Econ 205 - Slides from Lecture 7

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Linear Algebra: Main Theory

A *linear combination* of a collection of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a vector of the form $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i$ for some scalars $\lambda_1, \ldots, \lambda_k$.
$\mathbb{R}^n$ has the property that sums and scalar multiples of elements of $\mathbb{R}^n$ remain in the set. Hence if we are given some elements of the set, all linear combinations will also be in the set. Some subsets are special because they contain no redundancies:

Definition

A collection of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is *linearly independent* if $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i = \mathbf{0}$ if and only if $\lambda_i = 0$ for all i.

Definition

The *span* of a collection of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$, $S(X)$, is the set of all linear combinations of $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

$S(X)$ is the smallest vector space containing all of the vectors in X.

Theorem

If $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ *is a linearly independent collection of vectors and* $\mathbf{z} \in S(X)$*, then there are unique* $\lambda_1, \ldots, \lambda_k$ *such that* $\mathbf{z} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i$.

Proof.

Existence follows from the definition of span. Suppose that there are two linear combinations that of the elements of X that yield $\mathbf{z}$ so that

$$\mathbf{z} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i$$

and

$$\mathbf{z} = \sum_{i=1}^{k} \lambda_i' \mathbf{x}_i.$$

Subtract the equations to obtain:

$$\mathbf{0} = \sum_{i=1}^{k} \left(\lambda_i' - \lambda_i \right) \mathbf{x}_i.$$

By linear independence, $\lambda_i = \lambda_i'$ for all i, the desired result. □

Definition

The *dimension* of a vector space is N, where N is the smallest number of vectors needed to span the space.

$\mathbb{R}^n$ has dimension n.

Definition

A *basis* for a vector span V is any collection of linearly independent vectors than span V.

Theorem

If $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a set of linearly independent vectors that does not span V, then there exists $\mathbf{v} \in V$ such that $X \cup \{\mathbf{v}\}$ is linearly independent.

Proof.

Take $\mathbf{v} \in V$ such that $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \notin S(X)$. $X \cup \{\mathbf{v}\}$ is a linearly independent set. To see this, suppose that there exists λ_i $i = 0, \ldots, k$ such that at least one $\lambda_i \neq 0$ and

$$\lambda_0 \mathbf{v} + \sum_{i=1}^{k} \lambda_i \mathbf{x}_i. \tag{1}$$

If $\lambda_0 = 0$, then X are not linearly independent. If $\lambda_0 \neq 0$, then equation (1) can be rewritten

$$\mathbf{v} = \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_0} \mathbf{x}_i.$$

In either case, we have a contradiction. □

Definition

The *standard basis* for $\mathbb{R}^n$ consists of the set of N vectors e_i, $i = 1, \ldots, N$, where e_i is the vector with component 1 in the ith position and zero in all other positions.

1. Standard basis is a linearly independent set that spans $\mathbb{R}^n$.
2. Elements of the standard basis are mutually orthogonal. When this happens, we say that the basis is *orthogonal*.
3. Each basis element has unit length. When this also happens, we say that the basis is *orthonormal*.

Orthonormal Bases

1. We know an orthonormal basis for $\mathbb{R}^n$.
2. It is always possible to find an orthonormal basis for a vector space.
3. If $\{v^1, \ldots, \mathbf{v}^k\}$ is an orthonormal basis for V then for all $\mathbf{x} \in V$,

$$\mathbf{x} = \sum_{i=1}^{k} \mathbf{v}^i \left(\mathbf{x} \cdot \mathbf{v}^i \right)$$

(To prove this, multiple both sides by $\mathbf{v}^i$.)

Basis Properties

It is not hard (but a bit tedious) to prove that all bases have the same number of elements. (This follows from the observation that any system of n homogeneous equations and $m > n$ unknowns has a non-trivial solution, which in turn follows from "row-reduction" arguments.)

Eigenvectors and Eigenvalues

Definition

An *eigenvalue* of the square matrix $\mathbf{A}$ is a number λ with the property $\mathbf{A} - \lambda \mathbf{I}$ is singular. If λ is an eigenvalue of $\mathbf{A}$, then any $\mathbf{x} \neq \mathbf{0}$ such that $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$ is called an *eigenvector* of $\mathbf{A}$ associated with the eigenvalue λ.

- Eigenvalues are those values for which the equation

  $$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

  has a non-zero solution.
- Eigenvalues solve the equation $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.

Characteristic Polynomial

If $\mathbf{A}$ is an $n \times n$ matrix, then this *characteristic equation* is a polynomial equation of degree n. By the Fundamental Theorem of Algebra, it will have n (not necessarily distinct and not necessarily real) roots. That is, the characteristic polynomial can be written

$$P(\lambda) = (\lambda - r_1)^{m_1} \cdots (\lambda - r_k)^{m_k},$$

where $r_1, r_2 \ldots, r_k$ are the distinct roots ($r_i \neq r_j$ when $i \neq j$) and m_i are positive integers summing to n. We call m_i the *multiplicity* of root r_i. Eigenvalues and their corresponding eigenvectors are important because they enable one to relate complicated matrices to simple ones.

Diagonalization

Theorem

If $\mathbf{A}$ *is an* $n \times n$ *matrix that has* n *distinct eigen-values or is symmetric, then there exists an invertible* $n \times n$ *matrix* $\mathbf{P}$ *and a diagonal matrix* $\mathbf{D}$ *such that* $\mathbf{A} = \mathbf{PDP}^{-1}$*. Moveover, the diagonal entries of* $\mathbf{D}$ *are the eigenvalues of* $\mathbf{A}$ *and the columns of* $\mathbf{P}$ *are the corresponding eigenvectors.*

1. Certain square matrices are "similar to" a diagonal matrix.
2. The diagonal matrix has eigenvalues down the diagonal.
3. Similarity relationship:

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

Sketch of Proof

Suppose that λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ is an eigenvector. This means that $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$. If $\mathbf{P}$ is a matrix with column j equal to an eigenvector associated with λ_i, it follows that $\mathbf{AP} = \mathbf{PD}$. The theorem would follow if we could guarantee that $\mathbf{P}$ is invertible. When A is symmetric, one can prove that A has only real eigenvalues and that one can find n linearly independent eigenvectors even if the eigenvalues are not distinct. This result is elementary (but uses some basic facts about complex numbers).

The eigenvectors of distinct eigenvalues are distinct.

To see this, suppose that $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues and $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are associated eigenvectors. In order to reach a contradiction, suppose that the vectors are linearly dependent.

Without loss of generality, we may assume that $\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$ are linearly independent, but that $\mathbf{x}_k$ can be written as a linear combination of the first $k-1$ vectors. This means that there exists α_i $i = 1, \ldots, k-1$ not all zero such that:

$$\sum_{i=1}^{k-1} \alpha_i \mathbf{x}_i = \mathbf{x}_k. \tag{2}$$

Multiply both sides of equation (2) by $\mathbf{A}$ and use the eigenvalue property to obtain:

$$\sum_{i=1}^{k-1} \alpha_i \lambda_i \mathbf{x}_i = \lambda_k \mathbf{x}_k. \tag{3}$$

Multiply equation (2) by λ_k and subtract it from equation (3) to obtain:

$$\sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) \mathbf{x}_i = \mathbf{0}. \tag{4}$$

Since the eigenvalues are distinct, equation (4) gives a non-trivial linear combination of the first $k-1$ $\mathbf{x}_i$ that is equal to $\mathbf{0}$, which contradicts linear independence.