] We write $m < m'$ for m and $m' \in [0, 1]^N$ if $m \neq m'$ and $m_i \leqslant m'_i$ for $i = 1, ..., N$.

$\rho = (\rho_1, ..., \rho_N)$: $[0, 1] \to \mathbb{R}^N$ such that $\rho(0) = m$, $\rho(1) = m'$, and $\rho_i'(\tau) > 0$ for each i. To see this, let $\gamma(t, \rho(\tau))$ identically satisfy $u(t, \rho(\tau), \gamma(t, \rho(\tau))) \equiv u(t, m, a)$ for all $\tau \in [0, 1]$. Differentiation and the fact that $-(\partial u/\partial m_i)/(\partial u/\partial a)$ is decreasing in t shows that $u(t', \rho(\tau), \gamma(t, \rho(\tau)))$ is strictly increasing in τ for $t' > t$. Consequently, A4 follows from A1′.

Figure 1 provides a geometric description of our assumptions on the Sender's preferences. We have drawn indifference curves for two types of Sender. A1′ implies that higher values of a increase utility, independent of t. In Fig. 1, $u(t, m_1, a_1) = u(t, m_2, a_2)$, and $t' > t$. A4 requires that the indifference curve of t' through (m_1, a_1) must lie below t's indifference curve through (m_1, a_1) for all $m > m_1$. Note that the indifference curves in Fig. 1 are upward sloping, suggesting that increasing m with the action a fixed lowers the Sender's utility. This property holds in certain applications (for example labor market signaling where higher values of m represent larger investments in education), but is not needed in our analysis.

Lemma 4.1 describes some of the implications that our assumptions have for equilibria.

LEMMA 4.1. *Fix a sequential equilibrium in which the Sender of type t' sends the signal m with positive probability and receives utility $u^*(t') = u(t', m, a)$. Assume that* A2 *and* A4 *hold. If $m < m'$, then for all $t < t'$,*

(a) *$u(t, m, a) \leqslant u(t, m', a')$ implies that $u^*(t') < u(t', m', a')$;*

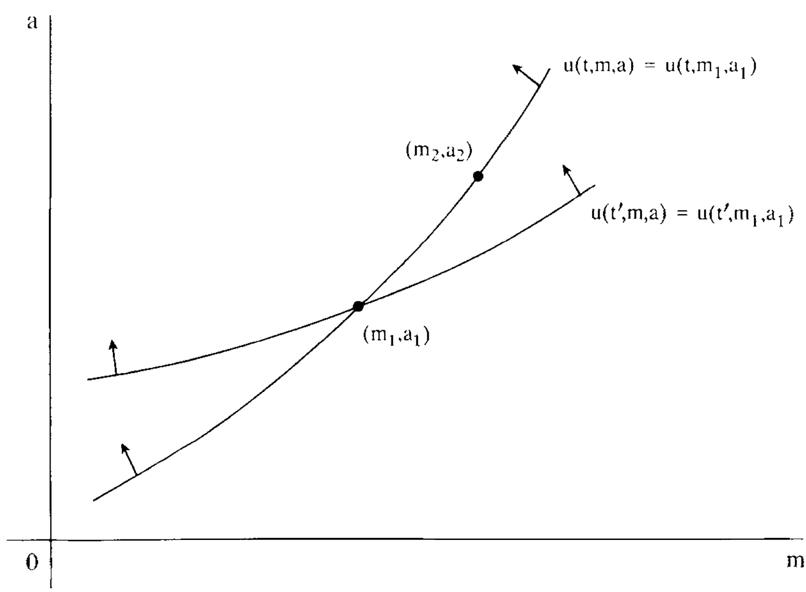

FIG. 1. The single-crossing property.

(b) $P(t \mid m') \cup P^0(t \mid m') \subset P(t' \mid m')$;

(c) *t sends the signal m' with probability zero in equilibrium; and*

(d) *if the equilibrium survives criterion* D1, *then it can be supported by beliefs that satisfy $\mu(t \mid m') = 0$ for all $t < t'$ and $m' > m$.*

Proof. Since $u^*(t') = u(t', m, a)$, part (a) follows directly from A4.
To prove part (b) it suffices to show that

$$u^*(t) \leqslant u(t, m', a') \qquad \text{implies that } u^*(t') < u(t', m', a') \tag{10}$$

because A2 implies that the Receiver's best-response set contains only pure strategies. However, since the Receiver responds to the signal m with the action a, t could obtain $u(t, m, a)$ by sending m. It follows that $u^*(t) \geqslant u(t, m, a)$. Hence (10) follows from part (a).

If the equilibrium response to signal m' is a', then it must be true that $u^*(t') \geqslant u(t', m', a')$. The contrapositive of part (a) then implies that $u^*(t) \geqslant u(t, m, a) > u(t, m', a')$. This inequality establishes part (c).

Finally, if $P(t' \mid m') \neq \varnothing$, then part (b) implies that any D1 equilibrium can be supported by beliefs such that $\mu(t \mid m') = 0$ for all $t < t'$. If $P(t' \mid m') = \varnothing$, then part (b) implies that $P(t \mid m') \cup P^0(t \mid m') = \varnothing$ for all $t < t'$. Therefore, if $P(s \mid m') \neq \varnothing$ for some s, then $s > t'$, and any D1 equilibrium can be supported by beliefs such that $\mu(t \mid m') = 0$ for all $t < t'$. Otherwise, if $P(s \mid m') = \varnothing$ for all s, then any beliefs given m' support the equilibrium. Thus, part (d) follows from part (b), and the proof is complete.

Lemma 4.1 guarantees that in any sequential equilibrium, if type t' uses the signal m with positive probability, then all types $t < t'$ never send signals $m' > m$. Thus, the set of signals used in equilibrium does not decrease with respect to t. In particular, if t' uses m in equilibrium and $m' > m$ is some signal used in equilibrium, then it must be true that $\mu(t \mid m') = 0$ for all $t < t'$. The only implication of criterion D1 that we need for our uniqueness result is that this monotonicity property holds off the equilibrium path as well. That is, if t' sends m with positive probability in equilibrium and $m' > m$ is any other signal (not necessarily sent with positive probability), then we can support the equilibrium with beliefs that satisfy $\mu(t \mid m') = 0$ for all $t < t'$. Observe that Lemma 4.1 holds without assuming A1$'$. Consequently, it identifies qualitative properties of equilibria in games where A1$'$ fails. Banks [1, 2, 3] provides interesting examples of non-monotonic signaling games in which Lemma 4.1 may be used to describe equilibria that survive refinements, even though the game in Banks [3] need not have a unique equilibrium outcome that survives D1.

Assumptions A0, A1$'$, A2–A4 are common to many general treatments of signaling models (see Riley [21]). We discuss models in which these assumptions fail in Section 6.

Fix a sequential equilibrium. We say that m is a pooled signal if more than one type of Sender uses m with positive probability. We use Lemma 4.1 to show that in any D1 equilibrium outcome, the highest type of the Sender in a pool could reveal his type if it were possible to increase his signal from the equilibrium level. Consequently, A1′ and A3 imply that pooling is possible only at $C = (c, ..., c)$.

PROPOSITION 4.1. *If* A0, A1′, *and* A2–A4 *hold, then* C *is the only possible pooled signal in a* D1 *equilibrium.*

Proof. Fix a D1 equilibrium outcome in which type t receives utility $u^*(t)$ and assume that there exists a signal $m < C$ that more than one type of Sender sends with positive probability in equilibrium. We argue to a contradiction. Let t' be the highest type that sends m with positive probability. Since there is pooling at m, A1′ and A3 imply that

$$u^*(t') < u(t', m, a(t', m)), \tag{11}$$

where $a(t', m)$ solves $\max_{a \in A} v(t', m, a)$. Lemma 4.1(d) implies that we can support the equilibrium with beliefs μ that satisfy $\mu(t \mid m') = 0$ for all $t < t'$ and $m' > m$. Therefore, the Receiver's equilibrium response to m', call it $a^*(m')$, satisfies $a^*(m') \geqslant a(t', m')$ for all $m' > m$ by A3. It follows from A1′ that the utility to t' of sending $m' > m$ is at least $u(t', m', a(t', m'))$. Consequently, we need

$$u^*(t') \geqslant u(t', m', a(t', m')) \text{ for } m' > m \tag{12}$$

in order for t' to be responding optimally. Since A0 and A2 imply that $u(t', m', a(t', m'))$ is continuous in m', (12) contradicts (11).

