# Econ 205 - Slides from Lecture 8

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# Computational Facts

1. $\det \mathbf{AB} = \det \mathbf{BA} = \det \mathbf{A} \det \mathbf{B}$
2. If $\mathbf{D}$ is a diagonal matrix, then $\det \mathbf{D}$ is equal to the product of its diagonal elements.
3. $\det \mathbf{A}$ is equal to the product of the eigenvalues of $\mathbf{A}$.
4. The *trace* of a square matrix $\mathbf{A}$ is equal to the sum of the diagonal elements of $\mathbf{A}$. That is, $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$. Fact: $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$, where $\lambda_i$ is the $i$th eigenvalue of $\mathbf{A}$ (eigenvalues counted with multiplicity).

# Symmetric Matrices Have Orthonormal E-vectors

### Theorem

*If* $\mathbf{A}$ *is symmetric, then we take take the eigenvectors of* $\mathbf{A}$ *to be orthonormal. In this case, the* $\mathbf{P}$ *in the previous theorem has the property that* $\mathbf{P}^{-1} = \mathbf{P}^t$*.*

# Why Eigenvalues?

1. They play a role in the study of stability of difference and differential equations.
2. They make certain computations easy.
3. They make it possible to define a sense in which matrices can be positive and negative that allows us to generalize the one-variable second-order conditions.

# Quadratic Forms

## Definition

A *quadratic form* in $n$ variables is any function $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ that can be written $Q(\mathbf{x}) = \mathbf{x}^t \mathbf{A} \mathbf{x}$ where $\mathbf{A}$ is a symmetric $n \times n$ matrix.

When $n = 1$ a quadratic form is a function of the form $ax^2$. When $n = 2$ it is a function of the form

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

(remember $a_{12} = a_{21}$). When $n = 3$, it is a function of the form

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

A quadratic form is second-degree polynomial that has no constant term.

# Classification

### Definition

A quadratic form $Q(\mathbf{x})$ is

1. *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$.
2. *positive semi definite* if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$.
3. *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$.
4. *negative semi definite* if $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x}$.
5. *indefinite* if there exists $\mathbf{x}$ and $\mathbf{y}$ such that $Q(\mathbf{x}) > 0 > Q(\mathbf{y})$.

# Interpretation

- Generalized "positivity."
- Check one-variable case.
- Lots of indefinite matrices when $n > 1$.
- Think about diagonal matrices for intuition. $Q(\mathbf{x}) = \mathbf{x}^t \mathbf{A} \mathbf{x} = \sum_{i=1}^n a_{ii} x_i^2$. This quadratic form is positive definite if and only if all of the $a_{ii} > 0$, negative definite if and only if all of the $a_{ii} < 0$, positive semi definite if and only if $a_{ii} \geq 0$, for all $i$ negative semi definite if and only if $a_{ii} \leq 0$ for all $i$, and indefinite if $\mathbf{A}$ has both negative and positive diagonal entries.

# Quadratic Forms and Diagonalization

- Assume $\mathbf{A}$ symmetric matrix. It can be written $\mathbf{A} = \mathbf{R}^t\mathbf{D}\mathbf{R}$, where $\mathbf{D}$ is a diagonal matrix with (real) eigenvalues down the diagonal and $\mathbf{R}$ is an orthogonal matrix.
- $Q(\mathbf{x}) = \mathbf{x}^t\mathbf{A}\mathbf{x} = \mathbf{x}^t\mathbf{R}^t\mathbf{D}\mathbf{R}\mathbf{x} = (\mathbf{R}\mathbf{x})^t\,\mathbf{D}\,(\mathbf{R}\mathbf{x})$.
- The definiteness of $\mathbf{A}$ is equivalent to the definiteness of its diagonal matrix of eigenvalues, $\mathbf{D}$.

# Theorem on Definiteness

Theorem

*The quadratic form* $Q(\mathbf{x}) = \mathbf{x}^t \mathbf{A} \mathbf{x}$ *is*

1. *positive definite if* $\lambda_i > 0$ *for all* $i$.
2. *positive semi definite if* $\lambda_i \geq 0$ *for all* $i$.
3. *negative definite if* $\lambda_i < 0$ *for all* $i$.
4. *negative semi definite if* $\lambda_i \leq 0$ *for all* $i$.
5. *indefinite if there exists* $j$ *and* $k$ *such that* $\lambda_j > 0 > \lambda_k$.

# Computational Trick

### Definition

A *principal submatrix* of a square matrix $\mathbf{A}$ is the matrix obtained by deleting any $k$ rows and the corresponding $k$ columns. The determinant of a principal submatrix is called the *principal minor* of $\mathbf{A}$. The *leading principal submatrix of order k* of an $n \times n$ matrix is obtained by deleting the last $n - k$ rows and column of the matrix. The determinant of a leading principal submatrix is called the *leading principal minor* of $\mathbf{A}$.

# Definiteness Tests

## Theorem

*A matrix is*

1. *positive definite if and only if all of its leading principal minors are positive.*
2. *negative definite if and only if its odd principal minors are negative and its even principal minors are positive.*
3. *indefinite if one of its $k$th order leading principal minors is negative for an even $k$ or if there are two odd leading principal minors that have different signs.*

The theorem permits you to classify the definiteness of matrices without finding eigenvalues.
The conditions make sense for diagonal matrices.

# Multivariable Calculus

Goal: Extend the calculus from real-valued functions of a real variable to general functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.
Raising the dimension of the range space: easy.
Raising the dimension of domain: some new ideas.

