Gale, David (1921-2008)*

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*I thank Harold Kuhn and Bernhard von Stengel for helpful comments. Sobel [16] is a more detailed overview of David Gale's research contributions.

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Abstract

The article reviews the research of David Gale, who made lasting contributions to game theory, general-equilibrium theory, and growth theory. In addition to his influence on the development of economic theory, his work has had important implications for many branches of mathematics and on mathematical education.
Keywords: assignment problem; competitive equilibrium; convexity; Debreu, G.; existence of competitive equilibrium; Kuhn, H.; matching; linear inequalities; von Neumann; Nikaido, H.; Ramsey, F.; Shapley, L.; Tucker, A. W.

1 Article

Gale, David 1921–2008

David Gale was born in New York on December 13, 1921 and died in Berkeley on March 7, 2008. He received an undergraduate degree from Swarthmore and a masters degree from the University of Michigan before earning a Ph.D. in Mathematics at Princeton. It was at Michigan, under the influence of Professor Norman Steenrod, that Gale decided to give up his study of physics and pursue a Ph.D. in Mathematics. He taught at Brown University from 1950 through 1965 and then joined the faculty at the University of California, Berkeley. His principal appointment was in the Mathematics Department, but he maintained affiliations with the departments of Economics and Industrial Engineering.

Gale won wide recognition for his research. His awards included a Fulbright Research Fellowship, two Guggenheim Fellowships, elections to the American Academy of Arts and Science and the National Academy of Science, the Lester Ford Prize (for outstanding mathematical exposition), the John von Neumann Theory Prize (for fundamental contributions to operations research), and the Pirelli Internetional Award (for the Internet Mathematics Museum "MathSite").

Mukul Majumdar [7] edited the volume *Equilibrium and Dynamics: Essays in Honour of David Gale*. The *International Journal of Game Theory*, Volume 36, Numbers 3-4, March, 2008 contains a collection of papers dedicated to David Gale on the occasion of his 85th birthday. This volume is edited by Marilda Sotomayor, who had also organized a scientific day in David's honor during the 18th Summer Festival on Game Theory in Stony Brook, July 12/13, 2007. Special issues of *Games and Economic Behavior* and *The Mathematical Intelligencer* are in preparation.

Gale lived in Berkeley, CA and Paris, France with his partner Sandra Gilbert, renowned feminist literary scholar and poet. Her 2000 book of poetry *Kissing the Bread* included a section of poems she wrote for Gale called "When she was kissed by the mathematician." He had three daughters and two grandsons. Julie Gale, his former wife and the mother of his daughters, died in February 2008.

2 Linear Inequalities

As a graduate student at Princeton, David Gale worked with classmate Harold Kuhn on a research project supervised by Professor Albert Tucker. At the time, there was considerable excitement about the new fields of zero-sum game theory and linear programming, but the mathematics of linear inequalities had not been developed. Existing proofs of the minimax theorem of zero-sum game theory required fixed point arguments and did not make the relationship between the theory and linear inequalities explicit. The project led to important results that identified the deep connections between the two new areas. Gale, Kuhn, and Tucker (1951) contained the first complete proof of the Duality Theorem of Linear Programming and used the theorem to prove the minimax theorem of zero-sum, two-person game theory. This paper uses convex analysis rather than fixed-point arguments to prove the minimax theorem and, implicitly, provided a computational foundation for equilibrium points in zero-sum games.

Gale (1960)'s book *The Theory of Linear Economic Models* contains central results on the theory of linear inequalities, including Gale, Kuhn, and Tucker (1951) and Gale (1956a)'s extension of von Neumann's model of an expanding economy. It discusses Dantzig's simplex algorithm and gives an economic interpretation to canonical problems. The book also contains a concise and elementary introduction to the theory of linear inequalities (including a proof of the separating hyperplane for convex polytopes), a chapter containing essential results on non-negative matrices, and a clean treatment of dynamic linear models of growth.