We now construct the candidate D1 equilibrium. First, we find the associated equilibrium payoff for the Sender. The construction is done inductively beginning with the lowest type $t = 1$. We ask: Does type $t = 1$ prefer the highest level of utility given that he reveals his type to being pooled with all higher types at the signal C? If the answer to this question is "no," then all types pool at C. If the answer is "yes," then $t = 1$ separates and sends a signal that maximizes his utility assuming that the Receiver believes the signal reveals the Sender's type to be $t = 1$. If all types $s < t$ separate, then we ask: Does t prefer the highest level of utility given that he reveals his type and no lower type wants to imitate him to being pooled with all higher types at C? Type t separates if and only if the answer to this question is "yes."

In order to develop this argument, we give explicit formulas for the Sender's utility level in a D1 equilibrium outcome. We denote these utility levels by $\bar{u}^*(t)$. We let $a(t, m)$ denote the Receiver's best response to the signal m if the Sender's type is t. That is, $a(t, m)$ is $\mathrm{BR}(\mu, m)$ where μ is concentrated on t. Also. we write $\pi|_{t \geqslant \bar{t}}$ for the prior π conditioned on $t \geqslant \bar{t}$. That is,

$$\pi|_{t \geqslant \bar{t}}(s) = \begin{cases} \pi(s) \Big/ \left[\sum_{t' = \bar{t}}^{T} \pi(t') \right] & \text{if} \quad s \geqslant \bar{t} \\ 0 & \text{if} \quad s \geqslant \bar{t}. \end{cases}$$

Note that $\pi|_{t \geqslant 1} = \pi$. For the construction, it is useful to note that A0, A1′, and A3 imply that $u(t, C, \mathrm{BR}(\alpha\pi|_{t' \geqslant t+1} + (1-\alpha)\pi|_{t' \geqslant t}, C))$ is a continuous strictly increasing function of α for all $\alpha \in [0, 1]$ and all t.

Now we begin the formal construction. Define $\bar{u}(1) = \max_{m \in M}[u(1, m, a(1, m))]$; $\bar{u}(1)$ is the maximum utility available to $t = 1$ if he separates and the Receiver responds optimally to his signal. If $\bar{u}(1) < u(1, C, \mathrm{BR}(\pi, C))$, then $t = 1$ prefers to pool at $m = C$ than to separate; we set $\bar{u}^*(t) = u(t, C, \mathrm{BR}(\pi, C))$ for all $t = 1, ..., T$. If $u(1, C, \mathrm{BR}(\pi, C)) \leqslant \bar{u}(1) < u(1, C, \mathrm{BR}(\pi|_{t' \geqslant 2}, C))$, then there is a unique convex combination μ of π and $\pi|_{t' \geqslant 2}$ satisfying $\bar{u}(1) = u(1, C, \mathrm{BR}(\mu, C))$. In this case, $t = 1$ sends $m = C$ with positive probability in equilibrium, the posterior probability of t given $m = C$ will be μ, and all Sender types $t > 1$ send $m = C$ with probability one. Thus, set $\bar{u}^*(t) = u(t, C, \mathrm{BR}(\mu, C))$ for $t = 1, ..., T$. But if $u(1, C, \mathrm{BR}(\pi|_{t' \geqslant 2}, C)) \leqslant \bar{u}(1)$, then $t = 1$ prefers to separate rather than to pool at the highest signal. In this case, set $\bar{u}^*(1) = \bar{u}(1)$. We have now completed the first step of the construction.

We continue inductively. Suppose that $\{\bar{u}^*(1), ..., \bar{u}^*(t-1)\}$ have been defined for $t > 1$, but that $\bar{u}^*(t)$ has not yet been defined. Let $\bar{u}(t)$ be the maximum value of the problem

$$\begin{aligned} & \max_{m \in M} u(t, m, a(t, m)) \\ & \text{subject to} \quad \bar{u}^*(s) \geqslant u(s, m, a(t, m)) \quad \text{for} \quad s = 1, ..., t-1. \end{aligned} \tag{$Q(t)$}$$

Thus, $\bar{u}(t)$ is the greatest level of utility available to type t if he separates, the Receiver responds optimally by choosing $a(t, m)$, and all $s < t$ would rather receive $\bar{u}^*(s)$ than imitate t's signal. There are three possible ways to continue. If $\bar{u}(t) < u(t, C, \mathrm{BR}(\pi|_{t' \geqslant t}, C))$, then type t would rather pool with all higher types than separate; we set $\bar{u}^*(s) = u(s, C, \mathrm{BR}(\pi|_{t' \geqslant t}, C))$ for $s = t, ..., T$. If $u(t, C, \mathrm{BR}(\pi|_{t' \geqslant t}, C)) \leqslant \bar{u}(t) < u(t, C, \mathrm{BR}(\pi|_{t' \geqslant t+1}, C))$, then there is a unique convex combination μ of $\pi|_{t' \geqslant t}$ and $\pi|_{t' \geqslant t+1}$ such that $\bar{u}(t) = u(t, C, \mathrm{BR}(\mu, C))$. In this case, type t sends $m = C$ with positive probability and all higher types pool at $m = C$. We define $\bar{u}^*(s) = u(s, C, \mathrm{BR}(\mu, C))$ for $s = t, ..., T$. Finally, if $u(t, C, \mathrm{BR}(\pi|_{t' \geqslant t+1}, C)) \leqslant \bar{u}(t)$, then t prefers to separate rather than pool with all higher types. We set

$\bar{u}^*(t) = \bar{u}(t)$, and continue the induction argument in order to compute $\bar{u}^*(t+1)$.

We need to confirm that the constraint set in the problem $Q(t)$ is nonempty whenever $u(t-1, C, \mathrm{BR}(\pi|_{t' \geq t}, C)) \leq \bar{u}(t-1) = \bar{u}^*(t-1)$. Otherwise we could not define $\bar{u}(t)$ when necessary. Lemma 4.2 establishes that $m = C$ is feasible for $Q(t)$, when type $t-1$ chooses to separate.

LEMMA 4.2. *Assume* A1′ *and* A2–A4 *hold. If* $\{\bar{u}^*(1), ..., \bar{u}^*(t-1)\}$ *have been constructed as above and* $u(t-1, C, \mathrm{BR}(\pi|_{t' \geq t}, C)) \leq \bar{u}(t-1) = \bar{u}^*(t-1)$, *then* $u(s, C, a(t, C)) \leq \bar{u}^*(s)$ *for all* $s < t$.

Proof. A3 implies that $\mathrm{BR}(\pi|_{t' \geq t}, C) \geq a(t, C)$. Hence, from A1′ and the given inequality,

$$u(t-1, C, a(t, C)) \leq u(t-1, C, \mathrm{BR}(\pi|_{t' \geq t}, C)) \leq \bar{u}^*(t-1). \tag{13}$$

If $m(t-1)$ is a signal that solves $Q(t-1)$, then we have $m(t-1) \leq C$ and $\bar{u}^*(s) \geq u(s, m(t-1), a(t-1, m(t-1)))$ for $s = 1, ..., t-1$. Consequently the lemma follows from the contrapositive of A4.

Lemma 4.2 guarantees that our procedure is well defined. The next result demonstrates that $\{\bar{u}^*(t)\}$ are the only possible Sender utilities in a D1 equilibrium.

PROPOSITION 4.2. *If* A0, A1′, *and* A2–A4 *hold, then in any* D1 *equilibrium the equilibrium expected utility for the type* t *Sender is* $\bar{u}^*(t)$.

