

# Econ 205 - Slides from Lecture 6

Joel Sobel

August 30, 2010

# Linear Algebra Basics

## Definition

$n$ -dimensional Euclidean Space:

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}$$

If  $X$  and  $Y$  are sets, then

$$X \times Y \equiv \{(x, y) \mid x \in X, y \in Y\}$$

so

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}, \forall i = 1, 2, \dots, n\}$$

## Interpretations:

- ▶ Calculus:  $x \in \mathbb{R}^n$  as a list of  $n$  real numbers;  $x = (x_1, \dots, x_n)$ .
- ▶ Linear algebra  $\mathbf{x} \in \mathbb{R}^n$  is column vector. When we do this we write

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

- ▶ Sometimes an element in  $\mathbb{R}^n$  is best thought of as a direction.

## Definition

The zero element:

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

## Definition (Vector Addition)

For  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  we have

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

Vector addition is commutative

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

## Definition (Scalar Multiplication)

For  $\mathbf{x} \in \mathbb{R}^n$ , and  $a \in \mathbb{R}$  we have

$$a\mathbf{x} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{pmatrix}$$

In other words every element of  $\mathbf{x}$  gets multiplied by  $a$ .

# Vector Spaces

## Definition (Vector Space)

A vector space is a set  $\mathcal{V}$  in which the operation of addition and multiplication by a scalar make sense, and in which operations are commutative and associative. In addition:

$\mathbf{0} + \mathbf{v} = \mathbf{v}$  (additive identity)

for each  $\mathbf{v}$  there is a  $-\mathbf{v}$  (additive inverse)

A subset of a vector space that is itself a vector space is called a subspace.

Euclidean Spaces ( $\mathbb{R}^n$ ) are the leading example of vector spaces.

We will need to talk about subsets of Euclidean Spaces that have a linear structure (they contain  $\mathbf{0}$ , and if  $\mathbf{x}$  and  $\mathbf{y}$  are in the set, then so is  $\mathbf{x} + \mathbf{y}$  and all scalar multiples of  $\mathbf{x}$  and  $\mathbf{y}$ ).

# Matrices

## Definition

An  $m \times n$  matrix is an element of  $\mathcal{M}^{m \times n}$  written as in the form

$$\mathbf{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = [\alpha_{ij}]$$

where  $m$  denotes the number of rows and  $n$  denotes the number of columns.

*Note* An  $m \times n$  matrix is just of a collection of  $nm$  numbers organized in a particular way. Hence we can think of a matrix as an element of  $\mathbb{R}^{m \times n}$ . The extra notation  $\mathcal{M}^{m \times n}$  makes it possible to distinguish the way that the numbers are organized.

*Note* Vectors are just a special case of matrices. e.g.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}^{n \times 1}$$

This notation emphasizes that we think of a vector with  $n$  components as a matrix with  $n$  rows and 1 column.



$$\mathbf{A}_{2 \times 3} = \begin{pmatrix} 0 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$$

## Definition

The *transpose* of a matrix  $\mathbf{A}$ , is denoted  $\mathbf{A}^t$ . To get the transpose of a matrix, we let the first row of the original matrix become the first column of the new (transposed) matrix.

$$\mathbf{A}^t = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{1n} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{1m} & \alpha_{2m} & \cdots & \alpha_{nm} \end{pmatrix} = [\alpha_{ji}]$$

## Definition

A matrix  $\mathbf{A}$  is *symmetric* if  $\mathbf{A} = \mathbf{A}^t$ .

So we can see that if  $\mathbf{A} \in \mathcal{M}^{m \times n}$ , then  $\mathbf{A}^t \in \mathcal{M}^{n \times m}$ .

Continuing the example, we see that

$$\mathbf{A}^t_{3 \times 2} = \begin{pmatrix} 0 & 6 \\ 1 & 0 \\ 5 & 2 \end{pmatrix}$$

# Matrix Algebra

[Addition of Matrices] If

$$\mathbf{A}_{m \times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = [\alpha_{ij}]$$

and

$$\mathbf{B}_{m \times n} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{pmatrix} = [\beta_{ij}]$$

then

$$\mathbf{A} + \mathbf{B} = \mathbf{D} = \begin{pmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} + \beta_{12} & \cdots & \alpha_{1n} + \beta_{1n} \\ \alpha_{21} + \beta_{21} & \alpha_{22} + \beta_{22} & \cdots & \alpha_{2n} + \beta_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} + \beta_{m1} & \alpha_{m2} + \beta_{m2} & \cdots & \alpha_{mn} + \beta_{mn} \end{pmatrix} = [\delta_{ij}]$$

$$\underbrace{\mathbf{A} + \mathbf{B}}_{m \times n} = \begin{pmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} + \beta_{12} & \cdots & \alpha_{1n} + \beta_{1n} \\ \alpha_{21} + \beta_{21} & \alpha_{22} + \beta_{22} & \cdots & \alpha_{2n} + \beta_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} + \beta_{m1} & \alpha_{m2} + \beta_{m2} & \cdots & \alpha_{mn} + \beta_{mn} \end{pmatrix} = [\alpha_{ij} + \beta_{ij}]$$

## Definition (Multiplication of Matrices)

If  $\mathbf{A}$  and  $\mathbf{B}$  are given, then we define

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{C} = [c_{ij}]$$

$m \times k$     $k \times n$     $m \times n$

such that

$$c_{ij} \equiv \sum_{l=1}^k a_{il} b_{lj}$$

so note above that the only index being summed over is  $l$ .

