Econ 205 - Slides from Lecture 5

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Univariate Optimization

$$\max f(x) \text{ subject to } x \in S$$

1. Objective function – what you are trying to optimize (f).
2. Constraint Set – possible values (S).
3. Solution: x^* such that (a) $x^* \in S$ and (b) if $x \in S$, then $f(x^*) \geq f(x)$.
4. Value: $f(x^*)$
5. Values unique; solutions not necessarily.
6. If you can solve max problems, you can solve min problems.

What you know

- Sufficient conditions for when solutions exist (continuity of f; compactness of S).
- First order necessary conditions for local optima.

I'll call the points where $f'(x) = 0$ critical points.

Secord-Order Conditions

Theorem (Second-Order Conditions)

Let f be twice continuously differentiable on an open interval (a, b) and let $x^* \in (a, b)$ be a critical point of f. Then

1. *If x^* is a local maximum, then $f^{''}(x^*) \leq 0$.*
2. *If x^* is a local minimum, then $f^{''}(x^*) \geq 0$.*
3. *If $f^{''}(x^*) < 0$, then x^* is a local maximum.*
4. *If $f^{''}(x^*) > 0$, then x^* is a local minimum.*

Comments

1. Conditions (1) and (3) are almost converses (as are (2) and (4)), but not quite.
2. Conditions parallel the results about first derivatives and monotonicity stated earlier.

Proof

Proof.

By Taylor's Theorem we can write:

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2}f^{''}(t)(x - x^*)^2 \qquad (1)$$

for t between x and x^*. If $f^{''}(x^*) > 0$, then by continuity of $f^{''}$, $f^{''}(t) > 0$ for t sufficiently close to x^* and so, by $f(x) > f(x^*)$ for all x sufficiently close to x^*. Consequently, if x^* is a local maximum, $f^{''}(x^*) \leq 0$, proving (1).

If $f^{''}(x^*) < 0$, then by continuity of $f^{''}$, there exists $\delta > 0$ such that if $0 <| x - t |< \delta$, then $f^{''}(t) < 0$. So if $0 <| x - x^* |< \delta$, then $f(x) < f(x^*)$, which establishes (3). □

If f is defined on an interval (and is twice continuously differentiable), the maximum (if it exists) must occur either at a boundary point or at a critical point x^* that satisfies $f(x^*) \leq 0$.

Convex Set

Definition

A set C is *convex* if $\forall x, y \in C$ and $\delta \in (0, 1)$, $\delta x + (1 - \delta) y \in C$.

This definition describes intervals if $C \subset \mathbb{R}$, but applies more generally.

Concavity

Definition

We say a function f is *concave* over an interval $X \subset \mathbb{R}$ if $\forall x, y \in X$ and $\delta \in (0, 1)$, we have

$$f(\delta x + (1 - \delta)y) \geq \delta f(x) + (1 - \delta)f(y) \tag{2}$$

f is *strictly concave* if the inequality above is strict (whenever $\delta \in (0, 1)$).

Properties

- The graph of the function always lies above segments connecting two points on the graph.
- The graph of the function always lies below its tangents (when the tangents exist).
- Linear functions are concave but not strictly concave.
- Concave functions have nicely behaved sets of local maximizers: if x and y are local maxima, then so are all of the points on the line segment connecting x to y.

Quasiconcavity

Definition

We say a function f is *quasi concave* over an interval $X \subset \mathbb{R}$ if $\forall x, y \in X$ and $\delta \in (0,1)$, we have $f(\delta x + (1-\delta)y) \geq \min\{f(x), f(y)\}$.

There is a strict version of the definition.

Properties

1. Concave functions are quasi-concave (not conversely).
2. If x_1 and x_2 solve: $\max f(x)$ subject to $x \in [a, b]$ and f is quasi-concave, then so does $\delta x_1 + (1 - \delta)x_2$ for $\delta \in (0, 1)$.
3. Set of maximizers of quasiconcave functions is convex.
4. Strictly quasiconcave functions have unique maximizers.

Convex function

Definition

We say a function f is *convex* over an interval $X \subset \mathbb{R}$ if $\forall x, y \in X$ and $\delta \in (0,1)$, we have

$$f(\delta x + (1-\delta)y) \leq \delta f(x) + (1-\delta)f(y)$$

Properties

Theorem

Let $f : X \longrightarrow \mathbb{R}$, *$X$ an open interval and* $f \in C^2$ *on* X.
$f''(x) \leq 0, \quad \forall x \in X$, *if and only if* f *is concave on* X.

If f is concave, then for all $\lambda \in (0,1)$

$$\frac{f(\lambda x + (1-\lambda)y) - f(x)}{1-\lambda} \geq \frac{f(y) - f(\lambda x + (1-\lambda)y)}{\lambda}.$$

The limit of the left-hand side (if it exists) as $\lambda \to 1$ is equal to $(y-x)f'(x)$ (note that $f(\lambda x + (1-\lambda)y) - f(x) = f(x + (1-\lambda)(y-x)) - f(x))$ and the limit of the right-hand side $\lambda \to 1$ is equal to $(y-x)f'(y)$. Hence f is concave and differentiable implies that

$$(y-x)f'(x) \geq (y-x)f'(y),$$

which in turn implies that f' is decreasing.

Conversely, if f is differentiable, then by the Mean Value Theorem

$$f(\lambda x + (1-\lambda)y) - f(x) = (1-\lambda)f'(c)(y-x)$$

for some c between x and $\lambda x + (1-\lambda)y$. So

$$f(\lambda x + (1-\lambda)y) - f(x) \geq (1-\lambda)f'(\lambda x + (1-\lambda)y)(y-x). \quad (3)$$

Similarly,

$$f(\lambda x + (1-\lambda)y) - f(y) = -\lambda f'(c)(y-x)$$

for some c between $\lambda x + (1-\lambda)y$ and y and if f' is decreasing, then

$$f(\lambda x + (1-\lambda)y) - f(y) \geq -\lambda f'(\lambda x + (1-\lambda)y)(y-x). \quad (4)$$

The result follows from adding λ times inequality (3) to $1-\lambda$ times inequality (4).

- If f' is decreasing, then $f(y) - f(x) \leq f'(x)(y - x)$. This inequality is the algebraic way of expressing the fact that the tangent line to the graph of f at $(x, f(x))$ lies above the graph.
- If $f^{''} < 0$ so that f is strictly concave, then f' is strictly decreasing. Hence f can have at most one critical point. Since $f^{''} < 0$, this critical point must be a local maximum. Local maxima of strictly concave functions are global maxima and these must be global maxima.