The appendix contains a proof of Proposition 4.2. This proposition specifies what the payoff to the Sender must be in a D1 equilibrium. It remains to show that we can actually construct an equilibrium that survives D1. We begin by presenting strategies for the Sender and responses and beliefs for the Receiver on the equilibrium path. Later, in Proposition 4.3, we demonstrate that there exist off-the-equilibrium-path strategies and beliefs that satisfy criterion D1 and support the equilibrium path.

Let $\bar{t}$ be the highest type for which $\bar{u}(t)$ is defined. If $t < \bar{t}$, let $m^*(t)$ be a value of m that solves $Q(t)$. If $t > \bar{t}$, let $m^*(t) = C$. Let $m^*(\bar{t})$ be the probability distribution that places weight α on a solution to $Q(\bar{t})$ and weight $1-\alpha$ on $m = C$. Set $\alpha = 0$ if $\bar{t}$ pools with probability one, and otherwise let α be the unique solution to

$$\bar{u}(\bar{t}) = u(\bar{t}, C, \mathrm{BR}(\alpha\pi|_{t' \geq \bar{t}+1} + (1-\alpha)\pi|_{t' \geq \bar{t}}, C)). \tag{14}$$

These strategies are unique if and only if $Q(t)$ has a unique solution whenever t separates with positive probability. The construction guarantees that the only signal sent by more than one type of Sender is C because if type

s sends $m \neq C$ with positive probability, then m is not feasible for $Q(t)$ for any $t > s$. Thus, if $m = m^*(s) < C$ for $s < \bar{t}$, or if $\bar{t}$ sends $m < C$ with positive probability, then we can unambiguously define

$$\mu^*(t \mid m) = \begin{cases} 1 & \text{if} \quad m^*(t) = m \text{ with positive probability} \\ 0 & \text{otherwise,} \end{cases}$$

and $a^*(m) = \mathrm{BR}(\mu^*(t \mid m), m)$. If $m = C$ is on the equilibrium path, then $\mu^*(t \mid C) = \alpha\pi|_{t' \geqslant \bar{t}+1}(t) + (1-\alpha)\pi|_{t' \geqslant \bar{t}}(t)$, where α is defined implicitly in (14), and $a^*(C) = \mathrm{BR}(\mu^*(t \mid C), C)$.

We have constructed the Receiver's on-the-equilibrium-path strategies as best responses to beliefs that are consistent with the Sender's strategy. It remains to construct the off-the-equilibrium-path strategies and beliefs of the Receiver, and to show that the Sender's strategy is actually an optimal response to the Receiver. Fix a signal m such that $m^*(t) \neq m$ for all t and let $r(t, m)$ solve

$$\bar{u}^*(t) = u(t, m, r(t, m)) \tag{15}$$

if $u(t, m, a(1, m)) \leqslant \bar{u}^*(t) \leqslant u(t, m, a(t, m))$. If $u(t, m, a(1, m)) > \bar{u}^*(t)$, then let $r(t, m) = a(1, m)$ and if $u(t, m, a(t, m)) < \bar{u}^*(t)$, then let $r(t, m) = a(t, m)$. A3 implies that $a(1, m) < a(t, m)$. Therefore, by A1$'$ there is at most one solution to (15) and therefore $r(t, m)$ is well defined. Next, let

$$\hat{t}(m) = \min\{t : r(t, m) = \min_{1 \leqslant t' \leqslant T} r(t', m)\},$$

$$\mu^*(t \mid m) = \begin{cases} 1 & \text{if} \quad t = \hat{t}(m) \\ 0 & \text{if} \quad t \neq \hat{t}(m)' \end{cases} \qquad \text{and} \qquad a^*(m) = a(\hat{t}(m), m).$$

We claim that these beliefs and actions satisfy criterion D1 and support the equilibrium path. Let us interpret the construction. For each t and m, $r(t, m)$ is the response (if one exists) to m that yields utility $\bar{u}^*(t)$ for type t. If $r(t, m) = a(1, m)$, then t would never settle for $\bar{u}^*(t)$ and if $r(t, m) = a(t, m)$, then t would never send m when he could obtain $\bar{u}^*(t)$. When A1$'$ holds, D1 requires that the off-the-equilibrium-path beliefs concentrate mass on those types most willing to send m—that is, on those types for which $r(t, m)$ is least.

Proposition 4.3. *If* A0, A1$'$, *and* A2–A4 *hold, then there exists an equilibrium that survives criterion* D1.

The proof of Proposition 4.3 is in the appendix. The complexity of the proof derives primarily from the possibility that signals are multidimensional. When $M = [0, c]$, it is straightforward to check that A4 implies that $m^*(s) \leqslant m^*(t)$ for $s < t$ with equality if and only if $m^*(s) = c$, and that D1 allows the Receiver to believe that the Sender is type t given $m \in (m^*(t), m^*(t+1))$. (If $u^*(t) > u(t, m^*(t+1), a^*(t+1))$, then other specifications of beliefs are consistent with D1.) Therefore, a one dimensional message space allows an easy characterization of the beliefs that satisfy D1 and support the equilibrium path. The example of Section 5 demonstrates that when M is multidimensional, the equilibrium signals need not be increasing in type. It is this possibility that complicates the proof of Proposition 4.3.

Proposition 4.2 characterizes the Sender's utility in any D1 equilibrium. However, it does not guarantee a unique outcome. In order to ensure uniqueness, there must be exactly one signal that solves $Q(t)$ for any t that separates with positive probability in equilibrium. Otherwise, the Receiver's equilibrium payoff need not be unique. When M is one dimensional, it is sufficient to assume the following as in Mailath [16].

A5. For each t, $u(t, m, a(t, m))$ is a strictly quasi-concave function of m.

Proposition 4.4 states the result.

PROPOSITION 4.4. *If* A0, A1′, *and* A2–A5 *hold and* $M = [0, c]$, *then there is a unique* D1 *equilibrium outcome.*

We do not have a simple condition that guarantees uniqueness in the multidimensional case; A5 does not appear to be sufficient. However, when $v(t, m, a)$ is independent of m, then the Receiver's payoff does not depend upon which solution to $Q(t)$ that the Sender selects. In this case, criterion D1 uniquely determines the equilibrium utility levels of both Sender and Receiver.

In order to guarantee that only separating equilibria survive D1, we need only assume that sufficiently costly signals exist. One such condition is A6.

A6. There exists a message m such that $u(T-1, m, a(1, m)) > u(T-1, C, a(T, C))$.

If condition A6 holds, then $t = T-1$ (and hence, by A4, all $t < T$) would prefer to send m and be treated like the lowest type rather than imitate T at the highest signal $m = C$. Hence, there will be no pooling.

PROPOSITION 4.5. *If* A0, A1′, A2–A4, *and* A6 *hold, then only separating equilibria survive criterion* D1.

No separating Nash equilibrium can yield type t more than $\bar{u}^*(t)$. This observation follows because $\bar{u}(t)$ is an upper bound for the payoff of the type t Sender in any separating equilibrium; indeed, the constraints in $Q(t)$ must hold in any equilibrium. Thus, criterion D1 selects the Pareto-dominating separating equilibrium.

In this section we assumed that A was a compact interval and that M was a product of compact intervals. We now give examples to show that multiple equilibria with qualitatively different properties could exist when either M or A is finite.

Suppose that there are two types, $\frac{2}{3}=\underline{t}<\bar{t}=1$, with utility functions $u(\underline{t}, m, a)=a-m/2$ and $u(\bar{t}\, m, a)=a-m/4$. Let the Receiver's utility function be $v(t, m, a)=-(a-t)^2$ and let $\pi(\underline{t})=\pi(\bar{t})=\frac{1}{2}$.