# Linear Structures

- In $\mathbb{R}^2$ three linear subspaces: point, lines, entire space.
- In general, we'll care about 1- and $(n-1)$-dimensional subsets of $\mathbb{R}^n$.

# Analytic Geometry

## Definition

A *line* is described by a point $x$ and a direction $v$. It can be represented as $\{z : \text{ there exists } t \in \mathbb{R} \text{ such that } z = x + tv\}$.

If we constrain $t \in [0, 1]$ in the definition, then the set is the line segment connecting $x$ to $x + v$. Two points still determine a line: The line connecting $x$ to $y$ can be viewed as the line containing $x$ in the direction $v$. You should check that this is the same as the line through $y$ in the direction $v$.

## Definition

A *hyperplane* is described by a point $x_0$ and a *normal* direction $p \in \mathbb{R}^n$, $p \neq 0$. It can be represented as $\{z : p \cdot (z - x_0) = 0\}$. $p$ is called the normal direction of the plane.

A hyperplane consists of all of the $z$ with the property that the direction $z - x_0$ is normal to $p$.

In $\mathbb{R}^2$ lines are hyperplanes. In $\mathbb{R}^3$ hyperplanes are "ordinary" planes.
Lines and hyperplanes are two kinds of "flat" subset of $\mathbb{R}^n$.
Lines are subsets of dimension one.
Hyperplanes are subsets of dimension $n - 1$ or *co-dimension* one.
You can have a flat subsets of any dimension less than $n$.
Lines and hyperplanes are not subspaces (because they do not contain the origin) you obtain these sets by "translating" a subspace that is, by adding the same constant to all of its elements.

## Definition

A *linear manifold* of $\mathbb{R}^n$ is a set $S$ such that there is a subspace $V$ on $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ with $S = V + \{x_0\}$.

In the above definition,
$V + \{x_0\} \equiv \{y : y = v + x_0 \text{ for some } v \in V\}$.
Officially lines and hyperplanes are linear manifolds and not linear subspaces.

# Review

Given two points $x$ and $y$, you can construct a line that passes through the points.

$$\{z : z = x + t(y - x) \text{ for some } t.\}$$

Two-dimensional version:

$$z_1 = x_1 + t(y_1 - x_1) \text{ and } z_2 = x_2 + t(y_2 - x_2).$$

If you use the equation for $z_1$ to solve for $t$ and substitute out you get:

$$z_2 = x_2 + \frac{(y_2 - x_2)(z_1 - x_1)}{y_1 - x_1}$$

or

$$z_2 - x_2 = \frac{y_2 - x_2}{y_1 - x_1}(z_1 - x_1),$$

which is the standard way to represent the equation of a line (in the plane) through the point $(x_1, x_2)$ with slope $(y_2 - x_2)(y_1 - x_1)$.

# Conclusion

The "parametric" representation is essentially equivalent to the standard representation in $\mathbb{R}^2$. (Why essentially?)
You need two pieces of information to describe a line:

- Point and Direction.
  or
- Two points. (Obtain direction by subtracting the points.)

# Hyperplane

- Given a point and (normal)direction.
- Three points in $\mathbb{R}^3$ ($n$ points in $\mathbb{R}^n$), provided that they are in "general position."
- Going from points to normal: Solve linear system (or remember computational trick).

# Example

Given $(1, 2 - 3), (0, 1, 1), (2, 1, 1)$ find $A, B, C, D$ such that

$$Ax_1 + Bx_2 + Cx_3 = D$$

$$\begin{array}{ccccccc} A & + & 2B & - & 3C & = & D \\ & & B & + & C & = & D \\ 2A & + & B & + & C & = & D \end{array}$$

Doing so yields $(A, B, C, D) = (0, .8D, .2D, D)$. (If you find one set of coefficients that work, any non-zero multiple will also work.) Hence an equation for the plane is: $4x_2 + x_3 = 5$ you can check that the three points actually satisfy this equation.

# Alternatively

- Look for a normal direction.
- A normal direction is a direction that is orthogonal to **all** directions in the plane.
- A direction in the plane is a direction of a line in the plane.
- You can get such a direction by subtracting any two points in the plane.
- A two dimensional hyperplane will have two independent directions.
- One direction can come from the difference between the first two points: $(1, 1, -4)$.
- The other can come from the difference between the second and third points $(-2, 0, 0)$.
- Now find a normal to both of them. That is, a $p$ such that $p \neq 0$ and $p \cdot (1, 1, -4) = p \cdot (-2, 0, 0) = 0$. This is a system of two equations and three variables. All multiples of $(0, 4, 1)$ solve the equations.

# Linear Functions

Definition

A function

$$L\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is *linear* if and only if

1. If for all $x$ and $y$, $f(x+y) = f(x) + f(y)$ and
2. for all scalars $\lambda, f(\lambda x) = \lambda f(x)$.

# Discussion

1. Linear functions are additive.
2. Linear functions exhibit constant returns to scale.
3. If $L$ is a linear function, then $L(0) = 0$ and, more generally, $L(x) = -L(-x)$.
4. Any linear function can be "represented" by matrix multiplication. Given a linear function, compute $L(e_i)$, where $e_i$ is the $i$th standard basis element. Call this $a_i$ and let $\mathbf{A}$ be the square matrix with $i$th column equal to $a_i$. $\mathbf{A}$ must have $n$ columns and $m$ rows and $L(x) = \mathbf{A}x$ for all $x$.
5. Multiplication of matrices is composition of functions.