Largely in recognition of their joint work, Gale, Kuhn, and Tucker won the 1980 von Neumann prize for work they began in the late 1940s. Their citation stated that they "played a seminal role in laying the foundations of game theory, linear and non-linear programming – work that continues to be of fundamental importance to modern operations research and management science."

3 Infinite Games

Early work on non-cooperative game theory concentrated on two-player, zero-sum games. When players had finitely many pure strategies, these games were well understood: All two-player finite zero-sum games have a value and by playing to maximize his minimum payoff, a player could guarantee this value independent of his opponent's strategy. Gale and Stewart 1953) studied a class of infinite zero-sum games and demonstrated that the minmax theorem need not hold in this more general setting. The paper examines the simplest possible infinite zero-sum game. In the two-player game of perfect information, the players take turns naming binary digits, which can be thought of as the binary expansion of a number between zero and one. The first player wins if the expansion is an element of a pre-specified set. Otherwise, the second player wins. Gale and Stewart show that basic results from finite games hold if the prespecified set is closed, but that in general the game does not have a value.

The class of infinite games introduced by Gale and Stewart have had broad implications for mathematics. Gale and Stewart's result led to research that identified a general set of games that do have values, culminating in a theorem of Martin [8]. The fact that some games do not have values led to developments in set theory (Mycielski [9]).

4 Growth

Gale (1956) and (1960, pages) generalize and simplify the von Neumann [10] (von Neumann [11] is the English translation) model of an expanding economy in what in now called the von Neumann-Gale model of growth. Gale made two essential additions to the original model. He substituted von Neumann's requirement that each production process involves each good in the economy (as either input or output), with a weaker, more plausible condition.

Gale (1967) provides the definitive treatment of the multi-good Ramsey problem, which is a generalization of the linear von Neumann-Gale model. An agent starts with a given endowment, which must be allocated between immediate consumption and investment. What the agent invests is transformed, via a given technology, into the next period's endowment, which again may be allocated between immediate consumption and investment. The process continues indefinitely. The agent cares about the (undiscounted) sum of utility received from consumption. The problem is to find the appropriate definition for optimality and to characterize optimal consumption paths. Gale presents an appropriate optimality condition, provides conditions under which optimal paths exist, and characterizes these paths in terms of "turnpike" properties. Roughly speaking, one can construct an optimal program with two phases: a bounded initial transition phase in which the state is built up to (approximate) a sustainable optimal steady state followed by a program that approximates the best steady state consumption.

5 General Equilibrium

Gale make several important contributions to the foundations of general-equilibrium theory. Indeed, he made basic contributions to the three central issues of the theory: existence, uniqueness, and stability.

Gale (1955) contains a result known as the Gale-Debreu-Nikaido Lemma (Debreu [2] and Nikaido [12]) that contains the essential mathematical result needed to prove existence of market equilibrium. Gale and Nikaido (1965) proves a theorem on the global univalence of differentiable mappings on $\mathbb{R}^n$. When translated into a general-equilibrium context, the theorem gives sufficient conditions for equilibrium prices to be unique (see, for example, Arrow and Hahn [1, Chapter 9]).

Gale (1963) provides an early robust example of global instability of the tâtonnement process in general equilibrium.

Gale and Mas-Colell (1975) provides an existence theorem in an economy without ordered preferences.

6 College Admissions and the Stability of Marriage

Gale's paper with Lloyd Shapley on the Stable Marriage Problem, Gale and Shapley (1962), is his most cited, and probably most influential work. Detailed overviews of the research appear in Knuth [5] (English translation: Knuth [6]), Gale [3], Roth [13], and Roth and Sotomayor [14].

The short, deceptively simple paper is important for several reasons. The motivation for the problem comes from the real world. Gale (2001) describes how the idea for the problem came from thinking about the college-application process. The translation of the practical problem into mathematics captures many important considerations but remains extraordinarily simple. The solution to the problem is not obvious, but is easy to understand. The framework lends itself to modifications that lead to insight into more complicated practical problems.