First, observe that if $M=A=[0, 1]$, then there is a unique equilibrium outcome that survives criterion D1. In this outcome the low type sends $m=0$, the high type sends $m=\frac{2}{3}$, the Receiver's beliefs satisfy $\mu(\underline{t} \mid m)=1$ when $m \in [0, \frac{2}{3})$, $\mu(\underline{t} \mid \frac{2}{3})=0$, and $\mu(t \mid m)$ is arbitrary for $m \in [\frac{2}{3}, 1]$. In order to respond optimally to these beliefs, the Receiver's equilibrium strategy $a(m)$ must satisfy $a(m)=\frac{2}{3}$ if $m \in [0, \frac{2}{3})$ and $a(\frac{2}{3})=1$. D1 places no restrictions on $a(m)$ when $m \in (\frac{2}{3}, 1]$. In order to support the equilibrium outcome, let $a(m) \in [\frac{2}{3}, 1]$ for $m \in (\frac{2}{3}, 1]$.

Now restrict attention to $M=\{0, 1\}$. Since the utility functions satisfy A0, A1$'$, and A2–A6 for $M=[0, 1]$, one might hope that the uniqueness result of this section would apply. However, the example has three sequential equilibria, all of which survive D1. In the first equilibrium both types of Sender send $m=0$ with probability one. The Receiver's beliefs are $\mu(\underline{t} \mid 0)=\mu(\bar{t} \mid 0)=\frac{1}{2}$ and $\mu(\underline{t} \mid 1)=0$, and the Receiver's strategy is $a(0)=\frac{5}{6}$ and $a(1)=1$. The Receiver's beliefs satisfy D1. There is pooling at the low signal because the cost of sending the high signal is so great that $\bar{t}$ does not want to send it even if he can reveal himself by so doing. The second equilibrium in this example is partially separating. This time, $\underline{t}$ sends $m=0$ with probability one; $\bar{t}$ sends $m=1$ with probability $\frac{2}{3}$, and $m=0$ with probability $\frac{1}{3}$. Since both messages are sent with positive probability, Bayes' Rule determines μ and D1 places no restrictions on beliefs. We find that $\mu(\underline{t} \mid 0)=\frac{3}{4}$, $\mu(\bar{t} \mid 0)=\frac{1}{4}$, $\mu(\underline{t} \mid 1)=0$, $\mu(\bar{t} \mid 1)=0$, and $\mu(\bar{t} \mid 1)=1$. As a result, the Receiver's responses are $a(0)=\frac{3}{4}$ and $a(1)=1$. The third equilibrium is a separating equilibrium in which $\bar{t}$ sends $m=1$ with probability one, and $\underline{t}$ sends $m=0$ with probability one. The Receiver's responses are $a(0)=\frac{2}{3}$ and $a(1)=1$. Again, D1 does not constrain beliefs.

This example does not require A to be an interval. If $A=\{\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, 1\}$, then all equilibria described above remain. ($\bar{t}$ need not randomize exactly as specified above in the hybrid equilibrium.) All three equilibria are (as singletons) strategically stable sets. Thus, this variation of the example demonstrates that the equivalence of D1 and strategic stability presented in Section 3 may hold even when equilibrium outcomes are not unique.

The Sender's payoffs are different in the three equilibria. In the pooling equilibrium $u^*(\underline{t})=u^*(\bar{t})=\frac{5}{6}$, in the semi-pooling equilibrium $u^*(\underline{t})=u^*(\bar{t})=\frac{3}{4}$, and in the separating equilibrium $u^*(\underline{t})=\frac{2}{3}$ and $u^*(\bar{t})=\frac{3}{4}$. While the fact that both types of Sender prefer the pooling equilibrium may argue in its favor, note that when $M=[0,1]$ the pooling outcome no longer survives criterion D1. In this case, the unique D1 equilibrium yields $u^*(\underline{t})=\frac{2}{3}$ and $u^*(\bar{t})=\frac{5}{6}$.

In order to demonstrate that our uniqueness results do not apply when A is discrete but $M=[0,1]$, consider the example with $A=\{\frac{1}{2},1\}$. Here there is a continuum of pooling equilibria that survive D1. Fix $m^* \geqslant 0$ and let $\mu(\underline{t} \mid m)=1$ and $\mu(\bar{t}\, m)=0$ for $m<m^*$ and let $\mu(\underline{t} \mid m)=\mu(\underline{t} \mid m)=\frac{1}{2}$ for $m \in [m^*, 1]$. It is a D1 equilibrium for both Sender types to pool at m^* and for the Receiver to respond to $m<m^*$ with $a=\frac{1}{2}$ and to $m \geqslant m^*$ with $a=1$. When $m^*=1$, $\underline{t}$ may randomize arbitrarily over $m=0$ and 1. (There are also sequential equilibria in which $\bar{t}$ randomizes, but these equilibria do not survive D1.)

The class of examples described above have multiple D1 equilibria because Proposition 4.1 does not hold. None of our other arguments depends on the continuous structure of strategy spaces in an essential way. When A is discrete, Proposition 4.1 may fail because A3 is false as stated; with a discrete action space, a small increase in μ (in the sense of first-order stochastic dominance) typically leaves the Receiver's best response unchanged. Consequently, a high type may have nothing to gain by separating from a pool with lower types. When M is discrete, the cost of sending the "next highest" signal may be so great that a Sender type would prefer to pool with lower types even if he could reveal his characteristic by sending a higher signal.

5. Example

Cho and Kreps [6] present a detailed discussion of how equilibrium refinements select outcomes in a game-theoretic version of the Spence [22] labor-market signaling model. Our analysis generalizes theirs in at least two ways. We allow signals to be multidimensional and we allow models in which pooling may occur in the (refined) equilibrium. This section contains an example that illustrates the more general features of our analysis and clarifies the construction of Section 4.

$M=[0,1]^2$, $A=[0,10]$, and there are three Sender types, which we represent as $t=1, 2$, and k. These types are equally likely. The preferences are: $u(1,m,a)=a-m_1-m_2$, $u(2,m,a)=a-m_1/2-m_2/4$, $u(k,m,a)=a-m_1/10-m_2/5$, and $v(t,m,a)=-(t-a)^2$. We use k as a parameter to describe several possible cases; we assume $2<k<10$. Provided that $k>2$, A0, A1′, and A2–A4 hold. The Receiver's preferences have the convenient property that for all m, $\mathrm{BR}(\mu, m)$ is equal to the expected value of t with respect to μ.

Let us characterize the outcomes that survive criterion D1. First, note that $\bar{u}(1)=1$; type $t=1$ attains this utility by sending $m_1=m_2=0$. The lowest type separates with probability zero if $\bar{u}(1)<u(1, C, \mathrm{BR}(\pi, C))$, where π is the (flat) prior and $C=(1, 1)$ is the highest possible signal. Therefore, $t=1$ always pools when $1<(1+2+k)/3-2$ or $k>6$. So when $k>6$ the D1 equilibrium requires all types to send the message $(1, 1)$; the Receiver's optimal response to this signal is $(1+2+k)/3$. It is straightforward to check that the only off-the-equilibrium-path beliefs that satisfy criterion D1 are $\mu^*(t \mid m)=\delta_1(t)$, where

$$\delta_s(t)=\begin{cases}1 & t=s\\ 0 & t\neq s.\end{cases}$$

Hence, the Receiver's equilibrium strategy, $a^*(m)$, is equal to one for all $m\neq(1, 1)$.

The second case occurs when $t=1$ pools with a positive probability less than one. For the lowest type to pool, it must be that $k\leqslant 6$ and $\bar{u}(1)<u(1, C, \mathrm{BR}(\pi|_{t'\geqslant 2}, C))=(2+k)/2-2$. Consequently, if $k>4$, then $t=1$ must randomize between $(0, 0)$ and $(1, 1)$. A straightforward computation reveals that these two signals yield the same utility to $t=1$ exactly when the Receiver responses to C with $a^*(C)=3$. This action is an optimal response for the Receiver provided that types $t=2$ and k play $m=C$ with probability one, and $t=1$ plays $m=(0, 0)$ with probability $(6-k)/2$ and $m=C$ with probability $(k-4)/2$. As in the first case, $\mu^*(t \mid m)=\delta_1(t)$ are the only beliefs that satisfy criterion D1 when $m\neq C$.