Let

$$\mathbf{A}_{2 \times 3} = \begin{pmatrix} 0 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$$

and

$$\mathbf{B}_{3 \times 2} = \begin{pmatrix} 0 & 3 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}$$

Then

$$\begin{aligned} \underbrace{\mathbf{A}_{2 \times 3} \cdot \mathbf{B}_{3 \times 2}}_{2 \times 2} &= \begin{pmatrix} 0 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3 \\ 1 & 0 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} (0 \times 0) + (1 \times 1) + (5 \times 2), & (0 \times 3) + (1 \times 0) + (5 \times 3) \\ (6 \times 0) + (0 \times 1) + (2 \times 2), & (6 \times 3) + (0 \times 0) + (2 \times 3) \end{pmatrix} \\ &= \begin{pmatrix} 11 & 15 \\ 4 & 24 \end{pmatrix} \end{aligned}$$

In general:

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

For example

$$\underset{2 \times 3}{\mathbf{A}} \cdot \underset{3 \times 4}{\mathbf{B}} \neq \underset{3 \times 4}{\mathbf{B}} \cdot \underset{2 \times 3}{\mathbf{A}}$$

(The product on the right is not defined.)

### Definition

Any matrix which has the same number of rows as columns is known as a square matrix, and is denoted  $\underset{n \times n}{\mathbf{A}}$ .



## Definition

There is a special square matrix known as the *identity matrix*. Any matrix multiplied by this *identity matrix* gives back the original matrix. The Identity matrix is denoted  $\mathbf{I}_n$  and is equal to

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

## Definition

A square matrix is called a *diagonal matrix* if  $a_{ij} = 0$  whenever  $i \neq j$ .

## Definition

A square matrix is called an *upper triangular matrix* (resp. *lower triangular*) if  $a_{ij} = 0$  whenever  $i > j$  (resp.  $i < j$ ).

Diagonal matrices are easy to deal with. Triangular matrixes are also somewhat tractable. You'll see that for many applications you can replace an arbitrary square matrix with a related diagonal matrix.

For any matrix  $\mathbf{A}$  we have the results that

$$\mathbf{A}_{m \times n} \cdot \mathbf{I}_n = \mathbf{A}_{m \times n}$$

and

$$\mathbf{I}_m \cdot \mathbf{A}_{m \times n} = \mathbf{A}_{m \times n}$$

## Definition

We say a matrix  $\mathbf{A}$  is *invertible* or *non-singular* if  $\exists \mathbf{B}$  such that

$$\underbrace{\mathbf{A} \cdot \mathbf{B}}_{n \times n} = \underbrace{\mathbf{B} \cdot \mathbf{A}}_{n \times n} = \mathbf{I}_n$$

If  $\mathbf{A}$  is invertible, we denote it's inverse as  $\mathbf{A}^{-1}$ .

So we get

$$\underbrace{\mathbf{A} \cdot \mathbf{A}^{-1}}_{n \times n} = \underbrace{\mathbf{A}^{-1} \cdot \mathbf{A}}_{n \times n} = \mathbf{I}_n$$

A square matrix that is not invertible is called *singular*.

## Definition

The *determinant* of a matrix  $\mathbf{A}$  (written  $\det \mathbf{A} = |\mathbf{A}|$ ) is defined inductively.

$$n = 1 \quad \mathbf{A} \\ (1 \times 1)$$

$$\det \mathbf{A} = |\mathbf{A}| \equiv a_{11}$$

$$n \geq 2 \quad \mathbf{A} \\ (n \times n)$$

$$\det \mathbf{A} = |\mathbf{A}| \equiv$$

$$a_{11}|\mathbf{A}_{-11}| - a_{12}|\mathbf{A}_{-12}| + a_{13}|\mathbf{A}_{-13}| - \cdots \pm a_{1n}|\mathbf{A}_{-1n}|$$

where  $\mathbf{A}_{-1j}$  is the matrix formed by deleting the first row and  $j$ th column of  $\mathbf{A}$ .

*Note*  $\mathbf{A}_{-1j}$  is an  $(n-1) \times (n-1)$  dimensional matrix.

## Examples

If

$$\mathbf{A}_{2 \times 2} = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
$$\implies \det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$$

If

$$\mathbf{A}_{3 \times 3} = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\implies \det \mathbf{A} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

# Determinant

The determinant is useful primarily because of the following result:

## Theorem

*A matrix is invertible if and only if its determinant  $\neq 0$ .*



## Definition

The *adjoint* of a matrix  $\mathbf{A}$  ( $\text{adj}(\mathbf{A})$ ) is the  $n \times n$  matrix with entry  $ij$  equal to

$$\text{adj } \mathbf{A} = (-1)^{i+j} \det \mathbf{A}_{-ij}$$

where  $\text{adj } \mathbf{A}$  is the adjoint of  $\mathbf{A}$ .