The basic problem is how to create an assignment of items from one group to items from another. The groups can be men and woman (the marriage problem), workers to jobs (labor-market matching), or students to universities (college admissions). For concreteness, consider the marriage problem, in which it is natural to impose the constraint that there are equal numbers of men and women and the desired matching is one to one. Assume everyone has preferences over potential partners (so each man can order the women from the most preferred to least preferred marriage partner and likewise each woman can order the men). Finding a match is simple. One can order them by age and match the youngest man to the youngest woman, the second youngest man to the second youngest woman, and so on. Gale and Shapley looked for a matching in which unmatched pair would prefer each other to their current partner. If this property failed, you would expect the matching to be unstable. Gale and Shapley show that stable matchings exist and present a simple algorithm that constructs stable matches.

Starting in 1951, eleven years before the publication of the Gale-Shapley paper, the National Intern Matching Program used an essentially equivalent algorithm to match graduating medical students to hospital residency programs (see Roth and Sotomayor [14, pages 169-170] and Roth [13, Appendix] for a discussion of the independent development of the matching algorithm). Practical problems in a wide variety of areas (from school assignment to kidney exchange) continue to stimulate the development of matching theory.

7 Assignment Markets and Auctions

Gale's work shows a sensitivity to computational issues. Knowing the connection between zero-sum, two-person game theory and the theory of linear programming combined with computational methods (like the simplex method) provides a tractable method for computing equilibria in two-player games. Demonstrating the equivalence between an equilibrium and the solution to a well behaved optimization problem is the reason that equilibria in linear economies studied by Eisenberg and Gale (1959) can be found efficiently. Gale's work on markets with indivisible goods is another example of a situation in which Gale adds just enough structure to a general model to obtain strong results.

Shapley and Shubik [15] introduce the assignment model, a market equilibrium model with indivisible goods. Their model has the structure of a matching game with the added feature that agents can exchange a divisible commodity, money. Demange and Gale (1985) show that this market inherits many properties from the college admissions problem. Demange, Gale, and Sotomayor (1986) applies the framework to the study of multi-unit auctions and show how to define variations of the Vickrey auction to the multi-good setting.

8 Other Contributions

Gale (1956b) contains a lasting contribution to the study of convex polyhedra, introducing what are now known as "Gale transforms" and "Gale diagrams" (see Grünbaum [4]).

Gale (2009) describes the board games invented by Gale and his contemporaries at Princeton. Gale's article provides an introduction to Bridg-It (or, as Martin Gardner called it, the "Game of Gale") and also John Nash's game of Hex. Gale (1974) invented the game of Chomp, a simple two-player game of perfect information in which it is easy to show one player has a winning strategy, but the winning strategy is hard to find in general.

9 Mathematical Explorations

Gale made examples of beautiful mathematical arguments accessible to a broad audience.

Between 1991 and 1996 he wrote a column entitled *Mathematical Explorations* for *The Mathematical Intelligencer*. The columns, collected in a book titled *Tracking the Automatic Ant* (Gale (1998)), are in the tradition of Martin Gardner's long-running "Mathematical Games" column in *Scientic American*. He also developed MathSite, a pedagogic website that uses interactive exhibits to illustrate important mathematical ideas. MathSite won the 2007 Pirelli Internetional Award for Science Communication in Mathematics.

10 See Also

- assignment games
- auctions

- convex analysis
- existence of market equilibrium
- Gale diagram
- Gale-Shapley algorithm
- Gale transform
- game theory
- global univalence
- linear inequalities
- linear programming
- matching models
- Mathsite
- polytope
- Ramsey problem
- stable-marriage problem
- turnpike theorem
- Vickery auction
- von Neumann-Gale model
- zero-sum game theory

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