When $k\leqslant 4$, $t=1$ separates with probability one. In order to compute the rest of the outcome we must find $\bar{u}(2)$, which is the value of

$$\max\, 2-m_1/2-m_2/4$$

subject to

$$u^*(1)=1\geqslant 2-m_1-m_2$$

$$0\leqslant m_1,\ m_2\leqslant 1.$$

This optimization problem has its unique solution at $(m_1, m_2)=(0, 1)$ and $\bar{u}(2)=\frac{7}{4}$. The value of k determines whether or not $t=2$ pools with the highest type. Since $u(2, C, \mathrm{BR}(\pi|_{t'\geqslant 2}, C))=(2+k)/2-3/4$, $t=2$ will pool at C with probability one whenever $(2+k)/2-3/4>7/4$ or $k>3$. Consequently, when $k\in(3, 4]$, a third distinct type of equilibrium exists. In the corresponding outcome, $t=1$ sends $m=(0, 0)$ with probability one, $t=2$ and k send $m=(1, 1)$ with probability one, and $a^*(0, 0)=1$, $a^*(1, 1)=(2+k)/2$. The off-the-equilibrium-path beliefs are more complicated than before. Since the two highest types pool at $m=C$, D1 (combined with A4) implies that $\mu^*(k \mid m)=0$ for $m\neq C$. However, whether μ^* places positive probability on $t=2$ depends on m. We need to compute $r(t \mid m)$, the value of the Receiver's action a that satisfies $u^*(t)=u(t, m, a)$. Since $u^*(1)=1$ and $u^*(2)=(2+k)/2-3/4$, we obtain $r(1 \mid m)=1+m_1+m_2$ and $r(2 \mid m)=1/4+k/2+m_1/2+m_2/4$. When $r(1 \mid m)<r(2 /\mid m)$, D1 requires that $\mu^*(t \mid m)=\delta_1(t)$. When $r(1 \mid m)>r(2 \mid m)$, D1 requires that $\mu^*(t \mid m)=\delta_2(t)$. If $r(1 \mid m)=r(2 \mid m)$, then D1 requires only that $\mu^*(t \mid m)$ place zero weight on $t=k$. Thus, beliefs consistent with criterion D1 have

$$\mu^*(t \mid m)=\delta_1(t) \qquad \text{if} \quad m \in \{(m_1, m_2): 2m_1+3m_2<2k-3\},$$

$$\mu^*(t \mid m)\delta_2(t) \qquad \text{if} \quad m \in \{(m_1, m_2): 2m_1+3m_2>2k-3\} \setminus \{(1, 1)\},$$

and

$$\mu^*(t \mid C)=\pi|_{t \geqslant 2}.$$

The Receiver's equilibrium strategy responds optimally to these beliefs.

When $k \leqslant 3$, $t=2$ separates with positive probability. Separation occurs with probability strictly less than one if $\bar{u}(2)=\frac{7}{4}<u(2, C, \mathrm{BR}(\pi|_{t' \geqslant 3}, C))=k-\frac{3}{4}$. Thus, if $k \in (\frac{5}{2}, 3]$, then $t=1$ sends $m=(0,0)$ with probability one; $t=2$ sends $m=(0,1)$ with probability $6-2k$ and sends $m=(1,1)$ with probability $2k-5$; and $t=k$ sends $m=(1,1)$ with probability one. We specified the strategy of type $t=2$ so that

$$\mu^*(t \mid (1,1))=\begin{cases} 0 & \text{if} \quad t=1 \\ (2k-5)/[2(k-2)] & \text{if} \quad t=2 \\ 1/[2(k-2)] & \text{if} \quad t=k. \end{cases}$$

In this way, $\mathrm{BR}(\mu^*(\cdot \mid (1,1)), (1,1))$ is equal to $\frac{5}{2}$. As a result, $t=2$ is indifferent between the two signals he sends with positive probability in equilibrium. We can find off-the-equilibrium-path beliefs that satisfy criterion D1 as we did when $k \in (3, 4]$. Computation reveals that

$$\mu^*(t \mid m)=\delta_1(t) \qquad \text{if} \quad 2m_1+3m_2<3,$$

and

$$\mu^*(t \mid m)=\delta_2(t) \qquad \text{if} \quad 2m_1+3m_2>3.$$

If $2m_1+3m_2=3$ and $(m_1, m_2) \neq (0,1)$, then $\mu^*(k \mid m)=0$, but otherwise μ^* is arbitrary along this line. If $(m_1, m_2)=(0,1)$, then $\mu^*(t \mid m)=\delta_2(t)$ since $(0,1)$ is on the equilibrium path.

Finally, when $2<k \leqslant \frac{5}{2}$, the equilibrium is completely revealing. Type $t=1$ sends $m=(0,0)$ with probability one; $t=2$ sends $m=(0,1)$ with probability one; and $t=k$ sends the signal that solves

$$\max k-m_1/10-m_2/5,$$

subject to

$$1 = u^*(1) = \bar{u}(1) \geqslant k - m_1 - m_2,$$

$$7/4 = u^*(2) = \bar{u}(1) \geqslant k - m_1/2 - m_2/4, \qquad \text{and} \qquad 0 \leqslant m_1,\ m_2 \leqslant 1.$$

If $k \geqslant \frac{7}{3}$, then the constraint involving $u^*(1)$ does not bind at the optimum; $t = k$ sends $m = (1, 4k - 9)$. If $k < \frac{7}{3}$, then the constraint involving $u^*(2)$ does not bind at the optimum; $t = k$ sends $m = (1, k - 2)$. A tedious computation reveals that for $m \notin D = \{(m_1, m_2)\colon 0 \leqslant m_1,\ m_2 \leqslant 1,\ m_1 + m_2 > k - 1, \text{ and } 2m_1 + m_2 > 4k - 7\}$,

$$\mu^*(t \mid m) = \begin{cases} \delta_1(t) & \text{whenever} \quad 2m_1 + m_2 < 3 \\ \delta_2(t) & \text{whenever} \quad 2m_1 + m_2 > 3. \end{cases}$$

If $2m_1 + 3m_2 = 3$ and $(m_1, m_2) \neq (0, 1)$, then any probability distribution $\mu^*(\cdot \mid m)$ with $\mu^*(k \mid m) = 0$ is allowed by D1. D1 places no restrictions on beliefs for $m \in D$ because in this event all Sender types prefer their equilibrium payoff to the best possible response given m $(a^*(m) = k)$.

Notice that in the separating equilibrium $t = k$ sends a signal that is not larger than the equilibrium signal of $t = 2$. Type $t = 2$ invests more heavily in the second component of the signal than $t = k$ because $t = 2$ has a comparative advantage in signal two. Also observe that when $k \in (2, \frac{7}{3})$ the constraint that guarantees that type $t = i$ does not want to imitate type $t = k$ binds when $i = 1$ but not when $i = 2$. Thus, in multidimensional signaling problems it is insufficient to check only that a type does not want to send the signal of the next higher type; verifying only these "local" constraints is sufficient when A4 holds and the signaling space is one dimensional.

The example has multiple equilibria that survive the "intuitive criterion" of Cho and Kreps [6]. In monotonic signaling games, this condition requires that $\mu^*(t \mid m) = 0$ for any unsent signal m and type t for which $P(t \mid m) \cup P^0(t \mid m) = \varnothing$ (provided that $P(t' \mid m) \cup P^0(t' \mid m) \neq \varnothing$ for some t'). The intuitive criterion is less restrictive than D1. Indeed, it is straightforward to check that the intuitive criterion places no restrictions on off-the-equilibrium-path beliefs whenever $k \geqslant \frac{9}{2}$. For $k \geqslant \frac{9}{2}$, the outcome in which all Sender types send the same signal m can be supported as a sequential equilibrium that survives the intuitive criterion provided that $m_1 + m_2 \leqslant k/3$ (if $m_1 + m_2 > k/3$, then the $t = 1$ Sender would prefer to send $(0, 0)$). Multiple equilibria that survive the intuitive criterion exist for all values of k.