## Definition

The Inverse of a matrix  $\mathbf{A}$  is defined as

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \cdot \text{adj } \mathbf{A}$$

where  $\text{adj } \mathbf{A}$  is the *adjoint* of  $\mathbf{A}$ .

If  $\mathbf{A}$  is a  $(2 \times 2)$  matrix and invertible then

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

# Inner Product

## Definition (Inner Product)

If  $\mathbf{x}, \mathbf{y} \in \mathcal{M}^{n \times 1}$ , then the *inner product* (or dot product or scalar product) is given by

$$\begin{aligned}\mathbf{x}^t \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \\ &= \sum_{i=1}^n x_i y_i\end{aligned}$$

Note that  $\mathbf{x}^t \mathbf{y} = \mathbf{y}^t \mathbf{x}$ . We will have reason to use this concept when we do calculus, and will write  $x \cdot y = \sum_{i=1}^n x_i y_i$ .

# Distance

## Definition (Euclidean Distance)

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

where

$$\begin{aligned}\|\mathbf{z}\| &= \sqrt{z_1^2 + z_2^2 + \cdots + z_n^2} \\ &= \sqrt{\sum_{i=1}^n z_i^2}\end{aligned}$$

Under the Euclidean metric, the distance between two points is the length of the line segment connecting the points. We call  $\|\mathbf{z}\|$ , which is the distance between  $\mathbf{0}$  and  $\mathbf{z}$  the *norm* of  $\mathbf{z}$ .

Notice that  $\|\mathbf{z}\|^2 = \mathbf{z} \cdot \mathbf{z}$ .

## Inner Product Properties

When  $\mathbf{x} \cdot \mathbf{y} = 0$  we say that  $\mathbf{x}$  and  $\mathbf{y}$  are *orthogonal/at right angles/perpendicular*.

Two vectors are perpendicular if and only if their inner product is zero. This fact follows rather easily from “The Law of Cosines.” The law of cosines states that if a triangle has sides  $A, B$ , and  $C$  and the angle  $\theta$  opposite the side  $c$ , then

$$c^2 = a^2 + b^2 - 2ab \cos(\theta),$$

where  $a, b$ , and  $c$  are the lengths of  $A, B$ , and  $C$  respectively. This means that:

$$(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} - 2\|\mathbf{x}\|\|\mathbf{y}\| \cos(\theta),$$

where  $\theta$  is the angle between  $\mathbf{x}$  and  $\mathbf{y}$ .

If you multiply everything out you get the identity:

$$\|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta) = \mathbf{x}^t \mathbf{y}. \quad (1)$$

1. The inner product of two non-zero vectors is zero if and only if the cosine of the angle between them is zero.
2. Since the absolute value of the cosine is less than or equal to one,

$$\|\mathbf{x}\| \|\mathbf{y}\| \geq | \mathbf{x}^t \mathbf{y} | .$$

# Systems of Linear Equations

Consider the system of  $m$  equations in  $n$  variables:

$$y_1 = \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n$$

$$y_2 = \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n$$

$$\vdots$$

$$y_i = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n$$

$$\vdots$$

$$y_m = \alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n$$

Here the variables are the  $x_j$ .

This can be written as

$$\underset{(m \times 1)}{\mathbf{y}} = \underbrace{\underset{(m \times n)}{\mathbf{A}} \cdot \underset{(n \times 1)}{\mathbf{x}}}_{(m \times 1)}$$

where

$$\underset{(m \times 1)}{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \quad \underset{(n \times 1)}{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\underset{m \times n}{\mathbf{A}} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = [\alpha_{ij}]$$

or, putting it all together



$$\mathbf{y}_{(m \times 1)} = \underbrace{\mathbf{A}_{(m \times n)} \cdot \mathbf{x}_{(n \times 1)}}_{(m \times 1)}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

# Facts about solutions to linear equations

## Definition

A system of equations of the form  $\mathbf{Ax} = \mathbf{0}$  is called a *homogeneous* system of equations.

A homogeneous system always has a solution ( $\mathbf{x} = \mathbf{0}$ ). The solution will not be unique if there are more unknowns than equations or if there are as many equations as unknowns and  $\mathbf{A}$  is singular.

## Theorem

When  $\mathbf{A}$  is square, the system  $\mathbf{Ax} = \mathbf{y}$  has a unique solution if and only if  $\mathbf{A}$  is nonsingular.

If defined,  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$ .

If not, then there is a nonzero  $\mathbf{z}$  such that  $\mathbf{Az} = \mathbf{0}$ . This means that if you can find one solution to  $\mathbf{Ax} = \mathbf{y}$  you can find infinitely many.

# General Theory

Using “elementary row operations” reduce general system of equations to canonical form that permits you to identify solution set easily.