6. Related Models

This section discusses several papers that analyze signaling models. We describe situations where the refinement concepts of this paper select an equilibrium similar to ours—even if our assumptions do not hold—and where our results do not apply. We do not provide a comprehensive survey of signaling games used in applications.

A4 is a strong assumption. Some form of this condition is necessary in order to obtain separating equilibria (see Mailath [16]); however, in certain economic applications, the assumption holds for only part of the parameter space. An example is the limit-pricing model of Milgrom and Roberts [17]. In this model, an incumbent monopolist with private information about its cost of production chooses an output level. This choice signals the value of the market to a potential entrant who, after observing the monopolist's output, decides whether or not to enter. There is a second period in which the incumbent earns duopoly or monopoly profit depending on whether there was entry. Milgrom and Roberts assume that the incumbent is uncertain about the potential entrant's costs. Payoffs depend on particular assumptions about the information revealed in the post-entry game and the nature of competition in the second period. However, if the incumbent knows the probability of entry, then it can compute the expected second-period profit. This formulation is more complicated than the games discussed in this paper but the model can be analyzed as a signaling game in which the incumbent firm's payoff depends upon its cost (type), its first-period output (signal), and the probability of entry. Under the Milgrom–Roberts assumption that private information is truthfully revealed in the post-entry game, A4 may fail at outputs higher than the complete information monopoly level. Consequently, D1 does not select a separating equilibrium in general. In fact, one can construct an example in which the unique D1 equilibrium is a pooling equilibrium that strictly Pareto dominates every separating equilibrium from the incumbent's point of view. Cho [5] drops the assumption that private information is revealed in the post-entry game. He shows that in this case A4 holds at outputs higher than the monopoly level, and uses arguments similar to those in Section 4 to demonstrate that D1 selects the Pareto-efficient separating equilibrium.

Unique stable outcomes may exist in signaling games when A4 holds only in the economically relevant portion of the strategy space. Yet sometimes A4 fails globally. Crawford and Sobel [7] analyze a model in which $u(t, m, a)$ is independent of m. Hence, A4 does not hold. Crawford and Sobel further restricted attention to a class of single-peaked utility functions for which A1′ does not hold. The Crawford–Sobel model generally has multiple sequential equilibria all of which involve partial pooling. The equilibrium refinement ideas discussed in this paper do not reduce the equilibrium set.

D1 may be used to select equilibria in signaling models in which A1′ fails. Here, as in the limit pricing model, economic considerations may guarantee that monotonicity holds on the equilibrium path even if it fails globally. One instance of such a global failure occurs in the widely studied models of settlement and litigation. For example, in Reinganum and Wilde [20] a plaintiff knows the value of damages that a court would award if there were a trial. The plaintiff makes a take-it-or-leave-it settlement demand to the defendant. If the offer is rejected, then the dispute goes to trial. Trial is costly to both litigants. The court requires the defendant to pay the true damages to the plaintiff.[5] The litigants seek to maximize their net total payment minus court costs. In this model, the plaintiff prefers that the defendant accept the settlement demand provided that it exceeds the value of going to trial. Therefore, A1 does not hold for all offers. Furthermore, the defendant has only two actions. Still there exists a unique outcome that survives criterion D1. This outcome is similar to the one characterized in Section 4. Since the analysis of these models is so close to our own, it is worthwhile to give conditions under which our results hold.

Let $A = \{\underline{a}, \bar{a}\}$, $M = [0, c]$, and the set of types be $\{1, 2, ..., T\}$. Make the following assumptions.

B0. The utility function $u(t, m, a)$ is continuous in m.

B1. There exist $d(1) \leqslant d(2) \leqslant \cdots \leqslant d(T) < 1$ such that $u(t, m, \bar{a}) - u(t, m, \underline{a}) \gtreqless 0$ for $m \gtreqless d(t)$

B2. The utility function $v(t, m, a)$ is continuous in m and $v(t, m, \bar{a}) - v(t, m, \underline{a}) = 0$ for finitely many t and m.

B3. $v(t, m, \bar{a}) - v(t, m, \underline{a})$ is a strictly increasing function of t.

B4. If $t < t'$, $d(t') < m'$, and $m < m'$, then $u(t, m, a) \leqslant u(t, m', a')$ implies that $u(t', m, a) < u(t', m', a')$.

B5. For all $m < c$ and $t > 1$, there exists $m' > m$ such that $m' > d(t)$ and $u(t, m', a') > u(t, m, a)$ for all $a' \in \mathrm{MBR}(\delta_t, m')$ and $a \in \mathrm{MBR}(\mu, m)$ where $\mu \neq \delta_t$ and $\mu(s) = 0$ for $s > t$.

Condition B1 weakens A1. It states that all types eventually prefer $\bar{a}$ to $\underline{a}$ and that $u(t, m, \bar{a}) - u(t, m, \underline{a})$ changes signs at most once. It also requires that if $u(t, m, \bar{a}) \geqslant u(t, m, \underline{a})$, then $u(s, m, \bar{a}) \geqslant u(s, m, \underline{a})$ for all $s \leqslant t$. B2 and B3 modify A2 and A3 to a two-point action space. The second restriction in B2 rules out degenerate cases in which the Receiver is indifferent between both pure strategies for an interval of signals. B4 weakens A4 by requiring that it hold for only a subset of signals. Finally, B5 guarantees that the highest type in a pool would prefer to send a higher signal provided that signal reveals his type. It is straightforward to verify that B5 holds if A1′ and A2–A4 hold and that B1–B5 hold in the ligitation model.

[5] Reinganum and Wilde [20] assume that there is a positive probability, independent of damages and the settlement demand, that the court will rule that no payment is needed. This modification does not alter the formal analysis.

Moreover, the arguments of Section 4 apply with minor modifications to characterize equilibria that survive D1.

There is no pooling except possibly at the highest signal. The utility $\bar{u}(t)$ of the type t Sender in a separating equilibrium satisfies $\bar{u}(t) = \max\{u(t, m, a) : a \in \mathrm{BR}(\delta_t, m),\ \ u(t-1, m, a) \leqslant \bar{u}(t-1)\}$, and type t separates if he prefers $\bar{u}(t)$ to being pooled with all higher types. If $t > 1$, then t never uses signals less than $d(t)$ in a D1 equilibrium.

We should note another model in which A1 need not hold. Laffont and Maskin [15] study a model where a monopolist provides goods of uncertain quality. The seller has private information about quality and sends a signal (price) to the buyers. The number of purchases places the role of the Receiver's action in this model. Laffont and Maskin assume that the monopolist values unsold products. Translating into our notation the seller's preferences can be written $u(t, m, a) = ma + f(t, 1-a)$, where m is price, a is the fraction of output purchased, and $f(\cdot)$ is the value of the unsold product. If the price is low enough (or marginal value of the first bit of unsold product sufficiently high), then $u(\cdot)$ need not be increasing in a. Therefore, A1′ does not hold in this model. Laffont and Maskin provide conditions under which separating equilibria exist in their model, and carry out a construction similar to ours. The single-crossing property A3 holds in their model and plays a crucial role in the construction. In addition, they provide arguments that select the equilibrium outcome which maximizes the ex ante profit of the seller. This outcome need not survive criterion D1.

Our analysis allows the possibility that there is an upper bound to signals. This restriction arises naturally in some economic models. Hoshi [11] presents a model in which the government's choice of current inflation rate influences people's expectations about future inflation. He assumes that the government cannot set an inflation rate lower than the level that makes the social cost of money equal to zero. For certain parameter values, his model's unique D1 equilibrium outcome involves pooling at the lowest allowed level of inflation. Ramey [18] presents a limit-pricing model in which no separating equilibrium exists for a range of parameter values. In this model the incumbent firm cannot charge negative prices.

7. Concluding Remarks

The refinement concept that we use is very powerful. One can construct convincing examples to demonstrate how *sensible* outcomes fail to be stable outcomes. It is certainly true that our equilibria are particularly sensitive to the details of the extensive form. For example, separating equilibrium strategies depend only on the support of the distribution of the Sender types. Therefore, the equilibria that we select may change dramatically if one perturbs the game by including a small probability that there is some other type of Sender. Fudenberg, Kreps, and Levine [9] demonstrate a related point in a more general setting. They show that the set of outcomes which are limits of *nice* equilibrium outcomes of *nearby* games may be very large. If these *nearby* games are accurate descriptions of the underlying economic situation, then the modeler should not be confident in predictions obtained by examining refinements of equilibria of a fixed game.

Nevertheless, we feel that we have a stronger intuition about the outcomes surviving criterion D1 than the typically smaller set of stable outcomes. Identifying a situation in which the full power of stability is not necessary to identify outcomes may be useful in applications. Moreover, the results in Section 4 provide a coherent justification for outcomes that often receive prominence in applications.

APPENDIX

We prove three preliminary facts before we establish Proposition 4.2. Fix a D1 equilibrium in which the Sender of type t obtains utility $u^*(t)$.

Fact 1. If there exists a type t who is pooled with other types, then every $t' > t$ must use $m = C$ with probability one.

Proof of Fact 1. Proposition 4.1 states that if t is pooled with other types, then he must send $m = C$ with positive probability. It follows from Lemma 4.1(c) that if $t' > t$, then t' sends $m = C$ with probability one.

Fact 2. Type $t < T$ separates with probability one if and only if $u^*(t) \geqslant u(t, C, \mathrm{BR}(\pi|_{t' \geqslant t+1}, C))$.

Proof of Fact 2. If type $t < T$ pools with positive probability, then from Fact 1 it follows that all higher types send $m = C$ with probability one, and from Proposition 4.1 that t sends $m = C$ with positive probability. Consequently, $u^*(t) = u(t, C, \mathrm{BR}(\mu, C))$, where μ is strictly first-order stochastically dominated by $\pi|_{t' \geqslant t+1}$. Hence, A1$'$ and A3 imply that $u^*(t) < u(t, C, \mathrm{BR}(\pi|_{t' \geqslant t+1}, C))$.

Conversely, if $t < T$ separates with probability one, then Lemma 4.1(c) implies that t sends $m = C$ with probability zero and, since t could always send $m = C$,

$$u^*(t) \geqslant u(t, C, \mathrm{BR}(\mu, C)), \tag{16}$$

where μ is the Receiver's belief given $m = C$. However, if the Sender sends $m = C$ with positive probability in equilibrium, then Fact 1 implies that μ weakly first-order stochastically dominates $\pi|_{t' \geqslant t+1}$; if the Sender sends $m = C$ with probability zero, then Lemma 4.1(d) implies that $\mu = \tau|_{t' \geqslant t+1}$ can support the equilibrium. In either case, $u^*(t) \geqslant u(t, C, \mathrm{BR}(\pi|_{t' \geqslant t+1}, C))$ follows from A1′, A3, and (16).

Fact 3. If $\tilde{t}$ is the lowest type that pools with positive probability, then $u^*(t) \geqslant \max\{u(t, m, a(t, m)) : u^*(s) \geqslant u(s, m, a(t, m))$ for $s < t\} \equiv \tilde{u}(t)$ for all $t \leqslant \tilde{t}$. If t separates with positive probability, then $u^*(t) = \tilde{u}(t)$.

Proof of Fact 3. If t separates with positive probability, then $u^*(t) \leqslant \tilde{u}(t)$ follows because no type $s \neq t$ can gain from imitating type t in equilibrium, and if t sends a separating signal m with positive probability, then the Receiver's equilibrium response must be $a(t, m)$.

It remains to show that $u^*(t) \geqslant \tilde{u}(t)$ for $t \leqslant \tilde{t}$. If $u^*(t) < \tilde{u}(t)$, then there exists a signal m such that

$$u^*(s) \geqslant u(s, m, a(t, m)) \qquad \text{for all} \quad s < t \tag{17}$$

and

$$u^*(t) < u(t, m, a(t, m)). \tag{18}$$

Now, no Sender of type s such that $s < t \leqslant \tilde{t}$ sends m with positive probability because this type separates with probability one and thus would receive $u(s, m, a(s, m))$ which, by A1′, A3, and (17), is strictly less than $u^*(s)$. Furthermore, A1′, (17), and (18) imply that $P(s \mid m) \cup P^0(s \mid m) \subset P(t \mid m)$ for $s < t$. Therefore, in any D1 equilibrium $\mu(s \mid m) = 0$ for $s < t$. Hence A1′ and A3 imply that if $a^*(m)$ is the Receiver's equilibrium response to m, then

$$u(t, m, a(t, m)) \leqslant u(t, m, a^*(m)). \tag{19}$$

Since the type t Sender is optimizing in equilibrium,

$$u(t, m, a^*(m)) \leqslant u^*(t). \tag{20}$$

However (19) and (20) contradict (18). Thus, $u^*(t) \geqslant \tilde{u}(t)$ for $t \leqslant \tilde{t}$ as claimed.

Proof of Proposition 4.2. Let $\tilde{t}$ be the lowest type that pools with positive probability (if there is no pooling, then put $\tilde{t} = T$). If $t < \tilde{t}$, then t separates with probability one. First, we will use induction to show that $u^*(t) = \bar{u}^*(t)$ for $t < \tilde{t}$ and $\tilde{u}(t) = \bar{u}(t)$ for $t \geqslant \tilde{t}$. Since $\tilde{u}(1) = \bar{u}(1)$, Fact 3 implies that $\tilde{u}(1) = u^*(1)$ if $1 < \tilde{t}$. It follows from Fact 2 that $\bar{u}^*(1) = \bar{u}(1)$ and therefore $\bar{u}^*(1) = u^*(1)$ if $1 < \tilde{t}$. Moreover, if $u^*(s) = \bar{u}^*(s) = \bar{u}(s) = \tilde{u}(s)$ for all $s < t \leqslant \tilde{t}$, then $\tilde{u}(t) = \bar{u}(t)$ from the definition of these values, and Fact 3 implies that $\tilde{u}(t) = u^*(t)$ if $t < \tilde{t}$. Therefore, $u^*(t) = \bar{u}(t)$ for $t < \tilde{t}$. It follows from Fact 2 and the construction of $\bar{u}^*(t)$ that $\bar{u}^*(t) = u^*(t)$. This completes the induction.

If $\tilde{t}$ separates with positive probability, then

$$u^*(\tilde{t}) = \tilde{u}(\tilde{t}) = u(\tilde{t}, C, \mathrm{BR}(\mu, C)) \tag{21}$$

and

$$u^*(t) = u(t, C, \mathrm{BR}(\mu, C)) \qquad \text{for} \quad t > \tilde{t}, \tag{22}$$

where the Receiver's beliefs given $m = C$ are μ. The first equality of (21) follows from Fact 3; the second equality holds since $\tilde{t}$ must be indifferent between pooling and separating if he randomizes in equilibrium; and (22) follows from Fact 1. If $\tilde{t} = T$, then $\tilde{u}^*(\tilde{t}) = \bar{u}(\tilde{t})$ and we are done. Otherwise, $u^*(\tilde{t}) < u(\tilde{t}, C, \mathrm{BR}(\pi|_{t' \geqslant \tilde{t}+1}, C))$ by Fact 2 and $\tilde{u}(\tilde{t}) = \bar{u}(\tilde{t})$, (21), (22), and the definition of $\bar{u}^*(t)$ imply that $u^*(t) = \bar{u}^*(t)$ for $t \geqslant \tilde{t}$ when $\tilde{t}$ separates with positive probability.

It remains to show that $u^*(t) = \bar{u}^*(t)$ for $t \geqslant \tilde{t}$ when $\tilde{t}$ pools with probability one. In this case, $u^*(t) = u(t, C, \mathrm{BR}(\pi|_{t' \geqslant \tilde{t}}, C))$ for $t \geqslant \tilde{t}$ by Fact 1. Consequently, since Fact 3 guarantees that $u(\tilde{t}, C, \mathrm{BR}(\pi|_{t' \geqslant \tilde{t}}, C)) \geqslant \tilde{u}(\tilde{t})$, the result follows from $\tilde{u}(\tilde{t}) = \bar{u}(\tilde{t})$ and the definition of $\bar{u}^*(t)$.

Proof of Proposition 4.3. We prove that the candidate equilibrium $(m^*(t), a^*(m), \mu^*(t \mid m))$ defined in the text really is an equilibrium that survives criterion D1. We have constructed a^* to respond optimally to μ^*, and μ^* to be consistent with m^*. We now show that μ^* satisfies criterion D1, and that the Sender responds optimally to a^* by using the strategy m^*.

When $r(\hat{t}(m), m) = a(t, m)$, $P(t \mid m) = \varnothing$ for all t and therefore D1 places no restrictions on $\mu^*(\cdot \mid m)$. Otherwise, D1 requires only that $\mu^*(t \mid m) = 0$ if $r(t, m) > r(\hat{t}(m), m)$. Hence, μ^* satisfies D1 since $\mu^*(t \mid m) = 0$ for all $t \neq \hat{t}(m)$.

We now demonstrate that the Sender optimally responds to a^* when constrained to signals on the equilibrium path. That is, we verify that for all $s \neq t$,

$$\bar{u}^*(t) \geqslant u(t, m^*(s), a^*(m^*(s))). \tag{IC(s, t)}$$

Let $\bar{t}$ be the lowest type that pools with positive probability. When $t < s < \bar{t}$, IC(s, t) follows from the constraints in $Q(t)$. When $\bar{t}$ separates with positive probability and $s \geqslant \bar{t} > t$, the construction guarantees that IC$(\bar{t}, t)$ holds and $\bar{u}^*(\bar{t}) = u(\bar{t}, C, a^*(C)) = \bar{u}(\bar{t}) = u(\bar{t}, m, a(\bar{t}, m))$ for some $m < C$, and $\bar{u}^*(t) \geqslant u(t, m, a(\bar{t}, m))$. Thus, the contrapositive of A4 implies that $\bar{u}^*(t) > u(t, C, a^*(C)) = u(t, m^*(s), a^*(m^*(s)))$ for $s > \bar{t}$, and therefore IC(s, t) holds. When $\bar{t} > 1$ pools with probability one, then for $s \geqslant \bar{t}$, IC$(s, \bar{t}-1)$ holds by construction, and IC(s, t) holds for $s \geqslant \bar{t} > \bar{t}-1 > t$ by the contrapositive of A4, since both IC$(s, \bar{t}-1)$ and IC$(\bar{t}-1, t)$ hold. Finally, when $s > t > \bar{t}$, s and t pool, so IC(s, t) holds trivially.

Next, we show that IC(s, t) holds when $s < t$. As a preliminary step, we claim that for all $s < t \leqslant \bar{t}$, there exists $\beta \in (0, 1]$ such that if

$$m' = (1-\beta)\, m^*(s) + \beta C, \tag{23}$$

then

$$\bar{u}^*(t') \geqslant u(t', m', a(t, m')) \qquad \text{for all} \quad t' < t, \tag{24}$$

and

$$\bar{u}^*(t'') = u(t'', m', a(t, m')) \qquad \text{for some} \quad t'' < t. \tag{25}$$

Recall from Lemma 4.2 that if $t' < t \leqslant \bar{t}$, then

$$\bar{u}^*(t') \geqslant u(t', C, a(t, C)). \tag{26}$$

Also, $s < t \leqslant \bar{t}$ implies that

$$\bar{u}^*(s) = u(s, m^*(s), a(s, m^*(s))) < u(s, m^*(s), a(t, m^*(s))), \tag{27}$$

where the inequality follows from A1′ because $a(s, m^*(s)) < a(t, m^*(s))$ by A3. Consequently, the claim follows from (26), (27), and the continuity of $u(t', m, a(t, m))$ in m (the latter is a consequence of A0 and A2).

The claim allows us to show that IC(s, t) holds for $s < t$. There is nothing to show when $t = 1$. Assume that IC(s, t') holds for $s \leqslant t' < t \leqslant \bar{t}$. Now we show that IC$(s, t)$ holds. Find m' to satisfy (24) and (25). By (24), m' is feasible for $Q(t)$. Thus,

$$\bar{u}^*(t) \geqslant \bar{u}(t) \geqslant u(t, m', a(t, m')). \tag{28}$$

Furthermore, there exists $t'' < t$ such that

$$u(t'', m^*(s), a^*(m^*(s))) \leqslant \bar{u}^*(t'') = u(t'', m', a(t, m')). \tag{29}$$

The inequality in (29) is IC(s, t''), which holds by the induction hypothesis if $s < t''$ (since $t'' < t$), and by earlier arguments if $s \geqslant t''$. Since $m' > m^*(s)$ by (23), it follows from A4, (28), and (29) that IC(s, t) holds for $s < t \leqslant \bar{t}$. If $s \leqslant \bar{t} < t$, then IC(s, t) follows from A4. And if $t > s > \bar{t}$, then s and t pool, trivially implying IC(s, t).

Finally, we check that for all m and t,

$$\bar{u}^*(t) \geqslant u(t, m, a^*(m)). \tag{30}$$

By the definition of $r(t, m)$ and A1', (30) holds if and only if $r(t, m) \geqslant a^*(m)$ for all t. Thus, we need only show that $r(\hat{t}(m), m) \geqslant a^*(m)$ for all m off the equilibrium path.

If $\hat{t}(m) = 1$, then $\bar{u}^*(1) \geqslant \bar{u}(1) \geqslant u(1, m, a(1, m))$ by the definition of $\bar{u}(1)$. Thus, $r(1, m) \geqslant a^*(m)$. If $1 < \hat{t}(m) \leqslant \bar{t}$ and if m satisfies the constraints of $Q(\hat{t}(m))$, then $\bar{u}^*(\hat{t}(m)) \geqslant \bar{u}(\hat{t}(m)) \geqslant u(\hat{t}(m), m, a(\hat{t}, m))$ and hence $r(\hat{t}(m), m) \geqslant a(\hat{t}(m), m)$. In order to obtain a contradiction assume that there exists $s < \hat{t}(m)$ such that

$$u^*(s) < u(s, m, a(\hat{t}(m), m)). \tag{31}$$

There exists $\beta \in (0, 1]$ such that if $m' = (1-\beta)m + \beta C$, then

$$u^*(t') \geqslant u(t', m', a(\hat{t}(m), m')) \qquad \text{for all} \quad t' < \hat{t}(m) \tag{32}$$

and

$$u^*(t) = u(t, m', a(\hat{t}(m), m')) \qquad \text{for some} \quad t < \hat{t}(m). \tag{33}$$

Such a β exists by Lemma 4.2 and (31) since $u(t', m, a(t', m))$ is continuous in m by A0 and A2. By (32), m' is feasible for $Q(\hat{t}(m))$. Therefore,

$$u(\hat{t}(m), m, r(\hat{t}(m), m)) = \bar{u}^*(\hat{t}(m)) \geqslant \bar{u}(\hat{t}(m)) \geqslant u(\hat{t}(m), m', a(\hat{t}(m), m')). \tag{34}$$

However, by (33) there exists $t < \hat{t}(m)$ such that

$$u(t, m, r(\hat{t}(m), m)) < u(t, m, r(t, m)) = \bar{u}^*(t) = u(t, m', a(\hat{t}(m), m')). \tag{35}$$

The inequality follows from A1' since $r(t, m) > r(\hat{t}(m), m)$ by the definition of $\hat{t}(m)$ and the first equality follows from the definition of $r(t, m)$. Since $t < \hat{t}(m)$ and $m < m'$, (34) and (35) contradict A4. Therefore, $a^*(m) \leqslant r(\hat{t}(m), m)$ for $\hat{t}(m) \leqslant \bar{t}$. To conclude the proof, we need only note that A4 guarantees that $r(t, m) \geqslant r(\bar{t}, m)$ for all $t > \bar{t}$. Hence, $a^*(m) \leqslant r(\hat{t}(m), m)$ for all $\hat{t}(m)$.

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